REDUCTION OF CONTROLLER COMPLEXITY IN NONLINEAR H_∞ CONTROL*

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Abstract

This paper deals with measurement feedback H_∞ control. Here one must solve a state estimator *on line* and this often is a driven PDE and so it is infinite dimensional. While in common constructions this appears to be a PDE on the state space of the plant, in this article we find that this PDE in practice sits on a space of much lower dimension k (often 1 or 2). Thus the on line computational burden for measurement feedback H_∞ control might not be prohibitive. More precisely if the plant is strictly hyperbolic and $k = k_{as}$ is the dimension of its unstable manifold, our construction produces a controller whose state dynamics is a driven PDE on $\mathbf{R}^{k_{as}}$. This extends results of Ball and Helton (discrete time) which "construct" finite dimensional H_∞ controllers for systems with stable plants.

1. Introduction

In the mixed sensitivity problem one is given plant P and wishes to find a controller K achieving a given H_∞ performance (see Figure 1).

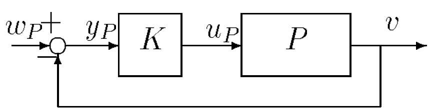

Figure 1: Mixed Sensitivity Setup

If the the weights are chosen correctly for a mixed sensitivity problem then one gets the standard two block problem of H_∞ control. We shall show under strong hypotheses on our setup:

> **Result** *If the plant P is strongly hyperbolic and has a $k = k_{as}$ dimensional antistable manifold and if the weights are well behaved (defined below), then the key dynamics of the two block problem are strongly hyperbolic and has a k dimensional antistable manifold.*
>
> *This allows a construction for a controller requiring on line computation having complexity of the same order as numerical solution of a PDE driven (by inputs) on $k < n$ dimensional space.*

Now it often happens in practice that plants P are stable or at worst have a low dimensional instability. Thus the numerics of constructing the central controller solving a measurement feedback problem might not be much more complex than that of constructing a state feedback controller. We caution the reader that the off line computation in designing an *uncompromised* H_∞ controller would still be huge.

The Result above was given for stable discrete time systems in **[BH1]**, **[BH2]**. Coupled with the discrete time version of singular controller **[BH2]**, this converts solution of the mixed sensitivity problem for stable plants to solving the usual HJBI difference analog of (4.7) (below). Corollary 5.2 in this paper gives these formulas in continuous time for the special class of systems we are study.

2. The Standard Problem of H_∞ Control

We consider nonlinear plants G of the form

$$G: \left\{ \begin{array}{rcl} \dot{x} &=& A(x) + B_1(x)w + B_2(x)u \\ z &=& C_1(x) + D_{12}(x)u \\ y &=& C_2(x) + w \end{array} \right. \tag{2.1}$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an output $y(t) \in \mathbf{R}^p$ is observed. The additional output quantity $z(t) \in \mathbf{R}^m$ is a performance measure, depending on the particular problem at hand. The control input is $u(t) \in \mathbf{R}^m$, while $w(t) \in \mathbf{R}^p$ is regarded as an opposing disturbance input. We assume that all of the functions appearing in (2.1) are as smooth and bounded as necessary, and that zero is an equilibrium: $A(0) = 0$, $C_1(0) = 0$ and $C_2(0) = 0$. We take $D_{12}(x)'D_{12}(x) = E_1(x) > 0$. This structure assumed of G is a nonlinear analogue of the (non-singular) *two block problem.*

The following signal spaces will be used: $L_2 = L_2([0,\infty), \mathbf{R}^q)$, $L_{2,T} = L_2([0,T], \mathbf{R}^q)$, and $L_{2,e} = \cup_{T>0} L_{2,T}$, where the dimension of the range space will not be stated explicitly but inferred from the context.

A *controller* K is a causal mapping $K : L_{2,e} \to L_{2,e}$ taking outputs y to inputs u. Such a controller will be termed *admissible* if the closed loop equations for G (2.1) with $u = K(y)$ are well defined in the sense that unique solutions exist in $L_{2,e}$. We will assume $K(0) = 0$. A controller K is said to solve the H_∞ control problem for G if the closed loop system (G, K) is (i) *internally stable*, and (ii) *dissipative*, in the sense that there exist $\beta(x) \geq 0$ with $\beta(0) = 0$ such that

$$\int_0^T |z(s)|^2 \, ds \;\leq\; \int_0^T |w(s)|^2 \, ds + \beta(x_0), \qquad (2.2)$$

for all $w \in L_{2,T}$, $T \geq 0$.

