Chapter 5

State Feedback H_∞ Control

This chapter is concerned with dissipative systems, the Bounded Real Lemmas, and the state feedback H_∞ control problem. Our objective is to provide a summary of some of the results available in the literature concerning the better-understood state feedback problem; this material is provided for use elsewhere in this book as well as to give the reader a familiar reference point to help interpret the output feedback results contained in other chapters of this book.

5.1 Dissipative Systems

The notion of dissipative dynamical system was introduced by Willems [Wil72], and the associated theory has important applications in the stability analysis of dynamical systems. The theory is a generalization of the behavior of passive electrical circuits, and other physical systems which dissipate energy. Central to this theory are the so-called storage functions, which satisfy an inequality known as the dissipation inequality, the infinitesimal version of which is a partial differential inequality (PDI). If a system possesses a storage function, then this storage function can be used as a (candidate) Lyapunov function in a stability analysis. In general, a dissipative system is stable provided it is observable or detectable. The framework provides a link between input-output and state space stability concepts. See the references [Wil72], [HM76], [HM77], [HM80a], [vdS96], etc, for more information.

The systems we consider are described by models of the form

$$\Sigma \ : \ \begin{cases} \dot{x} \ = \ A(x) + B(x)w, \quad x(0) = x_0, \\ \\ z \ = \ C(x), \end{cases} \tag{5.1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^s$, and $z(t) \in \mathbf{R}^r$.

Assumption 5.1.1 We assume that all of the functions appearing in (5.1) are as smooth with bounded first and second order partial derivatives, that B is bounded, and that zero is an equilibrium: $A(0) = 0$ and $C(0) = 0$.

Given a control $w : [0, \infty) \to \mathbf{R}^s$, the solution at time $t \geq 0$ with initial condition x_0 is denoted $x(t) = \Phi^w_{t,0}(x_0)$; the corresponding output is $z(t) = C(\Phi^w_{t,0}(x_0))$.

The system (5.1) with *supply rate*

$$\mathbf{s}(w, z) = \tfrac{1}{2}\left(\gamma^2|w|^2 - |z|^2\right) \tag{5.2}$$

is said to be *dissipative* if there exists a non-negative function $V : \mathbf{R}^n \to \mathbf{R}$, called a *storage function*, with $V(0) = 0$, such that

$$V(x) \geq \sup_{t\geq 0,\, w\in L_{2,T}} \left\{ V(x(t)) - \int_0^t \mathbf{s}(w(r), z(r))\, dr \ : \ x(0) = x \right\}. \tag{5.3}$$

Remark 5.1.2 Other choices of supply rate are possible, for example $\mathbf{s}(w, z) = w'z$. The choice (5.2) corresponds to L_2-gain (since $V \geq 0$):

$$\tfrac{1}{2}\int_0^T |z(t)|^2\, dt \leq \gamma^2 \tfrac{1}{2}\int_0^T |w(t)|^2\, dt + V(x_0) \tag{5.4}$$

for all $w \in L_{2,T}$ and all $T \geq 0$ (c.f. (2.10)). ∇

The relation (5.3) is known as the *dissipation inequality*, and expresses a constraint on the amount of (generalized) energy which can be extracted from the system. $V(x)$ is the amount of energy stored in the system when it is in state x, and is a candidate Lyapunov function. In general, a storage function is not uniquely defined, and in fact there is a continuum of storage functions for dissipative systems: $V_a \leq V \leq V_r$, where V_a is the available storage, and V_r is the required supply, [Wil72]. The *available storage* is defined by

$$V_a(x) = \sup_{T\geq 0, w\in L_{2,T}} \{-\tfrac{1}{2}\int_0^T [\gamma^2|w(s)|^2 - |z(s)|^2] ds\}, \tag{5.5}$$

where the integral is computed corresponding to a system (5.1) trajectory initialized at $x(0) = x$. The *required supply* is defined by

$$V_r(x) = \inf_{0\geq -T, w\in L_2[-T,0]} \{\tfrac{1}{2}\int_{-T}^0 [\gamma^2|w(s)|^2 - |z(s)|^2]\, ds\}, \tag{5.6}$$

where the integral is computed corresponding to a system trajectory starting at time $-T$ at $x(-T) = 0$ and terminated at time 0 at $x(0) = x$.

Storage functions can serve as Lyapunov functions in stability analysis. The following basic result is due to Willems [Wil72] (simplified).

Theorem 5.1.3 *Let V be a continuous storage function for the system (5.1) which is quadratically coercive: $V(x) \geq c_0|x|^2$ (for some constant $c_0 > 0$). Then the origin $x = 0$ is a stable equilibrium for the unforced system $\dot{x} = A(x)$.*

Proof. Set $w(\cdot) \equiv 0$ in the dissipation inequality (5.3) to get, integrating along the unforced trajectory,

$$V(x(t)) + \tfrac{1}{2}\int_0^t |z(s)|^2\, ds \leq V(x_0), \;\; t \geq 0. \tag{5.7}$$

This implies that $t \mapsto V(x(t))$ is non-increasing

$$V(x(t)) \leq V(x_0),$$

for all $t \geq 0$, and by quadratic coercivity we have

$$c_0|x(t)|^2 \leq V(x_0), \;\; \text{for all} \;\; t \geq 0.$$

This proves that the trajectory is bounded, and hence the unforced system is stable. Explicitly, the bound is

$$|x(t)| \leq \sqrt{\frac{V(x_0)}{c_0}} \;\; \text{for all} \;\; t \geq 0.$$

So if $\varepsilon > 0$ is given, select $\eta > 0$ such that $|x_0| \leq \eta$ implies $V(x_0) \leq c_0\varepsilon^2$. Then

$$|x(t)| \leq \sqrt{\frac{V(x_0)}{c_0}} \leq \sqrt{\frac{c_0\varepsilon^2}{c_0}} = \varepsilon \;\; \text{for all} \;\; t \geq 0.$$

□

Remark 5.1.4 The quadratic coercivity condition used in Theorem 5.1.3 can be relaxed to *properness*: the level sets $\{x \in \mathbf{R}^n \,:\, 0 \leq V(x) \leq c\}$ are compact for all $c > 0$. There are many criteria which imply this. ▽

Asymptotic stability requires stronger hypotheses. Recall that the system (5.1) is *strongly zero-state observable* if $w(t) = 0$, for all $t \geq 0$ implies there exists $T_1 > 0$, $c_1 > 0$ such that

$$\tfrac{1}{2}\int_0^T |C(\Phi^0_{t,0}(x_0))|^2\, dt \geq c_1|x_0|^2 \;\; \text{for all} \;\; T \geq T_1. \tag{2.27}$$

The following result is due to Hill and Moylan [HM80b] (simplified).

