AN INFORMATION STATE APPROACH TO NONLINEAR J-INNER/OUTER FACTORIZATION

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Abstract

In this paper we propose a new theory for obtaining J-inner/outer factorizations of nonlinear systems based on the recently developed information state framework for output feedback differential games and H_∞ control.

1. Introduction

In linear systems theory, there are strong connections between H_∞ control and J-inner/outer factorizations, see **[3]**. Essentially, the H_∞ problem for G below is solvable if and only if a related system $\tilde{G}$ admits a J-inner/outer factorization. This leads to a parameterization of all H_∞ controllers. Also when the H_∞ problem is solvable, it is solvable by a so-called central controller which is expressed explicitly in terms of a pair of Riccati equations. Formulas the J-inner/outer factorization can be obtained directly from solution of these Riccati equations. The purpose of this paper is to propose a new theory for obtaining parallel results for nonlinear systems.

Consider the following standard generalized regulator arrangement with nonlinear plant G and nonlinear controller K:

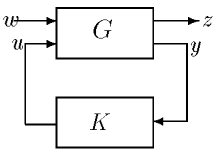

Figure 1.1: (G, K)

The information state approach to nonlinear output feedback differential games and H_∞ control (James-Baras **[9]**, **[10]**, **[11]**) gives analysis of and high level formulas for the central controller K^*.

In this paper we do two things.

(i) We "augment" the James-Baras formulas for central controller in a certain way to obtain formal formulas for the inner outer factor of a system $\tilde{G}$ cannonically related to G, viz., $\tilde{G} = \theta R$, where θ is to be the J-inner factor and R is the outer factor. It is important to note that the factors have *infinite dimensional* realizations in general. We are in the process of investigating the range of validity of this factoring and in this note list key properties which we have found so far. These formulas include as special cases all previously known results on J-inner/outer factorization for linear and nonlinear systems **[3]**, **[2]**. We use our formulas for factorizations to write down large numbers of solutions to the nonlinear H_∞ control problem. Whether or not this is all solutions remains open.

(ii) The second contribution of the paper is to tune the James-Baras solution to the nonlinear H_∞ control problem. Their solution and as well as certainty equivalence solutions (as introduced to H_∞ control in **[1]**) involves a PDE on the state space of the original system. We found that if one is not careful with how this PDE is initialized then one will obtain a closed loop system which while stable is not internally stable. Thus careful initialization is required to obtain a solution to the control problem. Improper initialization is analogous to linear situations where one builds a controller using the stablizing solution to the Doyle-Glover-Kargonekar-Francis X equation but a solution to the Y equation which while nonnegative is not stablizing. This typically leads to a "dissipative" closed loop system which is not internally stable (a practical disaster). Nonlinear H_∞ control papers to date have missed this mathematical subtlety. In this article we sketch the difficulty and characterize those initial conditions which produce internally stablizing controllers. Our characterization seems cumbersome at present and is in need of further development.

2. H_∞ Control

We consider nonlinear plants G of the form

$$G : \begin{cases} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= C_2(x) + D_{21}(x)w \end{cases} \tag{2.1}$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an output $y(t) \in \mathbf{R}^p$ is observed. The additional output quantity $z(t) \in \mathbf{R}^m$ is a performance measure, depending on the particular problem at hand. The control input is $u(t) \in \mathbf{R}^m$, while $w(t) \in \mathbf{R}^p$ is regarded as an opposing disturbance input. We assume that all of the functions appearing in (2.1) are as smooth and bounded as necessary, and that zero is an equilibrium: $A(0) = 0$, $C_1(0) = 0$ and $C_2(0) = 0$. In order to simplify the notation as much as possible, we take $D_{12}(x) = I_m$ and $D_{21}(x) = I_p$. This structure assumed of G is a nonlinear analogue of the *non-singular* case of linear systems (c.f. **[13]**). This restricts us to the "two block problem"; however our techniques are applicable more generally. The following signal spaces will be used: $L_2 = L_2([0,\infty), \mathbf{R}^q)$, $L_{2,T} = L_2([0,T], \mathbf{R}^q)$, and $L_{2,e} = \cup_{T>0} L_{2,T}$, where the dimension of the range space will not be stated explicity but infered from the context.