Also of importance is the "$A^\times$ system", with inputs u and y:

$$\dot{\xi} = A^\times(\xi) + B_1(\xi)y + B_2(\xi)u. \qquad (2.3)$$

Here, $A^\times = A - B_1 C_2$. Let $\Phi^{u,y}_{s,t}$ denote the semigroup or flow associated with (2.3), i.e., $\xi(s) = \Phi^{u,y}_{s,t}(x)$ is the solution of (2.3) with $\xi(t) = x$. A vector field f on $\mathbf{R}^n$ is called *strongly stable* if it is globally exponentially Lyapunov stable. Additionally, we say f is strongly *antistable* if $-f$ is strongly stable, and f is strongly *hyperbolic* if there exist global stable M_s and antistable M_{as} manifolds such that $\mathbf{R}^n = T_0 M_s \oplus T_0 M_{as}$ and f restricted to M_s is strongly stable and $-f$ restricted to M_{as} is strongly stable, and further that there exist coordinates such that $\mathbf{R}^n = M_s \oplus M_{as}$. We will assume that $A^\times$ is strongly hyperbolic.

3. Review of the Information State Solution to H_∞ Control

Define a function space $\mathcal{X} = C(\mathbf{R}^n)$ (say). We will need a "completion" $\mathcal{X}_e$ of this space which includes extended real valued functions such as

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M, \\ -\infty & \text{if } x \notin M, \end{cases}$$

where $M \subset \mathbf{R}^n$. Let $C(M)$ denote all continuous functions on a manifold M.

The information state theory is presented in **[J]**, **[JB]**, **[HJ1]**, **[HJ2]**. The main idea is that the central controller is obtained from and infinite dimensional PDE defined in terms of an *information state* $p_t(x)$, whose dynamics are a PDE, for fixed $u, y \in L_{2,e}$:

$$\dot{p}_t = F(p_t, u(t), y(t)), \qquad (3.1)$$

where $F(p, u, y)$ is the differential operator

$$F(p,u,y) = -\nabla_x p \cdot (A + B_1(y - C_2) + B_2 u)$$
$$+\tfrac{1}{2}|C_1 + D_{12}u| - \tfrac{1}{2}|y - C_2|^2. \qquad (3.2)$$

The central controller is obtained from the infinite dimensional PDE for the value function $W(p)$:

$$\inf_u \sup_y \nabla_p W(p)[F(p,u,y)] = 0, \qquad (3.3)$$

where $W(p)$ also satisfies $W(p) \geq \langle p \rangle$, $W(-\beta) = 0$. Here, $\langle p \rangle = \sup_x\{p(x)\}$ is the "sup-pairing", see **[JBE]**. Let $\mathbf{u}^*(p)$ denote the value of u which attains the infimum in (3.3); indeed,

$$\mathbf{u}^*(p) = (\nabla_p W(p)[E_1])^{-1} \nabla_p W(p)[-D_{12}'C_1 + B_2' \nabla_x p']. \qquad (3.4)$$

In this formula we have assumed that the matrix $\tilde{E}_1 = \nabla_p W(p)[E_1]$ is invertible (this will follow automatically if E_1 is independent of x).

These formulas define the *central controller* K^* by

$$K^*_{p_0} : \begin{cases} \dot{p} = F(p,u,y) \\ u = \mathbf{u}^*(p), \end{cases} \qquad (3.5)$$

where p_0 is suitably chosen in dom$W$. In **[JB]**, **[HJ2]** it is shown that under suitable hypotheses, *K^* solves the H_∞ control problem for G if and only if the H_∞ control problem for G is solvable.* The constructions here are limiting cases of those for which the general theory is proved, which means that much work (underway) is needed to establish all of the properties of the controllers here, see **[HJ2]**.