Theorem 5.1.5 *Let V be a continuous storage function for the system (5.1) which is strongly zero-state observable. Then the origin $x = 0$ is an asymptotically stable equilibrium for the unforced system $\dot{x} = A(x)$.*

Proof. Notice first that strong zero-state observability implies the quadratic coerciveness of storage functions (hence properness); thus by Theorem 5.1.3 the origin is stable.

As above, set $w(\cdot) \equiv 0$ in the dissipation inequality (5.3) to get, integrating along the unforced trajectory from $x(t)$ at time t to $x(t+T)$ at time $t+T$,

$$V(x(t+T)) + \tfrac{1}{2}\int_t^{t+T} |z(s)|^2\, ds \leq V(x(t)).$$

Strong zero-state observability implies

$$V(x(t+T)) + c_1|x(t)|^2 \leq V(x(t)). \tag{5.8}$$

Now $t \mapsto V(x(t))$ is monotone decreasing and bounded below by zero; therefore the limit $V_\infty = \lim_{t\to\infty} V(x(t))$ exists and is non-negative. Let $\varepsilon > 0$ be given. There exists $t_\varepsilon > 0$ such that $t \geq t_\varepsilon$ implies

$$V_\infty \leq V(x(t)) \leq V_\infty + c_1\varepsilon^2. \tag{5.9}$$

Now since $V(x(t+T)) \geq V_\infty$, we combine (5.8) and (5.9) to obtain

$$V_\infty + c_1|x(t)|^2 \leq V_\infty + c_1\varepsilon^2,$$

which implies

$$|x(t)| \leq \varepsilon \quad \text{for all} \quad t \geq t_\varepsilon.$$

$\square$

Remark 5.1.6 In the definition of strong zero-state detectability, the function $r \mapsto c_1r^2$ can be replaced by any strictly monotone function $\alpha : [0, +\infty) \to [0, +\infty)$ with $\alpha(0) = 0$. ∇

5.2 Bounded Real Lemma

The important Bounded Real Lemma for linear systems provides an algebraic test for the L_2-gain inequality (i.e. dissipativity, or $\| \Sigma \|_{H_\infty} < \gamma$) in terms of matrix inequalities, and storage functions can be computed by solving these matrix inequalities, see [AV73]. This result is useful in stability analysis and H_∞ control. The analogous characterization in the nonlinear context is a *partial differential inequality* (PDI).

The PDI associated with system (5.1) is

$$\sup_{w\in\mathbf{R}^s} \{\nabla_x V\cdot(A+Bw) + \tfrac{1}{2}(|C|^2 - \gamma^2|w|^2) \leq 0. \tag{5.10}$$

Evaluating the maximum gives

$$\nabla_x V\cdot A + \gamma^{-2}\tfrac{1}{2}\nabla_x VBB'\nabla_x V' + \tfrac{1}{2}|C|^2 \leq 0, \tag*{(5.10)$'$}$$

with optimal disturbance (it depends on the particular storage function V)

$$\mathbf{w}_V^*(x) = \gamma^{-2} B(x)' \nabla_x V(x)'. \tag{5.11}$$

The corresponding PDE is

$$\nabla_x V \cdot A + \gamma^{-2} \tfrac{1}{2} \nabla_x V B B' \nabla_x V' + \tfrac{1}{2} |C|^2 = 0, \tag{5.12}$$

with optimal disturbance $\mathbf{w}_V^*(x) = \gamma^{-2} B(x)' \nabla_x V(x)'$.

In general the PDI is satisfied by storage functions, while the PDE is satisfied by the available storage and required supply. Note that both the PDI and PDE can have many solutions, so when solving a PDI or PDE, one must be careful to select the desired solution. If one is seeking, e.g. the available storage, then the minimal non-negative solution should be sought. Another issue is that storage functions need not be differentiable, in which case the PDI and PDE can be interpreted in the viscosity sense ([Jam93], [BH96], Appendix B).

Given an interpretation of solutions to the PDI or PDE, we can state the following theorem.

Theorem 5.2.1 *The system (5.1) is dissipative if and only if there exists a solution* $V \geq 0$, $V(0) = 0$, *of the PDI (5.10) (viscosity sense).*

Proof. See, e.g., [Wil72], [HM76], [Jam93], [BH96]. □

Using observability conditions, as mentioned above, we can prove that a dissipative system is asymptotically stable. The system (5.1) is *zero-state observable* if $w(t) = 0$, $z(t) = 0$ for all $t \geq 0$ implies $x(t) = 0$ for all $t \geq 0$. The system (5.1) is *zero-state detectable* if $w(t) = 0$, $z(t) = 0$ for all $t \geq 0$ implies $x(t) \to 0$ as $t \to \infty$. A further result using these definitions is as follows (taken from [HM76], [vdS92], [vdS96] with some simplification).

Theorem 5.2.2 *Let* $V \geq 0$ *be a storage function for the system (5.1).*

(i) If (5.1) is zero-state observable, then $V(x) > 0$ *for all* $x \neq 0$ *(by definition* $V(0) = 0$*).*

(ii) Assume $V \in C^2$ *and* $V(x) > 0$ *for all* $x \neq 0$, $V(0) = 0$, $\nabla_x^2 V(0) \neq 0$, *and (5.1) is zero-state detectable. Then* $x = 0$ *is a locally asymptotically stable equilibrium of* $\dot{x} = A(x)$*. If also* V *is proper (i.e. the level sets* $\{x \in \mathbf{R}^n : 0 \leq V(x) \leq c\}$ *are compact for all* $c > 0$*), then* $x = 0$ *is an asymptotically stable equilibrium for* $\dot{x} = A(x)$*.*

Proof. To establish part (i), suppose $V(x_0) = 0$. Then by (5.4) we have, using $w(\cdot) = 0$,

$$0 \leq \int_0^T |z(t)|^2 \, dt \leq V(x_0) = 0, \quad \text{for all } T \geq 0.$$

This implies $z(\cdot) = 0$, and by zero-state observability it follows that $x(t) = 0$ for all $t \geq 0$. Therefore $x_0 = x(0) = 0$, and $V(x_0) > 0$ if $x_0 \neq 0$.