A *controller* K is a causal mapping $K : L_{2,e} \rightarrow L_{2,e}$ taking outputs y to inputs u. Such a controller will be termed *admissible* if the closed loop equations for G (2.1) with $u = K(y)$ are well defined in the sense that unique solutions exist in $L_{2,e}$. We will assume $K(0) = 0$. A controller K is said to solve the H_∞ control problem for G if the closed loop system (G, K) is (i) *internally stable* (the precise definition will be left open in the present discussion, except in §6), and (ii) *dissipative*, in the sense that there exist $\beta(x) \geq 0$ with $\beta(0) = 0$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_0^T |z(s)|^2 \, ds \;\leq\; \int_0^T |w(s)|^2 \, ds + \beta(x_0) \\ \text{for all } \; w \in L_{2,T} \text{ and all } T \geq 0. \end{array} \right. \tag{2.2}$$

The function β depends on K: $\beta = \beta^K$. Let $\mathcal{B}$ denote the class of functions of this type. (Sometimes *strict* dissipativity is required, in which case the RHS of (2.2) is multiplied by a factor $(1 - \varepsilon^2) < 1$.)

A system $\tilde{G}$ is obtained from G by reversing the w and y arrows; the equations defining $\tilde{G}$ are as follows:

$$\tilde{G} : \left\{ \begin{array}{lcl} \dot{\xi} & = & A(\xi) + B_1(\xi)(y - C_2(\xi)) + B_2(\xi) u \\ z & = & C_1(\xi) + u \\ w & = & -C_2(\xi) + y. \end{array} \right. \tag{2.3}$$

3. Information State Solution

Define a function space $\mathcal{X} = C(\mathbf{R}^n)$ (say). We will need a "completion" $\mathcal{X}_e$ of this space which includes extended real valued functions such as

$$\delta_x(z) = \left\{ \begin{array}{ll} 0 & \text{if } z = x, \\ -\infty & \text{if } z \neq x. \end{array} \right.$$

For $p \in \mathcal{X}_e$ and a controller K define the cost functional

$$J_{p_0}(K) = \sup_{T \geq 0} \sup_{w \in L_{2,T}} \sup_{x_0 \in \mathbf{R}^n} \left\{ p_0(x(0)) + \tfrac{1}{2} \int_0^T \left[|z(s)|^2 - |w(s)|^2 \right] \, ds \right\}.$$

Clearly, (G, K) is dissipative if and only if

$$J_{-\beta}(K) \leq 0, \text{ some } \beta \in \mathcal{B}. \tag{3.1}$$

For fixed $u, y \in L_{2,e}$, the *information state* $p_t \in \mathcal{X}_e$ is defined by

$$\begin{array}{l} p_t(x) = p_0(\xi(0)) + \frac{1}{2} \displaystyle\int_0^t \left[|C_1(\xi(s)) + u(s)|^2 \right. \\ \qquad \left. - |y(s) - C_2(\xi(s))|^2 \right] ds, \end{array} \tag{3.2}$$

where $\xi(\cdot)$ is the solution of (2.3) with *terminal* condition $\xi(t) = x$. If $p_0 \in \mathcal{X}$, then $p_t \in \mathcal{X}$ for all $t \geq 0$. (It may happen that $\lim_{t\rightarrow\infty} p_t \in \mathcal{X}_e \backslash \mathcal{X}$.) Using the information state, the dissipative property can be represented in terms of a new completely observed state, viz.

$$J_{p_0}(K) = \sup_{T \geq 0} \sup_{y \in L_{2,T}} \left\{ (p_T, 0) \right\}, \tag{3.3}$$

where $(p, q) \triangleq \sup_x (p(x) + q(x))$ is the "sup-pairing" **[11]**. Thus (G, K) is dissipative if and only if

$$\sup_{T \geq 0} \sup_{y \in L_{2,T}} \left\{ (p_T, 0) \right\} \leq 0, \tag{3.4}$$

where $p_0 = -\beta$ for some $\beta \in \mathcal{B}$.

The dynamics for p_t is a partial differential equation: for fixed $u \in L_{2,e}$ and $y \in L_{2,e}$ we have

$$\dot{p}_t = F(p_t, u(t), y(t)), \tag{3.5}$$

where $F(p, u, y)$ is the differential operator

$$\begin{array}{l} F(p, u, y) = -\nabla_x p \cdot (A + B_1(y - C_2) + B_2 u) \\ \qquad + \frac{1}{2}|C_1 + u| - \frac{1}{2}|y - C_2|^2. \end{array} \tag{3.6}$$

The central controller is obtained by finding the controller which minimizes $J(K)$. If the H_∞ problem for G is solvable, then the function

$$W(p) = \inf_K J_p(K) \tag{3.7}$$

is finite on a domain $\mathrm{dom} W \subset \mathcal{X}_e$. By dynamic programming, W satisfies the (infinite dimensional) PDE

$$\left\{ \begin{array}{l} \inf_u \sup_y \langle \nabla_p W(p), F(p, u, y) \rangle = 0, \\ (p, 0) \leq W(p), \quad W(-\beta) = 0. \end{array} \right. \tag{3.8}$$