In general, the information state solution to the (nonlinear) H_∞ control problem is *infinite dimensional*, since in general it is not possible to compute $p_t(x)$ using a finite set of ODE's. Thus in general, p_t evolves in the infinite dimensional space $\mathcal{X}_e$. (For finite dimensional cases, see **[JY]**.) In the following sections we show that the computational complexity can be reduced for a range of interesting cases.

4. Reduced Dimensional Central Controller

In this section we discuss what in general appears to be the simplest solution to the one or two block problem when the $A^\times$ dynamics is strongly hyperbolic, and the central controller is initialized at the corresponding *singular* p_e:

$$p_e = \delta_{M_{as}} + \phi, \tag{4.1}$$

where $\phi \in C(M_{as})$ is given by **[HJ2]**:

$$\phi(x) = \lim_{t\to+\infty} \tfrac{1}{2}\int_0^t [|C_1(\Phi^{0,0}_{s,t}(x))|^2 - |C_2(\Phi^{0,0}_{s,t}(x))|^2 ds.$$

The function p_e is a solution of the *steady state information state equation*

$$0 = F(p_e, 0, 0).$$

We refer to $K^*_{p_e}$ with initialization (4.1) as the *canonical central controller*. The form of this controller will be discussed in detail below.

In the case that $A^\times$ is strongly hyperbolic, we expect that a reduction in complexity is attained when one initializes with $p_0 = p_e$. What is critical is the dimension of the antistable manifold M_{as}. If $k_{as} = \dim M_{as} < n$, we expect that one only needs to compute the information state on k dimensional manifolds, a much easier task. Thus the on line computational burden in running the controller will be reduced if $k_{as} < n$, a striking fact! If $k_{as} = n$, we do not expect any reduction in complexity. The canonical central controller is the direct generalization of the $A^\times$ stable example in **[BH2]**, **[HJ2]** to the hyperbolic case (see below).

Geometrical Description. Now we give a geometric picture of the information state dynamics (which, for those who find comfort in technical language, evolves on a *fibre bundle*). This describes the dynamics of the canonical central controller. Let $\mathcal{M}$ denote all (maximal) $k = k_{as}$ dimensional smooth manifolds embedded in $\mathbf{R}^n$ (i.e. there exists coordinates such that $M = (x_1, \ldots, x_k, 0\ldots, 0)$).

Lemma 4.1 *If p_0 is of the form (4.1), then for any $u, y \in L_{2,e}$, then p_t is also of the form (4.1) for all $t \geq 0$, i.e.*

$$\begin{aligned} p_t(x) &= \delta_{M_t}(x) + \phi(\xi(0)) \\ &+ \tfrac{1}{2}\int_0^t [|C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2 \\ &-|y(s) - C_2(\xi(s))|^2]ds, \end{aligned} \tag{4.2}$$

where $\xi(\cdot)$ is the solution of (2.3), i.e.,

$$\dot{\xi} = A^\times(\xi) + B_1(\xi)y + B_2(\xi)u, \tag{4.3}$$

with terminal condition $\xi(t) = x \in M_t$, and M_t is the image of $M = M_{as}$ under the flow of (2.3), i.e., $M_t = \{\xi(t) : \xi(0) \in M\}$.

The following corollary gives a more explicit geometric picture.

Corollary 4.2 *The state space for the information state with initialization (4.1) is contained in*

$$\mathcal{S} \triangleq \{(M, \phi) : M \in \mathcal{M}, \phi \in C(M)\}.$$

(This set is a fibre bundle over $\mathcal{M}$.) The dynamics of the information state for the canonical central controller with initialization (4.1) is given by

$$M_t = \Phi^{u,y}_{t,0}(M_{as}), \tag{4.4}$$

where $\Phi^{u,y}_{t,0}$ is the flow associated with (4.3), and

$$\begin{aligned}\phi_t(x) = \tfrac{1}{2}\int_0^t &[|C_1(\Phi^{u,y}_{s,0}(x_0)) + D_{12}(\xi(s))u(s)|^2 \\ &-|y(s) - C_2(\Phi^{u,y}_{s,0}(x_0))|^2]\, ds + \phi(x_0),\end{aligned} \tag{4.5}$$