Part (ii). Since 0 is a strict minimum for V and since V is smooth, it follows from the mean value theorem that V is locally quadratically coercive:

$$V(x) \geq c_0|x|^2 \text{ if } |x| < r$$

(for some $c_0 > 0$, $r > 0$). The argument in the proof of Theorem 5.1.3 can be applied locally to show that trajectories starting near 0 remain bounded and that 0 is a stable equilibrium (see, e.g. [Vid93, pp158-159] for details).

Now it follows from (5.10) (with $w = 0$) that

$$\nabla_x V(x) \cdot A(x) \leq -\tfrac{1}{2}|C(x)|^2 \leq 0,$$

so applying the invariance principle (see, e.g. [Vid93, pp177-179]) that for x_0 sufficiently close to 0 we have

$$\Phi^0_{t,0}(x_0) \to M \text{ as } t \to \infty,$$

where M is the largest invariant subset of $\{x \in \mathbf{R}^n : \nabla_x V(x) \cdot A(x) = 0\}$. Thus local asymptotic stability will have been proven once it is shown that $M = \{0\}$. To see this, let $x_0 \in M$. Since M is invariant, by definition we get

$$V(\Phi^0_{t,0}(x_0)) = V(x_0) \text{ for all } t \geq 0. \tag{5.13}$$

This implies, via the dissipation inequality (5.7) with $w(t) = 0$ for all $t \geq 0$, that $z(t) = 0$ for all $t \geq 0$. Zero-state detectability implies that $\Phi^0_{t,0}(x_0) \to 0$ as $t \to \infty$, and combining this with (5.13), since V is continuous, we get

$$0 = V(0) = \lim_{t\to\infty} V(\Phi^0_{t,0}(x_0)) = V(x_0).$$

This implies $x_0 = 0$, and hence $M = \{0\}$.

Finally, if V is proper, it follows that 0 is an asymptotically stable equilibrium. □

An important link with the linear theory is given in [vdS91], [vdS92]. Consider the linearization of (5.1) at $x = 0$:

$$\Sigma_{lin} : \begin{cases} \dot{x} = A_{lin}x + B_{lin}w, \quad x(0) = x_0, \\ z = C_{lin}x, \end{cases} \tag{5.14}$$

where the constant matrices A_{lin}, B_{lin} and C_{lin} are given by

$$A_{lin} = \frac{\partial}{\partial x}A(x)|_{x=0}, \quad B_{lin} = \frac{\partial}{\partial x}B(x)|_{x=0}, \quad C_{lin} = \frac{\partial}{\partial x}C(x)|_{x=0}.$$

Theorem 5.2.3 *([vdS91], [vdS92, Theorems 8, 10]) Assume that A_{lin} is asymptotically stable and $\| \Sigma_{lin} \|_{H_\infty} < \gamma$. (i) Then there exists a neighborhood $\mathcal{W}$ of $x = 0$ and a smooth function $V^- \geq 0$ on $\mathcal{W}$ such that (5.12) holds in $\mathcal{W}$ and the vector field*

$$A + \gamma^{-2} B B' \nabla_x V^{-\prime}$$

is locally exponentially stable on $\mathcal{W}$. Moreover,

$$\tfrac{1}{2} \int_0^T |z(t)|^2 \, dt \leq \gamma^2 \tfrac{1}{2} \int_0^T |w(t)|^2 \, dt + V^-(x_0)$$

for all $w \in L_{2,T}$, $T \geq 0$ for which the trajectory $\Phi^w_{t,0}(x_0)$ remains in $\mathcal{W}$ for $0 \leq t \leq T$. (ii) If (A_{lin}, B_{lin}) is controllable, then there exists a neighborhood $\tilde{\mathcal{W}}$ of $x = 0$ smooth function $V^+ \geq 0$ on $\tilde{\mathcal{W}}$ such that the vector field

$$-(A + \gamma^{-2} B B' \nabla_x V^{+\prime})$$

is locally exponentially stable on $\tilde{\mathcal{W}}$.

It turns out that

$$V_a = V^-, \quad V_r = V^+ \text{ locally,}$$

or globally under an assumption concerning the existence of global stable and unstable invariant manifolds of the associated Hamiltonian vector field, see [vdS92, Theorem 13]. This means that

$$A + \gamma^{-2} B B' \nabla_x V_a' \quad \text{is (locally) exponentially stable}$$

and

$$-(A + \gamma^{-2} B B' \nabla_x V_r') \quad \text{is (locally) exponentially stable.}$$

So the available storage V_a is a *stabilizing* solution of the PDE (5.12), whereas the required supply V_r is an *antistabilizing* solution of the PDE (5.12). The vector fields $A + \gamma^{-2} B B' \nabla_x V_a' = A + B(\gamma^{-2} B' \nabla_x V_a')$ and $A + \gamma^{-2} B B' \nabla_x V_r' = A + B(\gamma^{-2} B' \nabla_x V_r')$ are obtained by using the optimal disturbances $\gamma^{-2} B' \nabla_x V_a'$ and $\gamma^{-2} B' \nabla_x V_r'$ respectively (these are particular instances of (5.11)). Thus the stabilizing/antistabilizing properties provide important characterizations of solutions to the PDE (5.12) and as well give information on the stability of the optimal-disturbance closed loop system. Notice that a strict inequality $\| \Sigma_{lin} \|_{H_\infty} < \gamma$ was required in Theorem 5.2.3. These are important themes in H_∞ control and will be discussed in more detail in the next section which deals with the Strict Bounded Real Lemma.

5.3 Strict Bounded Real Lemma

As we have seen, the Bounded Real Lemma provides a characterization of the dissipation property in terms of the solvability of a PDI (or PDE), and by making use of observability conditions, asymptotic stability of the uncontrolled system can be inferred. The Strict Bounded Real Lemma (SBRL) [PAJ91] applies to a linear system

$$\begin{cases} \dot{x} &= Ax + Bw \\ \\ z &= Cx \end{cases} \tag{5.15}$$

and says that the following three statements are equivalent

(i) A is *asymptotically stable* and

$$\| \, C(sI - A)^{-1}B \, \|_{H_\infty} < 1;$$

(ii) there exists a matrix $\tilde{P} > 0$ such that

$$A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + C'C < 0; \tag{5.16}$$

(iii) the Riccati equation

$$A'P + PA + PBB'P + C'C = 0 \tag{5.17}$$

has a solution $P \geq 0$ with $A + BB'P$ *asymptotically stable*.