(For the purpose of this paper, we treat this equation in a formal sense; for more details, see **[10]**.) Let $\mathbf{u}^*(p)$ and $\mathbf{y}^*(p)$ denote the values of u and y which attain the infimum and supremum in (3.8). Indeed, a direct calculation gives

$$\begin{array}{l} \mathbf{u}^*(p) = \langle \nabla_p W(p), -C_1 + B_2' \nabla_x p \rangle \\ \mathbf{y}^*(p) = \langle \nabla_p W(p), C_2 - B_1' \nabla_x p \rangle. \end{array} \tag{3.9}$$

These formulas define the *central controller* K^* by

$$K^*[y](t) = \mathbf{u}^*(p_t[y]), \tag{3.10}$$

where p_0 is suitably chosen in dom$W$. The particular choice of p_0 is a very important issue (see below). In **[9]** it is shown (in discrete-time) that under suitable hypotheses, *K^* solves the H_∞ control problem for G if and only if the H_∞ control problem for G is solvable.*

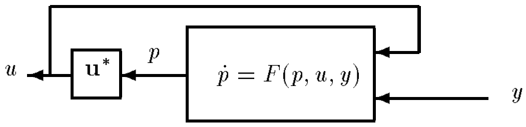

Figure 3.1: Central controller K^*.

The non-negative function $e : \mathbf{R}^n \times \mathcal{X}_e \to \mathbf{R}$ defined by the simple formula

$$e(x,p) = -p(x) + W(p) \tag{3.11}$$

is a *storage function* **[15]** for the closed loop system. This follows because $\nabla_x e(x,p) = -\nabla_x p(x)$ and $\nabla_p e(x,p) = -E_x + \nabla_p W(p)$ (where E_x is the evaluation operator $\langle E_x, f\rangle = f(x)$), which impies that $e(x,p)$ satisfies the PDE

$$\begin{aligned} &\sup_y\{\nabla_x e \cdot (A + B_1(y - C_2) + B_2\mathbf{u}^*) + \\ &\langle \nabla_p e, F(p,\mathbf{u}^*,y)\rangle + \tfrac{1}{2}|C_1 + \mathbf{u}^*|^2 - \tfrac{1}{2}|y - C_2|^2\} = 0, \end{aligned} \tag{3.12}$$

where the supremum is attained at $y = \mathbf{y}^*(p)$. Integrating yields the *dissipation inequality*

$$\begin{aligned} &e(x(t),p(t)) + \tfrac{1}{2}\int_0^t |z(s)|^2\,ds \\ &\quad \le e(x_0,p_0) + \tfrac{1}{2}\int_0^t |w(s)|^2\,ds \end{aligned} \tag{3.13}$$

for any $w \in L_{2,t}$, for all $t \ge 0$.

In general, the information state solution to the (nonlinear) H_∞ control problem is *infinite dimensional*, since in general it is not possible to compute $p_t(x)$ using a finite set of ODE's. Thus in general, p_t evolves in the infinite dimensional space $\mathcal{X}_e$. Even in typically infinite dimensional cases, it can happen that for certain choices of initial state p_0, the resulting trajectory p_t stays in a finite dimensional subset of $\mathcal{X}_e$, whereas for other choices of p_0 it does not. Consequently, the dimension and structure of the set of states reachable from p_0 can vary dramatically. Indeed, *if $p_0 = \delta_{x_0}$, then for any $u, y \in L_{2,e}$*

$$\begin{aligned} p_t = \delta_{\xi(t)} + \tfrac{1}{2}\int_0^t &[|C_1(\xi(s)) + u(s)|^2 \\ &-|y(s) - C_2(\xi(s))|^2]\,ds, \end{aligned} \tag{3.14}$$

where $\xi(\cdot)$ is the solution of (2.3) with initial condition $\xi(0) = x_0$. Consequently, the set of reachable information states from δ_{x_0} is an n-dimensional subset of $\mathcal{X}_e$. Moreover, if $\delta_x \in$ dom$W$, then

$$W(\delta_x) = V(x), \tag{3.15}$$

where $V(x)$ satisfies the PDE

$$\nabla_x V\,(A - B_2C_1) + \tfrac{1}{2}\nabla_x V\left(B_1B_1' - B_2B_2'\right)\nabla_x V' = 0. \tag{3.16}$$

This recaptures the solution to the measurement feedback H_∞ problem for nonlinear stable plants first given in **[2a]**. Moreover, when this initialization is used with the forthcoming formulas for J-inner/outer factorization it gives the the solution to the "stable" case obtained in **[2]**. Also note that the function $V(x)$ corresponds to state feedback H_∞ control.