where $x_0 = \Phi^{u,y}_{0,t}(x) \in M_{as}$, or in semigroup notation,

$$\phi_t = \hat{\Phi}^{u,y}_{t,0}(\phi).$$

This geometric picture while strikingly graphic has lots of things moving at once: the information state evolves in a fibre bundle with base point M_t and ϕ_t belonging the the fibre $C(M_t)$. Equation (4.5) gives a less dizzying interpretation by "pulling back" onto the initial antistable manifold M_{as}. Indeed, the formula (4.5) shows that ϕ_t is computable via characteristics starting in the k_{as}-dimensional manifold M_{as}. The remaining $n - k_{as}$ dimensions are irrelevant. This is how the reduction in complexity is obtained.

Computational Complexity. This evolution of functions ϕ on the $M \equiv \mathbf{R}^{k_{as}}$ together with the evolution (4.3) constitute the "reduced dimensional" picture of the compensator dynamics. Suppose one wishes to approximately compute the compensator state (M_t, ϕ_t) via a flow-based method for solving hyperbolic PDE. This entails using a finite grid on $M_{as} \equiv \mathbf{R}^{k_{as}}$. If N is the number of grid points one wishes to use in each dimension, then the memory requirements scale like

$$\mathrm{O}(N^{k_{as}}),$$

which is much less than $\mathrm{O}(N^n)$ if one had to solve the PDE (3.1) in $\mathbf{R}^n$.

Formula for the controller. While we have appealing formulas for the controller dynamics interperting the high level formula (3.4) for controller output in this situation requires technical interpretation, since it involves singular p of the form (4.1). This causes little trouble with $W(p)$. However, interpretation of $\nabla_p W(p)$ and $u^*(p)$ is difficult (see Appendix).

Using Corollary 4.2, we can write down the canonical central controller for $A^\times$ strongly hyperbolic:

$$K^*_{p_e} : \begin{cases} M_t = \Phi^{u,y}_{t,0}(M_{as}), \quad \phi_t = \hat{\Phi}^{u,y}_{t,0}(\phi) \\ u = \mathbf{u}^*(\delta_{M_t} + \phi_t), \end{cases} \tag{4.6}$$

where $\mathbf{u}^*(\delta_{M_t} + \phi_t)$ is given by formula (3.4). (See the Appendix for an interpretation of $\nabla_p W(\delta_M + \phi)$.) Note that the PDE underlying (4.6) sits on a space of dimension $k_{as} = \dim M_{as} << n$, in general.

Example. If $A^\times$ is strongly stable, it turns out that the canonical central controller is *finite dimensional* (see **[BH2]**, **[HJ2]**), a very dramatic reduction in complexity! We now write down the controller in this case. Here, $k_{as} = \dim M_{as} = 0$, and $p_e = \delta_0$ (corresponding to $Y_\infty = 0$ in the linear case).

Let $V(x)$ be the state feedback value function solving the finite dimensional PDE

$$\begin{aligned} &\nabla_x V(A - B_2 E_1^{-1} D_{12} C_1) \\ &\quad + \tfrac{1}{2}\nabla_x V(B_1 B_1' - B_2 E_1^{-1} B_2')\nabla_x V' \\ &\qquad + \tfrac{1}{2} C_1'(I - D_{12} E_1^{-1} D_{12}') C_1 = 0. \end{aligned} \tag{4.7}$$

The optimal state feedback controller is

$$u_s^*(x) = -E_1(x)^{-1}(D_{12}(x)' C_1(x) + B_2(x)' \nabla_x V(x)'). \tag{4.8}$$

Then the canonical central controller for $A^\times$ strongly stable is

$$K_{\delta_0}^* : \begin{cases} \dot{\xi} = A^\times(\xi) + B_1(\xi) y + B_2(\xi) u \\ \\ u = u_s^*(\xi). \end{cases} \tag{4.9}$$