Notice that the SBRL does not make any reachability/observability assumptions, but rather cleverly exploits the *strictness* of the H_∞ inequality and asymptotic stability.

The purpose of this section is to generalize this SBRL to the class of nonlinear systems (5.1) Precise statements to this effect are given in the following subsection. These results are used elsewhere in this book, and are special cases of results in the paper [YJH98]. Since in general solutions to nonlinear PDI and PDE need not be smooth globally (although they may be locally), one of our aims is to emphasize this point by indicating where smoothness is and is not needed. In the nonlinear case, there are numerous definitions of stability, and sometimes it is necessary to use rather strong forms of stability, and so we emphasize this point also.

5.3.1 Main Results

The system defined by (5.1) is said to have L_2 gain *strictly less than* γ if there exists a finite function $\beta(x) \geq 0$ with $\beta(0) = 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$\tfrac{1}{2}\int_0^T |z(t)|^2 dt \leq (1-\varepsilon^2)\gamma^2 \tfrac{1}{2}\int_0^T |w(t)|^2 dt + \beta(x_0) \tag{5.18}$$

for all $w \in L_2[0,T]$ and all $T \geq 0$. G has L_2 gain *less than* γ if (5.18) holds at least for $\varepsilon = 0$.

The generalization of the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) will be divided into three theorems, as follows. The definition of "viscosity sense" is given in Appendix B.

Theorem 5.3.1 *Assume (A, B) is L_2 stable and the system (5.1) has L_2 gain strictly less than γ. Then there exists a lower semicontinuous (l.s.c.) function $\tilde{V}(x) > 0$ for $x \neq 0$ with $\tilde{V}(0) = 0$ such that*

$$\sup_{w \in \mathbf{R}^s} \{\nabla_x \tilde{V} \cdot (A + Bw) - \tfrac{1}{2}[\gamma^2 |w|^2 - |C|^2]\} + \eta^2 \tfrac{1}{2}|x|^2 \leq 0 \tag{5.19}$$

in the viscosity sense (for some $\eta > 0$).

A function $\tilde{V}$ is *locally bounded* if the supremum of $|\tilde{V}(x)|$ is finite as x ranges over any compact subset.

Theorem 5.3.2 *Assume there exists a (locally bounded) l.s.c. function $\tilde{V}(x) \geq 0$ with $\tilde{V}(0) = 0$ satisfying the strict PDI (5.19) in the viscosity sense. Then there exists a (locally bounded) function $V(x) \geq 0$ with $V(0) = 0$ satisfying the PDE*

$$\sup_{w \in \mathbf{R}^s} \{\nabla_x V \cdot (A + Bw) - \tfrac{1}{2}[\gamma^2 |w|^2 - |C|^2]\} = 0 \tag{5.20}$$

in the viscosity sense (the available storage is such a function.) Further, if $\tilde{V}$ is continuous, V is smooth and $V \in \mathcal{X}_{q,2,b}$, then the vector field

$$A^*(x) \stackrel{\triangle}{=} A(x) + \gamma^{-2} B(x) B(x)' \nabla_x V(x)' \tag{5.21}$$

is asymptotically L_2 stable.

Theorem 5.3.3 *Assume there exists a smooth function $V(x) \geq 0$ with $V(0) = 0$ and $V \in \mathcal{X}_{q,2,b}$ satisfying the PDE (5.20) in the classical sense, and that the vector field $A^*(x)$ given by (5.21) is monotone stable. Then the vector field A is asymptotically L_2 stable and the system (5.1) has L_2 gain strictly less than γ.*

The circle of ideas embodied in these theorems is not as complete nor as elegant as in the linear case. This is because in the nonlinear case one does not necessarily have smoothness of solutions to PDEs and PDIs, and different types of stability arise naturally (some statements are not true without a sufficiently strong form of stability). However, these theorems contain the overall thrust of the SBRL, and it is likely that they can be sharpened with further effort.

5.3.2 Proofs of Main Results

In the following proofs we set $\gamma = 1$ for simplicity.

Proof of Theorem 5.3.1. The strict gain property (5.18) of (5.1) implies, for $0 < \varepsilon \leq \varepsilon_0$,

$$\tfrac{1}{2} \int_0^T [|z(t)|^2 + \varepsilon^2 |w(t)|^2] dt \leq \tfrac{1}{2} \int_0^T |w(t)|^2 dt + \beta(x_0) \tag{5.22}$$

for all $w \in L_2[0,T]$ and all $T \geq 0$. The L_2 stability of A implies there exist $\tilde{\gamma} > 0$ and $\tilde{\beta}(x) \geq 0$, $\tilde{\beta}(0) = 0$ such that

$$\tfrac{1}{2}\int_0^T |x(t)|^2 dt \leq \tilde{\gamma}^2 \tfrac{1}{2}\int_0^T |w(t)|^2 dt + \tilde{\beta}(x_0) \tag{5.23}$$

for all $w \in L_2[0,T]$ and all $T \geq 0$ [FJ96]. Together, (5.22) and (5.23) imply that the system

$$\tilde{\Sigma} : \begin{cases} \dot{x} \;=\; A(x) + B(x)w \\ \tilde{z} \;=\; \begin{pmatrix} C(x) \\ \dfrac{\varepsilon}{\tilde{\gamma}} x \end{pmatrix} \end{cases} \tag{5.24}$$

has L_2 gain less than one:

$$\tfrac{1}{2}\int_0^T |\tilde{z}(t)|^2 dt \leq \tfrac{1}{2}\int_0^T |w(t)|^2 dt + \bar{\beta}(x_0) \tag{5.25}$$

where $\bar{\beta}(x) = \beta(x) + \frac{\varepsilon}{\tilde{\gamma}}\tilde{\beta}(x)$. By the Bounded Real Lemma (Theorem 5.2.1) there exists a l.s.c. (storage) function $\tilde{V} \geq 0$ with $\tilde{V}(0) = 0$ solving the PDI

$$\sup_{w \in \mathbf{R}^s} \{\nabla_x \tilde{V} \cdot (A + Bw) - \tfrac{1}{2}[|w|^2 - (|C|^2 + \eta^2|x|^2)]\} \leq 0$$

where $\eta \stackrel{\triangle}{=} \frac{\varepsilon}{\tilde{\gamma}}$. However, this is just (5.19).