4. J-Inner/Outer Factorization

We want to obtain a J-inner/outer factorization of the "reversed arrow" plant $\tilde{G}$ given by (2.3): $\tilde{G} = \theta R$, where θ is J-inner and R is outer. We shall use the central controller K^* to obtain a promising formula and state those properties of it which we have obtained.

We say that $\theta : (v_1, v_2) \mapsto (z, w)$ is *J-inner* provided it is:

- *J-dissipative*, i.e.

$$\left\{\begin{aligned} &\int_0^T [|z(s)|^2 - |w(s)|^2]\,ds \\ &\le \int_0^T [|v_2(s)|^2 - |v_1(s)|^2]\,ds, \end{aligned}\right. \tag{4.1}$$

  for all v_1, v_2 in $L_{2,T}$, $T > 0$, and

- *J-lossless*, i.e., (4.1) holds with equality for $T = \infty$ for all v_1, v_2 in $L_2(\theta)$, which consists of those inputs in L_2 for which the outputs $z, w \in L_2$.

We call $R : (u, y) \mapsto (v_1, v_2)$ *outer* if both R and R^{-1} are stable.

As in **[2]**, **[3]**, we will construct this factorization by selecting R^{-1} so that $\theta = \tilde{G}R^{-1}$ is J-inner:

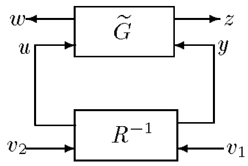

Figure 4.1: Inner factor θ.

The definition of the system R^{-1} and the recipe below are based on the *augmented information state* which is defined to be the solution of

$$\begin{aligned} \dot{p}_t = F(p_t, &\mathbf{u}^*(p_t) + v_2(t), \mathbf{y}^*(p_t) + v_1(t)) \\ &-\tfrac{1}{2}(|v_2(t)|^2 - |v_1(t)|^2). \end{aligned} \tag{4.2}$$

RECIPE FOR FACTORS:

(i) Construct R^{-1} by augmentation:

$$R^{-1}: \begin{cases} \dot{p} &= F(p, \mathbf{u}^*(p) + v_2, \mathbf{y}^*(p) + v_1) \\ & \quad -\frac{1}{2}(|v_2|^2 - |v_1|^2) \\ u &= \mathbf{u}^*(p) + v_2 \\ y &= \mathbf{y}^*(p) + v_1 \end{cases} \tag{4.3}$$

(ii) Take the inverse of R^{-1} to give the outer factor:

$$R: \begin{cases} \dot{p} &= F(p, u, y) \\ & \quad -\frac{1}{2}(|\mathbf{u}^*(p) - u|^2 - |\mathbf{y}^*(p) - y|^2) \\ v_2 &= -\mathbf{u}^*(p) + u \\ v_1 &= -\mathbf{y}^*(p) + y \end{cases} \tag{4.4}$$

(iii) The J-inner factor θ is given by combining (2.3) and (4.3). □

Since θ is not necessarily be defined on all L_2 (rather, it is defined on a space $L_2(\theta) \subset L_2$), the corresponding reversed arrow system (G, Q) is often easier to work with:

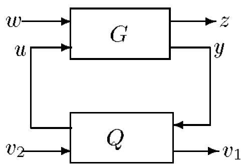

Figure 4.2: Reversed-arrow system (G, Q).

Here, the system $Q : (v_2, y) \mapsto (u, v_1)$ is given by

$$Q: \begin{cases} \dot{p} &= F(p, \mathbf{u}^*(p) + v_2, y) \\ & \quad -\frac{1}{2}(|v_2|^2 - |y - \mathbf{y}^*(p)|^2) \\ u &= \mathbf{u}^*(p) + v_2 \\ v_1 &= -\mathbf{y}^*(p) + y. \end{cases} \tag{4.5}$$

The properties of the recipe which are verified at this time are as follows:

Theorem 4.1 *Assume that there exists a solution W of (3.8). Let $x_0 = 0$ and p_0 be a ground state (i.e. $e(0, p_0) = 0$) for (G, Q). Then (G, Q) is dissipative. If $e(x(t), p_t) \to 0$ as $t \to \infty$, then (G, Q) is lossless.*

Proof. We show that e defined in equation (3.11) is a "lossless" storage function for (G, Q). This in turn is based on the observation that the value function W satisfies (3.8) if and only if W solves

$$\begin{cases} \langle \nabla_p W(p), F(p, \mathbf{u}^*(p) + v_2, \mathbf{y}^*(p) + v_1) \\ \quad -\frac{1}{2}(|v_2|^2 - |v_1|^2) \rangle = 0, \quad \text{for all } v_1, v_2 \\ (p, 0) \le W(p), \quad W(-\beta) = 0. \end{cases} \tag{4.6}$$