Certainty equivalence. Under certain assumptions, the certainty equivalence controller solves the H_∞ control problem; see **[BB]**, **[JB2]**, **[JBE]**. In general, the certainty equivalence controller is defined by

$$u_{CE}(t) = u_s^*(\bar{x}(t)),$$

where $\bar{x}(t) = \operatorname{argmax}_{x \in \mathbf{R}^n} \{p_t(x) + V(x)\}$, assumed uniquely defined, and $u_s^*(x)$ is the optimal state feedback given by (4.8). The significance of this is that the controller is determined by an n-dimensional PDE (4.7), and in general the controller dynamics is also n-dimensional, since it requires the solution of the PDE (3.1). However, in the light of our results, the controller dynamics dimension can be reduced to $k = k_{as} < n$ when $A^\times$ is strongly hyperbolic. Indeed, in this case the reduced complexity certainty equivalence controller takes the form:

$$K_{p_e}^{CE} : \begin{cases} M_t = \Phi_{t,0}^{u,y}(M_{as}), \quad \phi_t = \hat{\Phi}_{t,0}^{u,y}(\phi) \\ \\ u = u_{CE}(\delta_{M_t} + \phi_t), \end{cases} \tag{4.10}$$

where

$$u_{CE}(\delta_M + \phi) = u_s^*(\bar{x}(M, \phi)),$$

and

$$\bar{x}(M, \phi) = \operatorname*{argmax}_{x \in M} \{\phi(x) + V(x)\}.$$

Note that when $A^\times$ is stable, the certainty equivalence controller (4.10) agrees with the finite dimensional canonical controller (4.9) (since $M = \{\xi\}$, ϕ a constant, $\bar{x}(\xi, \phi) = \xi$).

5. Mixed Sensitivity

Now we apply the solution given in the previous section to the mixed sensitivity problem. The G in standard problem of H_∞ control which we treat in this paper is typically not the plant in classical control problems, such as mixed sensitivity (see Figure 1, §1). Rather, G is built from the plant P together with weights.

Notation and other details. We treat the mixed sensitivity problem for a nonlinear plant P having state space representation

$$P : \begin{cases} \dot{x}_P &= A_P(x_P) + B_P(x_P) u_P \\ \\ v &= C_P(x_P). \end{cases} \tag{5.1}$$

The mixed sensitivity problem is to design the compensator K so that (i) the following dissipation inequality holds

$$\begin{aligned} &\tfrac{1}{2}\int_0^T [|\mathcal{W}_1(y_P)(s)|^2 + |\mathcal{W}_2(v)(s)|^2]\, ds \\ &\qquad \le \tfrac{1}{2}\int_0^T |w_P(s)|^2\, ds + \beta_P(x_P(0)), \end{aligned} \tag{5.2}$$

for all $w_P \in L_{2,T}$, $T \ge 0$, for some finite $\beta_P(x_P(0)) \ge 0$, and, (ii) the closed loop system is internally stable (see Figure 1). Here $\mathcal{W}_1$ and $\mathcal{W}_2$ are systems which weight the signals in the closed loop system. The relevant system of equations is: $u_P = y_P + P(u_P)$, $K(y_P) = u_P$.

To set the problem in the framework of the standard H_∞-problem for G as in (2.1), we should make the following identifications (Figure 1) $w = w_P$, $u = u_P$, $y = y_P$, $z = (z_1', z_2')'$, with $z_1 = \mathcal{W}_1(y_P)$, $z_2 = \mathcal{W}_2(v)$. To write down the state space equations for G, we need to specify the form of the state space realizations of the weights. A key issue is the proper choice of weights.

Choosing the weights. In the linear case, physically correct choices of the frequency domain weights weights dictate that for complex numbers s we have $\mathcal{W}_1(s) \to 0$ and $\mathcal{W}_2(s) P(s) K(s) \to$ *constant* $\neq 0$ as $s \to \infty$. (c.f. **[HM]**) To pick weights correctly, one must make an *a priori* decision on the rate at which the compensator $K(s)$ must go to 0 at ∞. For simplicity of exposition we make the choice that $K(s) \to c \neq 0$ as $s \to \infty$. One can select other asymptotics by extracting them from the K and placing them in P. Reasonable nonlinear analogues of this with a "proper" plant are given by the following.