Now the function $\tilde{V}$ satisfies the integral representation (dissipation inequality)

$$\begin{aligned} \tilde{V}(x) \geq \sup_{T \geq 0, w \in L_2[0,T]} &\{\tilde{V}(x(T)) + \tfrac{1}{2}\int_0^T [|z(t)|^2 \\ &+\eta|x(t)|^2 - |w(t)|^2] dt : x(0) = x\} \end{aligned} \tag{5.26}$$

If $\tilde{V}(x_0) = 0$ for some x_0, then (5.26) implies with $T > 0$ and $w = 0$ that

$$0 = \tilde{V}(x_0) \geq \tfrac{1}{2}\int_0^T \eta|x(t)|^2 dt \geq 0.$$

Hence $x(t) = 0$, $0 \leq t \leq T$. By uniqueness of solutions to ODEs, we must have $x_0 = 0$. □

Proof of Theorem 5.3.2. By Theorem 3.1 of [Jam93], any viscosity solution $\tilde{V} \geq 0$ of the PDI (5.19) is a storage function for the system (5.1), and hence satisfies the integral representation

$$\begin{aligned} \tilde{V}(x) \;\geq\; &\sup_{T \geq 0, w \in L_2[0,T]} \{\tilde{V}(x(t)) + \tfrac{1}{2}\int_0^T [|z(t)|^2 \\ &\quad +\eta|x(t)|^2 - |w(t)|^2] dt : x(0) = x\} \\ \geq\; &\sup_{T \geq 0, w \in L_2[0,T]} \{\tfrac{1}{2}\int_0^T [|z(t)|^2 + \eta|x(t)|^2 - |w(t)|^2] dt : x(0) = x\} \end{aligned} \tag{5.27}$$

Now define $V(x)$ by the formula

$$V(x) = \sup_{T \geq 0, w \in L_2[0,T]} \{ \tfrac{1}{2} \int_0^T [|z(t)|^2 - |w(t)|^2] dt \; : \; x(0) = x \}. \tag{5.28}$$

Then from (5.27) and (5.28) we have $0 \leq V(x) \leq \tilde{V}(x)$ and, so $V(x)$ exists (is not ∞ for any x). $V(x)$ is called the available storage. The proof of the fact that V is a viscosity solution of (5.20) follows from [Sor96], [McE95b].

Now we suppose that V is a *smooth* solution of the PDE (5.20), and let

$$\mathbf{w}^*(x) = B(x)' \nabla_x V(x)'.$$

Define $S(x) = \tilde{V}(x) - V(x)$ (it is *not* assumed that $\tilde{V}$ is smooth). Then $S(x) \geq 0$ and $S(0) = 0$ (actually $S(x) > 0$ if $x \neq 0$ see below).

Set $w = \mathbf{w}^*$ in (5.27) to get

$$\tilde{V}(x(0)) - \tilde{V}(x(T)) \geq \tfrac{1}{2} \int_0^T [|z(t)|^2 + \eta |x(t)|^2 - |w^*(t)|^2] dt, \tag{5.29}$$

where $w^*(t) = \mathbf{w}^*(x(t))$. Integrate (5.20) to get

$$V(x(0)) - V(x(T)) = \tfrac{1}{2} \int_0^T [|z(t)|^2 - |w^*(t)|^2] dt. \tag{5.30}$$

Subtract the previous two equations to get

$$S(x(T)) \leq -\tfrac{1}{2} \eta^2 \int_0^T |x(s)|^2 ds + S(x(0)), \tag{5.31}$$

so $S(x(T))$ is strictly decreasing. Moreover, since $S(x(T)) \geq 0$ this inequality implies that any initial $x(0)$ produces a trajectory $x(\cdot)$ for which $x(\cdot)$ is in $L_2[0, \infty)$.

Since $V \in \mathcal{X}_{q,2,b}$, it follows that

$$|\dot{x}| \leq c|x|,$$

for some constant $c \geq 0$, and so $\dot{x}(\cdot) \in L_2[0, \infty)$. Now this bound also shows

$$\begin{aligned} |x(t)|^2 &= |x(s)|^2 + 2 \int_s^t \dot{x}(r)' x(r) dr \\ &\leq |x(s)|^2 + c \int_s^t |x(r)|^2 dr, \end{aligned} \tag{5.32}$$

for some constant $c \geq 0$. This inequality and (5.31) together with the continuity of S prove stability and asymptotic stability for A^*.

Now we prove strict positivity of S. Suppose $S(x_0) = 0$ for some x_0. Then (5.31) implies

$$\tfrac{1}{2} \eta^2 \int_0^T |x(s)|^2 ds \leq 0,$$

hence $x(s) = 0$, $s \geq 0$. By uniqueness of solutions to ODE, we must have $x_0 = 0$. □

Proof of Theorem 5.3.3. To show that A is asymptotically stable, we write

$$\dot{x}(t) = A(x(t)),\ x(0) = x_0,$$

$$\dot{\xi}(t) = A^*(\xi(t)),\ \xi(0) = \xi_0,$$

and $\zeta(t) = \xi(t) - x(t)$. If $V(x)$ is a solution of (5.20), then with $w = \mathbf{w}^*$ we have

$$\nabla_x V \cdot A + \tfrac{1}{2}|B'\nabla_x V'|^2 + \tfrac{1}{2}|C|^2 = 0,$$

which implies

$$\nabla_x V \cdot A + \tfrac{1}{2}|B'\nabla_x V'|^2 \leq 0.$$

Hence

$$V(x(t)) + \tfrac{1}{2}\int_0^t |B(x(s))'\nabla_x V(x(s))'|^2 ds \leq V(x_0),$$

which implies $B(x(\cdot))'\nabla_x V(x(\cdot))' \in L_2[0,\infty)$.

Next, by monotone stability,

$$\begin{aligned}
\tfrac{d}{dt}\tfrac{1}{2}|\zeta|^2 &= \zeta'\tfrac{d}{dt}\zeta = (A^*(\xi) - A^*(x))\zeta + \zeta' B(x)B(x)'\nabla_x V(x)' \\
&\leq -c|\zeta|^2 + \zeta' B(x)B(x)'\nabla_x V(x)' \\
&\leq -c|\zeta|^2 + |\zeta| \parallel B \parallel_{L_\infty} |B(x)'\nabla_x V(x)'| \\
&\leq -c\tfrac{1}{2}|\zeta|^2 + c_1|B(x)'\nabla_x V(x)'|^2
\end{aligned}$$

for constants $c > 0$ and $c_1 = \parallel B \parallel_{L_\infty} /2c$ (using $ab \leq (\alpha/2)a^2 + (1/2\alpha)b^2$ for any $\alpha > 0$). Integrating, we get

$$\tfrac{1}{2}|\zeta(t)|^2 \leq e^{-ct}\tfrac{1}{2}|\zeta_0|^2 + \tfrac{1}{2}c_1\int_0^t e^{-c(t-s)}|B(x(s))'\nabla_x V(x(s))'|^2 ds.$$

Therefore $\zeta(\cdot)$ is in $L_2[0,\infty)$. This implies $x(\cdot) \in L_2$ since $\xi(\cdot) \in L_2$. Also, since $|\dot{x}(\cdot)| \leq c_2|x(\cdot)|$ (some $c_2 \geq 0$), $\dot{x}(\cdot)$ is in L_2, and it follows that A is asymptotically stable.