This implies that the storage function $e(x, p)$ defined by (3.11) automatically satisfies the following energy balance PDE for the system (G, Q):

$$\begin{aligned} &\nabla_x e \cdot (A + B_1(w) + B_2 u) \\ &\quad + \langle \nabla_p e, F(p, u, y) - \tfrac{1}{2}(|v_2|^2 - |v_1|^2) \rangle \\ &\quad + \tfrac{1}{2}(|z|^2 + |v_1|^2) - \tfrac{1}{2}(|v_2|^2 + |w|^2) \\ &= 0 \quad \text{for all } v_2, w. \end{aligned} \tag{4.7}$$

Integration yields the *energy balance equation* for (G, Q):

$$\begin{aligned} &e(x(t), p_t) + \tfrac{1}{2} \int_0^t [|z(s)|^2 + |v_1(s)|^2] \, ds \\ &= e(x_0, p_0) + \tfrac{1}{2} \int_0^t [|w(s)|^2 + |v_2(s)|^2] \, ds, \end{aligned} \tag{4.8}$$

for all $w, v_2 \in L_{2,t}$, $t \ge 0$. This energy balance relation can be used to analyze input-output properties and internal stability. Since $e \ge 0$ and $e(0, p_e) = 0$ the dissipation property follows:

$$\tfrac{1}{2} \int_0^T [|z(s)|^2 + |v_1(s)|^2] ds \le \tfrac{1}{2} \int_0^T [|w(s)|^2 + |v_2(s)|^2] ds, \tag{4.9}$$

for all w, v_2 in $L_{2,T}$, and all $T > 0$.

If (G, Q) is stable, and if for $x_0 = 0$, $e(0, p_0) = 0$, $w, v_2 \in L_2$ we have $e(x(t), p_t) \to 0$ as $t \to \infty$, then (G, Q) is lossless:

$$\int_0^\infty [|z(s)|^2 + |v_1(s)|^2] ds = \int_0^\infty [|w(s)|^2 + |v_2(s)|^2] ds, \tag{4.10}$$

for all w, v_2 in L_2. □

Our only theorem (at present) regarding the J-inner/outer factorization is the following result.

Theorem 4.2 *If R and R^{-1} given by (4.4) and (4.3) are BIBO stable wrt L_2, under the assumptions of Theorem 4.1, $\tilde{G} = \theta R$ is a J-inner/outer factorization.*

This construction is (at least) a formal prescription of the structure of the J-inner and outer factors in the nonlinear context. It subsumes all known examples such as the linear case **[3]** and the "stable" case **[2]**. The main gap is in proving that R is outer. Certainly, R is stable provided that the maps $t \mapsto \mathbf{u}^*(p_t), \mathbf{y}^*(p_t)$ are in L_2 whenever $u, y \in L_2$. Also, R is incrementally stable **[5]** if a related differential is bounded. Similar statements apply to R^{-1}.

APPLICATION:

The factorization framework allows us to produce a large class of controllers (possibly all) K solving the H_∞ problem for G via a natural parameterization involving a dissipative "load" $L : v_1 \mapsto v_2$ attached to the additional terminals of Q. This was first analysed in **[2]** and improved in **[5]**:

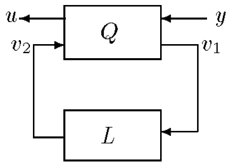

Figure 4.3: Loading.

Assume that $\tilde{G} = \theta R$ is a J-inner incrementally outer **[5]** *factorization, where the factors are defined by the recipe. Then any dissipative load $L : v_1 \mapsto v_2$ defines a solution K^L to the H_∞ problem for G:*

$$K^L : \begin{cases} \dot{p} &= F(p, \mathbf{u}^*(p) + L(-\mathbf{y}^*(p) + y), y) \\ & \quad -\frac{1}{2}(|L(-\mathbf{y}^*(p) + y)|^2 - |y - \mathbf{y}^*(p)|^2) \\ u &= \mathbf{u}^*(p) + L(-\mathbf{y}^*(p) + y) \end{cases} \tag{4.11}$$

Conversely, assume that K solves the H_∞ problem for G. Then the load L^K defined by

$$L^K : \begin{cases} \dot{p} &= F(p, K(\mathbf{y}^*(p) + v_1), \mathbf{y}^*(p) + v_1) \\ & \quad -\frac{1}{2}(|K(\mathbf{y}^*(p) + v_1) - \mathbf{u}^*(p)|^2 - |v_1|^2) \\ v_2 &= K(\mathbf{y}^*(p) + v_1) - \mathbf{u}^*(p) \end{cases} \tag{4.12}$$

defines a representation for K as in (4.11) provided it is dissipative. Open is the issue of whether or not L^K is dissipative.