We treat the nonlinear analog of the case where $P(s)$ has a first order zero at $s = \infty$ and we desire a compensator satisfying $K(s) \to c \neq 0$. Then $\mathcal{W}_2(s)$ must be asymptotic to s as $s \to \infty$. This motivates the state space realizations

$$\mathcal{W}_1 : \begin{cases} \dot{x}_1 &= A_{\mathcal{W}_1}(x_1) + B_{\mathcal{W}_1}(x_1) y \\ \\ z_1 &= C_{\mathcal{W}_1}(x_1), \end{cases} \tag{5.3}$$

$$\mathcal{W}_2 : \begin{cases} \dot{x}_2 &= A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)v \\ z_2 &= C_{\mathcal{W}_2}(x_2) + D_{\mathcal{W}_2}(x_2)\frac{\partial v}{\partial t}. \end{cases} \quad (5.4)$$

Write $n_P = \dim x_P$, $n_1 = \dim x_1$, $n_2 = \dim x_2$, $n = n_P + n_1 + n_2$, and $x = (x_P, x_1, x_2)$. We say that the weights *well behaved* provided that $\mathcal{W}_1$ and

$$\dot{x}_2 = A^{\mathcal{W}_2}(x_2) + B^{\mathcal{W}_2}(x_2)v, \; z_2 = C^{\mathcal{W}_2}(x_2) + D^{\mathcal{W}_2}(x_2)v$$

are asymptotically stable in a very strong sense. (This is not the same as saying $\mathcal{W}_1$ and $\mathcal{W}_2$ stable since the $\frac{d}{dt}v$ destablizes $\mathcal{W}_2$.)

Standard form. Using such weights the extended system G (2.1) with state variables $x = (x_P, x_1, x_2)'$ corresponding to (5.1), (5.3), (5.4) satisfies G:

$$\begin{pmatrix} \dot{x}_P \\ \dot{x}_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_P(x_P) + B_P(x_P)u \\ (A_{\mathcal{W}_1}(x_1) \\ \quad + B_{\mathcal{W}_1}(x_1)(-C_P(x_P) + w)) \\ A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)C_P(x_P) \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} C_{\mathcal{W}_1}(x_1) \\ (C_{\mathcal{W}_2}(x_2) + D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)) \\ \quad .(A_P(x_P) + B_P(x_P)u)) \end{pmatrix}$$

$$y = -C_P(x_P) + w. \quad (5.5)$$

Of course, $u = K(y)$. ¿From (5.5) it is evident that

$$A^{\times}(x_P, x_1, x_2) = \begin{pmatrix} A_P(x_P) \\ A_{\mathcal{W}_1}(x_1) \\ A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)C_P(x_P) \end{pmatrix}, \quad (5.6)$$

and the "D" terms of G satisfy $D_{11}(x) = 0$, $D_{12}(x) = (0, [D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)]B_P(x_P)]')'$, $D_{21}(x) = I$, $D_{22}(x) = 0$. Thus $D_{21}(x) = I_p$, and if

$$D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)]B_P(x_P)$$

is full rank, then $E_1(x) = D_{12}(x)'D_{12}(x) > 0$ is invertible, and indeed we have produced the *two block problem*.

The stability of the weights insure that the instabilities of this system are in perfect correspondence with the instabilities of P.

Theorem 5.1 *Assume that the vector field A_P is strongly hyperbolic with $\dim M^P_{as} = k_{as}$ and that the vector fields $A_{\mathcal{W}_1}$, $A_{\mathcal{W}_2}$ are strongly stable. Then the vector field $A^{\times}$ defined by (5.6) is strongly hyperbolic with antistable manifold M_{as} of dimension k_{as}.*

PROOF. By the definition of (global) hyperbolicity, we assume that after a coordinate transformation that $\mathbf{R}^{n_P} = M^P_a \times M^P_{as}$, and $x_P = (x_{Ps}, x_{Pas})$. Define

$$M_s = M^P_s \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2},$$

and

$$M_{as} = M^P_{as} \times \{0_{n_1}\} \times \{0_{n_2}\}.$$

The hypotheses imply that if $x(0) \in M_s$, then $x(t) \to 0$ as $t \to \infty$, where $x(\cdot)$ is the solution to the $A^{\times}$ dynamics. Similarly, if $x(0) \in M_{as}$, then $x(t) \to 0$ as $t \to -\infty$. By construction, $\dim M_{as} = k_{as} = \dim M^P_{as}$, as required. □