It follows easily that the system (5.1) has L_2 gain less than one, since $V(x)$ serves as a storage function. To see that it is *strictly* less than one needs a bit more work, as follows (c.f. [GL95] in the linear case). Consider the auxiliary system

$$L : \begin{cases} \dot{x} = A(x) + B(x)w \\ v = w - \mathbf{w}^*(x), \end{cases}$$

where $\mathbf{w}^*(x) = B(x)'\nabla_x V(x)'$. The inverse of L is

$$L^{-1} : \begin{cases} \dot{x} = A(x) + B(x)\mathbf{w}^*(x) + B(x)v \\ w = \mathbf{w}^*(x) + v. \end{cases}$$

Since $A^* = A + B\mathbf{w}^*$, (A^*, B) is incrementally L_2 exponentially stable, there exists $\hat{\gamma} > 0$ and a constant $c_2 \geq 0$ such that

$$\tfrac{1}{2}\int_0^T |w(t)|^2 dt \leq \hat{\gamma}^2 \tfrac{1}{2}\int_0^T |v(t)|^2 dt + c_2 |x(0)|^2$$

for all $v \in L_{2,T}$, for all $T \geq 0$.

Fix $T \geq 0$, and $w \in L_{2,T}$, and input this into (5.1), giving trajectory $x(\cdot)$ and $z(\cdot)$. Write $w(t) = \mathbf{w}^*(x(t)) + v(t)$; then $v \in L_{2,T}$ and the previous display holds. That is, on rearranging,

$$-\tfrac{1}{2}\int_0^T |v(t)|^2 dt \leq -\hat{\gamma}^{-2}\tfrac{1}{2}\int_0^T |w(t)|^2 dt + c_2\hat{\gamma}^{-2}|x(0)|^2.$$

By completion of squares, for any $w \in L_{2,loc}$ we have

$$\begin{aligned} V(x(T)) \; + \tfrac{1}{2}\int_0^T [|z(s)|^2 - |w(s)|^2] ds &= -\tfrac{1}{2}\int_0^T |w(s) - \mathbf{w}^*(x(s))|^2 ds + V(x(0)) \\ &\leq \hat{\gamma}^{-2}(-\tfrac{1}{2}\int_0^T |w(s)|^2 ds + c_2|x(0)|^2) + V(x(0)). \end{aligned}$$

This implies that the system (5.1) has finite gain strictly less than one. □

5.4 State Feedback H_∞ Control

In the state feedback case the controller is a function of the state x, i.e. $u = K_{state}(x)$. For simplicity, we assume this to be static state feedback. It is not necessary to use an information state; all that is required is an analog of the value function $W(p)$ and corresponding dynamic programming equation (4.16), and formulas for the optimal state feedback control and disturbance. In §5.4.5 connections with output feedback control will be discussed.

For an excellent and comprehensive account of the state feedback theory, see [vdS92], [vdS96].

5.4.1 The State Feedback Problem

In the state feedback context the plant G reads

$$G : \begin{cases} \dot{x} = A(x) + B_1(x)w + B_2(x)u \\ \\ z = C_1(x) + D_{12}(x)u \end{cases} \tag{5.33}$$

This model is obtained from (2.2) by omitting the output y. The control u is given by static state feedback, $u = K_{state}(x)$. Such a controller is *admissible* if the closed-loop trajectories exist and are unique. This will be the case, e.g., if K_{state} is Lipschitz continuous.

The problem is to find a state feedback controller K^*_{state} such that the closed-loop system (G, K^*_{state}) is γ-dissipative and internally stable (weak internal stability and internal stability are equivalent in the state feedback case).

5.4.2 A State Feedback H_2 Assumption

We consider an H_2 state feedback assumption. This assumption is stronger than the condition (1.6) for linear systems, and is very useful for stability. The H_2 assumption concerns the PDE

$$\begin{aligned}\nabla_x V_{H_2}\left(A - B_2 E_1^{-1} D_{12} C_1\right) - \tfrac{1}{2}\nabla_x V_{H_2} B_2 E_1^{-1} B_2' \nabla_x V_{H_2}' \\ + \tfrac{1}{2} C_1'(I - D_{12} E_1^{-1} D_{12}')C_1 = 0\end{aligned} \tag{5.34}$$

and the vector field

$$A^*_{H_2} \stackrel{\triangle}{=} A - B_2 E_1^{-1} D_{12} C_1 - B_2 E_1^{-1} B_2' \nabla_x V_{H_2}'. \tag{5.35}$$

> Assumption 5.4.1 The state feedback H_2 PDE (5.34) has a smooth solution V_{H_2} which is non-negative, vanishes at zero, is quadratically coercive
> $$V_{H_2}(x) \geq c_{H_2}|x|^2, \tag{5.36}$$
> with $c_{H_2} > 0$, and the optimal closed loop vector field $A^*_{H_2}$ given by (5.35) is incrementally L_2 exponentially stable.

The function V_{H_2} is called the H_2 *state feedback value function* and is given by

$$\begin{aligned}V_{H_2}(x) = \inf_{K_{state}} \{\tfrac{1}{2}\int_0^\infty |C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2\, ds \; : \; \xi(0) = x, \\ \lim_{t\to\infty}\xi(t) = 0,\; u(s) = K_{state}(\xi(s)),\; 0 \leq s \leq t\}\end{aligned} \tag{5.37}$$

where $\xi(\cdot)$ solves

$$\dot{\xi} = A(\xi) + B_2(\xi)u. \tag{5.38}$$

5.4.3 Necessity

A state feedback value function $V(x)$ can be defined as follows:

$$V(x) = \inf_K \sup_{T\geq 0,\, w\in L_{2,T}} \{\tfrac{1}{2}\int_0^T [|z(s)|^2 - \gamma^2|w(s)|^2]\, ds \; : \; x(0) = x\}, \tag{5.39}$$

where the infimum is taken over the class of all admissible static state feedback controllers $K_{state}(x)$ achieving dissipation and internal stability. (Under suitable conditions, the value of the infimum is the same if one uses the class of all causal full information controllers $u(t) = K(x_{0,t}, w_{0,t})$.)