5. Example

To illustrate the above recipe and stability issues, consider the following one-dimensional linear system:

$$\begin{cases} \dot{x} &= ax + b_1 w + b_2 u \\ z &= c_1 x + u \\ y &= c_2 x + w, \end{cases} \tag{5.1}$$

where a, b_1, b_2, c_1 and c_2 are scalars. The information state is given **[12]** explicitly by

$$p_t(x) = -\tfrac{1}{2}(x - \hat{x})^2 Y^{-1} + \phi, \tag{5.2}$$

where

$$\dot{\hat{x}} = (a - b_1 c_2 + Y(c_1^2 - c_2^2))\hat{x} + (b_1 + Y c_2)y + (b_2 + Y c_1)u, \tag{5.3}$$

$$\dot{Y} = 2Y(a - b_1 c_2) + Y^2(c_1^2 - c_2^2), \tag{5.4}$$

$$\dot{\phi} = \tfrac{1}{2}(c_1 \hat{x} + u)^2 - \tfrac{1}{2}(y - c_2 \hat{x})^2, \tag{5.5}$$

with initial data p_0 specified by $\hat{x}_0 = 0$, $Y_0 \geq 0$, $\phi_0 = 0$ ($Y_0 = 0$ corresponds to the choice $p_0 = \delta_0$). The value function is:

$$W(\hat{x}, Y, \phi) = \tfrac{1}{2}\hat{x}^2 X_\infty (1 - Y X_\infty)^{-1} + \phi, \tag{5.6}$$

where $X_\infty \geq 0$ is required to be a stabilizing solution of the Riccati equation

$$0 = 2X_\infty(a - b_2 c_1) + X_\infty^2(b_1^2 - b_2^2). \tag{5.7}$$

Note that $\mathrm{dom} W$ consists of those points $(\hat{x}, Y, \phi)$ for which $Y X_\infty < 1$ whenever $\hat{x} \neq 0$. The optimal control and observation are:

$$\mathbf{u}^*(\hat{x}, Y, \phi) = -(c_1 + (b_2 + Y c_1) X_\infty (1 - Y X_\infty)^{-1})\hat{x},$$

$$\mathbf{y}^*(\hat{x}, Y, \phi) = (c_2 + (b_1 + Y c_2) X_\infty (1 - Y X_\infty)^{-1})\hat{x}.$$

The recipe produces the R (outer) factor which is:

$$R : \begin{cases} \dot{\hat{x}} &= (a - b_1 c_2 + Y(c_1^2 - c_2^2))\hat{x} \\ & \quad +(b_1 + Y c_2) y + (b_2 + Y c_1) u \\ v_2 &= -\mathbf{u}^*(\hat{x}, Y, \phi) + u \\ v_1 &= -\mathbf{y}^*(\hat{x}, Y, \phi) + y, \end{cases} \tag{5.8}$$

and the formal J-inner factor is given by:

$$\theta : \begin{cases} \dot{\xi} &= (a - b_1 c_2)\xi + b_1(\mathbf{y}^*(\hat{x}, Y, \phi) + v_1) + \\ & \quad b_2(\mathbf{u}^*(\hat{x}, Y, \phi) + v_2) \\ \dot{\hat{x}} &= (a - b_1 c_2 + Y(c_1^2 - c_2^2))\hat{x} + \\ & \quad (b_1 + Y c_2)(\mathbf{y}^*(\hat{x}, Y, \phi) + v_1) + \\ & \quad (b_2 + Y c_1)(\mathbf{u}^*(\hat{x}, Y, \phi) + v_2) \\ z &= c_1 \xi + (\mathbf{u}^*(\hat{x}, Y, \phi) + v_2) \\ w &= -c_2 \xi + (\mathbf{y}^*(\hat{x}, Y, \phi) + v_1). \end{cases} \tag{5.9}$$

It is well known that (under the standard assumptions) the H_∞ problem for G is solvable (with norm strictly less that 1) if and only if (i) there exists $X_\infty \geq 0$ solving (5.7) with $(a - b_2 c_1 + X_\infty(b_1^2 - b_2^2)) < 0$, (ii) there exists $Y_\infty \geq 0$ solving (5.4) with $(a - b_1 c_2 + Y_\infty(c_1^2 - c_2^2)) < 0$, and (iii) $X_\infty Y_\infty < 1$, see **[13]**; these are essentially the conditions needed for the J-inner/outer factorization $\tilde{G} = \theta R$, see **[3]**. If these conditions are satisfied one usually uses the time independent solution $Y = Y_\infty$ in the above formulas. This corresponds to a specific initialization p_0 of p_t.