Formula for the controller. To get an explicit formula for the controller one merely combines Corollary 4.2 and formulas (4.6) describing the controller for the two block problem with formulas (5.6) and (5.5) give the two block problem in terms of the mixed sensitivity problem. We reiterate that the dynamics of the controller has the numerical complexity of a PDE on a $k = \dim M^P_{as} << n$ dimensional space. Writing them out in $A_{\mathcal{W}}$ notation is not particularly informative, so we shall not. However, we will be somewhat more explicit in the special case that $A^{\times}$ is strongly stable. Indeed, see **[BH2]**, **[HJ2]**:

Corollary 5.2 *Assume*

(i) $B_1(x_P, x_1, x_2) = (0, B_{\mathcal{W}_1}(x_1)', 0)'$ and $B_2(x_P, x_1, x_2) = (B_P(x_P)', 0, 0)'$ are independent of x_P, x_1.

(ii) $A_P(x_P)$ is strongly stable.

(iii) There exists a (smooth) solution $V(x_P, x_1, x_2)$ solving the PDE (4.7) with $A^*(x_P, x_1, x_2) = A - B_2C_1 + (B_1B_1' - B_2B_2')\nabla_x V'$ strongly stable.

Then the controller $K^*_{\delta_0}$ defined by (4.9) solves the mixed sensitivity control problem for P with internal stability. This controller is finite dimensional, or from the information state PDE viewpoint, it is given by a PDE on 0 dimensional space.

6. References

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Appendix

Regularity of $W(p)$. We say that W is *regular* if W is smooth for $p \in \mathcal{X}$ (see **[JB1]**, **[HJ2]**) and satisfies the following properties for $p \in \mathcal{X}_e$ of the form (4.1):

(i) For any continuous and bounded (above) function $h \in C(M)$, W at $p = \delta_M + \phi$ is differentiable in the direction h and

$$\nabla_p W(\delta_M + \phi)[h] = \Gamma(M, \phi)[h],$$

for some operator $\Gamma : C(M) \to \mathbf{R}$ depending on M, ϕ.

(ii) If M_t and ϕ_t are defined as in Corollary 4.2, then

$$\frac{d}{dt} W(\delta_{M_t} + \phi_t) = \Gamma(M_t, \phi_t)[\Omega(\cdot, M_t, \phi_t) \cdot A^\times(\cdot)]. \tag{7.7}$$

Here $\Omega(\cdot, M, \phi)$, is a continuous n-vector valued function on $\mathbf{R}^n$, and Γ is an extension of the operator in (i).

This gives the interpretation

$$\nabla_p W(\delta_M + \phi)[\nabla_x p \cdot A^\times] = \Gamma(M, \phi)[\Omega(\cdot, M, \phi) \cdot A^\times]; \tag{7.8}$$

whence, from (3.4),

$$\begin{aligned} \mathbf{u}^*(\delta_M + \phi) &= (\Gamma(M, \phi)[E_1])^{-1} \\ &.\Gamma(M, \phi)[-D_{12}' C_1 - B_2' \cdot \Omega(\cdot, M, \phi)]. \end{aligned} \tag{7.9}$$

Example. If $A^\times$ is strongly stable, $M = \{\xi\}$, and ϕ is a constant. Then

$$\Gamma(\xi, \phi)[h] = E_\xi[h] = h(\xi),$$

(evaluation map), and $\Omega(x, \xi, \phi) = \nabla_x V(x)$, where $V(x)$ is the state feedback value function, see (4.7).

Certainty equivalence. The certainty equivalence case corresponds to the function $\tilde{W}(p) = \langle p + V \rangle$ solving the infinite dimensional PDE (3.3). If $\bar{x}(p) = \operatorname{argmax}_{x \in \mathbf{R}^n} \{p(x) + V(x)\}$ is unique, then for non-singular $p \in \mathcal{X}$ ($M = \mathbf{R}^n$, $\phi = p$) corresponding to $A^\times$ strongly antistable,

$$\nabla_p \tilde{W}(p)[h] = E_{\bar{x}(p)}[h] = h(\bar{x}(p)),$$

and $\Omega(x, \mathbf{R}^n, p) = \nabla_x V(x)$.