We begin with a lemma that relates the H_2 and H_∞ state feedback value functions.

Lemma 5.4.2 *For all $x \in \mathbf{R}^n$ we have*

$$V(x) \geq V_{H_2}(x), \tag{5.40}$$

and in view of the coercivity of V_{H_2} the value function V is coercive:

$$V(x) \geq c_{H_2}|x|^2, \quad (c_{H_2} > 0). \tag{5.41}$$

Indeed, any bias will be coercive: $\beta_{K_{0,state}}(x) \geq c_{H_2}|x|^2$, where $K_{0,state}$ is any controller solving the state feedback H_∞ control problem.

Proof. For any stabilizing controller K_{state} we have

$$\tfrac{1}{2}\int_0^T |z(s)|^2\,ds \leq \sup_{w(\cdot)} \tfrac{1}{2}\int_0^T [|z(s)|^2 - \gamma^2|w(s)|^2]\,ds$$

where the dynamics are given by (5.33) with $x(0) = x$, and $w(\cdot) = 0$ in the LHS. Since K is stabilizing, this inequality and the definition of V_{H_2} implies

$$V_{H_2}(x) \leq \tfrac{1}{2}\int_0^\infty |z(s)|^2\,ds \leq \sup_{T\geq 0,\, w(\cdot)} \tfrac{1}{2}\int_0^T [|z(s)|^2 - \gamma^2|w(s)|^2]\,ds.$$

Now minimize the RHS over K to obtain (5.40). □

Theorem 5.4.3 *Assume that the state feedback γ-dissipative control problem for G is solved by a controller $u = K_{0,state}(x)$. Then the value function defined by the formula (5.39) enjoys the following properties:*

(i) *$V(x)$ is finite.*

(ii) *$V(x) \geq 0$ for all $x \in \mathbf{R}^n$, and $V(0) = 0$.*

(iii) *If K^ε_{state} is ε-optimal, then for any disturbance input $w \in L_{2,loc}$,*

$$V(x) + \varepsilon \geq V(x(t)), \text{ where } u(t) = K^\varepsilon_{state}(x(t)),\ t \geq 0, \tag{5.42}$$

where $x(\cdot)$ is the solution of (5.33) with $x(0) = x$. Thus any almost-optimal state feedback controller results in a state trajectory along which $V(x(t))$ is almost-decreasing.

(iv) *The dynamic programming principle holds: for any $t \geq 0$,*

$$V(x) = \inf_{K_{state}} \sup_{w \in L_{2,t}} \{V(x(t)) + \tfrac{1}{2}\int_0^t [\gamma^2|w(s)|^2 - |z(s)|^2]\,ds\} \tag{5.43}$$

where $x(\cdot)$ is the solution of (5.33) with $x(0) = x$ and $u = K_{state}(x)$.

(v) *If an admissible optimal state feedback controller K^*_{state} exists, then along any resulting optimal closed loop state trajectory, for any disturbance input $w \in L_{2,loc}$,*

$$V(x) + \tfrac{1}{2}\gamma^2 \int_0^t |w(s)|^2 \, ds \geq V(x(t)) + \tfrac{1}{2}\int_0^t |z(s)|^2 \, ds, \qquad (5.44)$$

$$\text{where } u(t) = K^*_{state}(x(t)), \ t \geq 0,$$

where $x(\cdot)$ is the solution of (5.33) with $x(0) = x$.

Therefore the closed-loop system (G, K^*_{state}) is γ-dissipative.

(v-a) *If (G, K^*_{state}) is z-detectable, then the closed-loop system is internally stable.*

(v-b) *If Assumption 5.4.1 holds, then the closed-loop system is asymptotically stable.*

Proof. By the definition of V and the hypotheses, we see immediately that

$$0 \leq V(x) \leq \beta_{K_0}(x),$$

where β_{K_0} is the bias for the γ-dissipative controller K_0. This proves items (i) and (ii).

Items (iii) and (iv) follow from dynamic programming considerations, see [McE95b], [Sor96], [vdS92].

The dissipation inequality (5.44) of item (v) follows from (5.43) using the optimal controller stated. Item (v-a) follows from Theorem 2.1.3. Item (v-b) follows as in the proof of Theorem 5.1.5, since by Lemma 5.4.2 the value function V is coercive. □

Remark 5.4.4 We see that the state feedback H_2 Assumption 5.4.1 allows us to deduce stability of the closed-loop system without having to invoke the z-detectability assumption. ▽

The PDE satisfied by V is

$$\begin{aligned} &\nabla_x V \left(A - B_2 E_1^{-1} D_{12} C_1\right) + \tfrac{1}{2}\nabla_x V \left(\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2'\right) \nabla_x V' \\ &\qquad + \tfrac{1}{2} C_1' (I - D_{12} E_1^{-1} D_{12}') C_1 = 0. \end{aligned} \qquad (5.45)$$

This equation holds in general in the viscosity sense. If V is sufficiently smooth, the optimal control and disturbance are given by

$$\mathbf{u}^*_{state}(x) = -E_1(x)^{-1}(D_{12}(x)' C_1(x) + B_2(x)' \nabla_x V(x)'), \qquad (5.46)$$

and

$$\mathbf{w}^*_{state}(x) = \gamma^{-2} B_1(x)' \nabla_x V(x)'. \qquad (5.47)$$

In general the value function is not globally smooth. However, the following fact follows from [Sor96], [McE95b].

Theorem 5.4.5 *Assume that the state feedback γ-dissipative control problem for G is solved by a controller $u = K_{0,state}(x)$. Then the value function defined by the formula (5.39) is a viscosity solution of the PDE (5.45).*

We shall assume that V is smooth in order to define the *state feedback central controller* K^*_{state}:

$$u(t) = K^*_{state}(x(t)) \stackrel{\triangle}{=} \mathbf{u}^*_{state}(x(t)). \tag{5.48}$$

This is illustrated in Figure 5.1.