The main implication of this example for the general nonlinear case is that it is not clear how p_0 should be chosen (see **[9]** for some conditions which p_0 must satisfy). In this example, one may try to use other values of p_0 corresponding to initial conditions $Y_0 \geq 0$ (the dynamics for the information state involve more than the equation for $\hat{x}$). Difficulties will be encountered if Y_0 is chosen incorrectly. For instance, suppose that $(a - b_1 c_2) < 0$ and $c_1^2 > c_2^2$. Then (5.4) has two solutions: $Y_\infty^s = 0$ which is stabilising and $Y_\infty^u = -2(a - b_1 c_2)/(c_1^1 - c_2^2) > 0$ which is not. Setting $Y_0 \geq Y_\infty^u$ will result in a central controller yielding a closed loop system which is not internally stable although it may be dissipative; consquently the J-inner/outer factorization can therefore fail if not initialized correctly. If

$Y_0 > Y_\infty^u$, then the coupling condition $Y(t)X_\infty < 1$ will eventually be violated even if $Y_\infty^u X_\infty < 1$ and $Y_0 X_\infty < 1$, that is, Y_O is in dom$W$. (The initialization $0 < Y_0 < Y_\infty^u$ will yield $p_t \to \delta_0 + const.$ as $t \to \infty$, which is fine.) Similar difficulties are encountered if $(a - b_1 c_2) > 0$ and $c_1^2 < c_2^2$ in which case $Y_\infty^s > 0$. Thus Y_0 must be chosen such that (i) $Y_0 X_\infty < 1$, and (ii) Y_0 is in the domain of attraction of Y_∞^s. These remarks have important consequences for the general nonlinear case. For example, one can have initializations p_0 in dom$W$ which guarantee dissipation but do not produce a solution to the H_∞ problem. The next section addresses ways around this issue.

6. Internal Stability

As one suspects from the example in the previous section internal stability turns out to be a surprisingly deep issue which in earlier papers is treated under strong hypotheses. What we saw was that the information state equations for a particular control problem (even in the linear case) can have multiple solutions corresponding to a set $\mathcal{P}_0$ of possible initial conditions. While many initial conditions lead to controllers achieving the desired performance, some produce internally stable systems while some do not. To select which solutions yield internally stablizing controllers we introduce a measure ν of internal stablity. Corresponding to $\nu \neq 0$ is an information state equation IS_ν with provably well behaved solutions and IS_0 is the original information state equation. For small ν one can use a controller produced by IS_ν because it is internally stable and produces nearly the same performance as IS_0. This produces one satisfactory solution to the control problem. We suspect that solutions to IS_ν converge in some weak sense to solutions of IS_0 which also give internally stablizing controllers thus this limiting processes selects the useful solutions to IS_0.

This is reminiscent of a situation with other equations such as the Euler equations of fluid flow where the viscosity ν is a key parameter. When ν is 0 the equations have multiple solutions most of which do not occur physically. However, the physical solutions can be selected as a limit of solutions to viscous fluid flow equations as the viscosity goes to 0. Our goal here is to select physically suitable initial conditions from a variety of choices.

Various definitions of *internal stability* of (G, K) are possible, depending on the context. We do not go into this but select one very strong definition and introduce our parameter ν as a quantitative measure of how internally stable a system is.

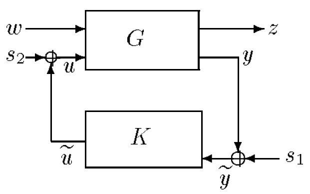

Figure 6.1: Internal stability.

Refering to Figure 6.1, we say that (G, K) is *strongly internally stable* if for any $x_0 \in \mathbf{R}^n$ and $w, s_1, s_2 \in L_2$ then $z, y, u \in L_2$.

It will be convenient to define a property which encompasses dissipativity and strong internal stability. We say that (G, K) is *ν_0-stable and dissipative* if for any $x_0 \in \mathbf{R}^n$ and $w, s_1, s_2 \in L_2$ then

$$\begin{aligned} \int_0^T [\nu|x(s)|^2 + |z(s)|^2]\, ds \;\le\; \int_0^T [|w(s)|^2 + \\ (C/\nu)(|s_1(s)|^2 + |s_2(s)|^2)] ds + \beta(x_0), \end{aligned} \tag{6.1}$$

for all $w, s_1, s_2 \in L_{2,T}$ and all $T \ge 0$, $\nu \le \nu_0$, where $C > 0$ is a (perhaps very large) constant, $\beta \in \mathcal{B}$, and $\nu_0 > 0$. Intuitively this says (provided we take ν small) that Figure 6.1 viewed as a system from $(w, s_1, s_2) \to (z, x)$ is dissipative in a metric which is near 0 on x and near infinity on (s_1, s_2).

A necessary and sufficient analysis of the nonlinear H_∞ control problem which includes internal stability can be given in terms of ν. It follows readily from the results cited in §3 above.

An *ν-information state* naturally associated with Figure 6.1 is defined by the equation

$$\begin{aligned} p_t^\nu(x) = \sup_{s_1, s_2, x_0} \{ p_0(x(0)) + \tfrac{1}{2} \int_0^t [|z(s)|^2 + \nu|x(s)|^2 \\ -|\tilde{y}(s) - s_1(s) - C_2(\xi(s))|^2 \\ -\tfrac{C}{\nu}(|s_1(s)|^2 + |s_2(s)|^2)]\, ds \}, \end{aligned} \tag{6.2}$$

where $\xi(\cdot)$ is the solution of (2.3) with terminal condition $\xi(t) = x$. The corresponding dynamics are

$$\dot{p}_t^\nu = F^\nu(p_t^\nu, \tilde{u}(t), \tilde{y}(t)), \tag{6.3}$$

where

$$\begin{aligned} F^\nu(p, \tilde{u}, \tilde{y}) = \\ \sup_{s_1, s_2} \{ -\nabla_x p \cdot (A + B_1(\tilde{y} - s_1 - C_2) + B_2(\tilde{u} + s_2)) \\ + \tfrac{1}{2}|C_1 + \tilde{u} + s_2| - \tfrac{1}{2}|\tilde{y} - s_1 - C_2|^2 \\ - \tfrac{1}{2}(C/\nu)(|s_1|^2 + |s_2|^2) \}, \end{aligned} \tag{6.4}$$

and the equation for $W^\nu(p)$ is

$$\begin{cases} \inf_u \sup_y \langle \nabla_p W^\nu(p), F^\nu(p, u, y) \rangle = 0, \\ (p, 0) \le W^\nu(p), \quad W^\nu(-\beta^\nu) = 0. \end{cases} \tag{6.5}$$

Necessity: *Assume there exists a controller K which solves the H_∞ problem for G in the sense of (6.1), i.e. (G, K) is ν_0-stable and dissipative. Then there exists a solution $W^\nu(p)$ of the PDE (6.5) with non-empty domain* domW^ν.

Conversely, using the function $W^\nu(p)$ obtained above, we can construct a controller K^ν via the minimizer $\mathbf{u}^\nu(p)$ and maximizer $\mathbf{y}^\nu(p)$ for (6.5). Thus

$$K^\nu[y](t) = \mathbf{u}^\nu(p_t^n u[y]). \tag{6.6}$$

Also, the energy function $e^\nu(x,p) = -p(x) + W^\nu(p)$ is well-defined, and satisfies the analogue of (3.12) and

$$e^\nu(x(t), p^\nu(t)) + \tfrac{1}{2}\int_0^t [|z(s)|^2 + \nu|x(s)|^2]\, ds$$

$$\leq e^\nu(x_0, p_0^\nu) + \tfrac{1}{2}\int_0^t [|w(s)|^2 + \frac{C}{\nu}(|s_1(s)|^2 + |s_2(s)|^2)]\, ds. \tag{6.7}$$

Thus, sufficiency: *Assume: (i) there exists a solution $W^\nu(p)$ of (6.5) for some $\beta^\nu \in \mathcal{B}$ with non-empty domain* $\mathrm{dom}W^\nu$*, and (ii) there exists $\mathcal{P}_0^\nu \subset \mathrm{dom}W^\nu$ such that $p_0^\nu \in \mathcal{P}_0^\nu$ implies $p_t^\nu \in \mathrm{dom}W^\nu$ for all $t \geq 0$ and all y, where u is the output of K^ν defined above, see (6.6). Then (G, K^ν) is ν_0-stable and dissipative. Consequently, K^ν solves the H_∞ problem for G.*

We expect that as $\nu \to 0$ the functions p_t^ν, $W^\nu(p)$ and K^ν converge in a weak sense. In that case we might think of the limit K^* as a central controller. Then the controller K^ν is a perturbation of K^*. The parameter $\nu > 0$ introduces a regularizing effect, in the sense that (G, K^ν) enjoys stability properties without assuming additional controllability/observability conditions. We leave open the question of using these facts to prove similar properties for the central controller K^*.

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