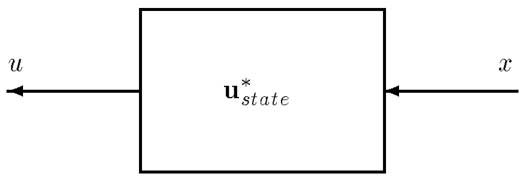


Figure 5.1: State feedback central controller K^*_{state}.

To be precise, the amount of smoothness required of V is C^1 with $\mathbf{u}^*_{state}$ (which depends on $\nabla_x V$) globally Lipschitz, so that solutions to the state equation in (5.33) exists and are unique (i.e. K^*_{state} is admissible in the sense that solutions to the closed-loop differential equations exist and are unique).

Theorem 5.4.6 *Assume that the state feedback γ-dissipative control problem for G is solved by a controller $u = K_{0,state}(x)$. Assume that the value function $V(x)$ defined by (5.39) is smooth and the state feedback central controller defined from V via (5.48) is admissible. Then (5.44) holds, so the closed-loop (G, K^*_{state}) is γ-dissipative. If (G, K^*_{state}) is z-detectable, then the closed-loop system is internally stable. If Assumption 5.4.1 holds, then the closed-loop system is asymptotically stable.*

Proof. By Theorem 5.4.3, we know that V is finite, and the conclusions of that theorem hold. If V is smooth, dynamic programming shows that the optimal control is given by (5.48). □

5.4.4 Sufficiency

For dissipation, the dissipation inequality is all that is required. In this subsection we will construct state feedback controllers from smooth solutions of the *state feedback dissipation PDI*:

$$\begin{aligned} &\nabla_x \hat{V}\left(A - B_2 E_1^{-1} D_{12} C_1\right) + \tfrac{1}{2}\nabla_x \hat{V}\left(\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2'\right)\nabla_x \hat{V}' \\ &\qquad + \tfrac{1}{2} C_1'(I - D_{12} E_1^{-1} D_{12}') C_1 \le 0. \end{aligned} \tag{5.49}$$

Of course, the state feedback value function V is a particular (and minimal, $V \leq \hat{V}$) solution. The optimal control and disturbance corresponding to $\hat{V}$ are

$$\mathbf{u}^*_{state,\hat{V}}(x) = -E_1(x)^{-1}(D_{12}(x)'C_1(x) + B_2(x)'\nabla_x \hat{V}(x)'), \tag{5.50}$$

and

$$\mathbf{w}^*_{state,\hat{V}}(x) = \gamma^{-2} B_1(x)'\nabla_x \hat{V}(x)'. \tag{5.51}$$

In this way the function $\hat{V}$ solving the PDI determines a controller $K_{state}^{\mathbf{u}^*_{state,\hat{V}}}$.

Theorem 5.4.7 *Assume that there exists a smooth solution $\hat{V}(x) \geq 0$, $\hat{V}(0) = 0$, of the PDE (5.49) and that the controller $K_{state}^{\mathbf{u}^*_{state,\hat{V}}}$ defined by $\mathbf{u}^*_{state,\hat{V}}$ is admissible. Then for any disturbance input $w \in L_{2,loc}$,*

$$\begin{aligned} &\hat{V}(x) + \tfrac{1}{2}\gamma^2 \int_0^t |w(s)|^2\, ds \geq \hat{V}(x(t)) + \tfrac{1}{2}\int_0^t |z(s)|^2\, ds, \\ &\qquad \text{where } u(t) = K_{state}^{\mathbf{u}^*_{state,\hat{V}}}(x(t)),\ t \geq 0, \end{aligned} \tag{5.52}$$

where $x(\cdot)$ is the solution of (5.33) with $x(0) = x$. So the closed-loop $(G, K_{state}^{\mathbf{u}^*_{state,\hat{V}}})$ is γ-dissipative. If $(G, K_{state}^{\mathbf{u}^*_{state,\hat{V}}})$ is z-detectable, then the closed-loop system is weakly internally stable. If Assumption 5.4.1 holds, then the closed-loop system is asymptotically stable.

Proof. Once the controller is constructed from $\hat{V}$, the proof follows the proofs of Theorems 5.4.3 and 5.4.6. □

Theorem 5.4.8 *Assume there exists a smooth solution $\hat{V}(x) \geq 0$ of (5.45) with $\hat{V}(0) = 0$ and that the vector field*

$$A^*_{state,\hat{V}} \stackrel{\triangle}{=} A - B_2 E_1^{-1} D_{12} C_1 + (\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2')\nabla_x \hat{V}'$$

is monotone stable. Assume that the state feedback controller $K_{state}^{\mathbf{u}^*_{state,\hat{V}}}(x) = \mathbf{u}^*_{state,\hat{V}}(x)$ is admissible. Then the closed loop system $(G, K_{state}^{\mathbf{u}^*_{state,\hat{V}}})$ is strictly γ-dissipative and asymptotically L_2 stable.

Proof. Apply Theorem 5.3.3 to the system

$$\dot{x} = A(x) - B_2(x)E_1^{-1}(x)D_{12}(x)C_1(x) - B_2(x)E_1(x)^{-1}B_2(x)\nabla_x \hat{V}(x)' + B_1(x)w$$

which has optimal disturbance $\mathrm{w}^* = \gamma^{-2} B_1 \nabla_x \hat{V}'$. (See also [YJH98].) □

5.4.5 State Feedback and its Relation to Output Feedback

Consider the output feedback H_∞ control problem for the plant (2.2), as described in Chapter 2. As described above, the corresponding state feedback problem is obtained by ignoring the measurement equation $y = C_2(x) + D_{21}(x)w$ and providing the controllers with full state access.

Theorem 5.4.9 *Assume that the state feedback value function V defined by (5.39) is a smooth solution of the PDE (5.45) with $\mathbf{u}^*_{state}$ stabilizing. Then*

$$W(p) \geq \langle p + V \rangle, \tag{5.53}$$

where W is the output feedback value function for the system (2.2) defined by (4.4).

Proof. W is defined to be optimal relative to the class of admissible output feedback controllers achieving dissipation and weak internal stability, while V is optimal over the class of admissible state feedback controllers achieving dissipation and internal stability. Consider the union of these two classes of controllers. Since $\mathbf{u}^*_{state}$ is stabilizing, we see that it attains the infimum over this union. Hence

$$\begin{aligned} W(p) &= \inf_{K_{output}} \sup_{T\geq 0,\, w(\cdot),\, x_0} \{p(x_0) + \tfrac{1}{2}\int_0^T [|z(s)|^2 - \gamma^2 |w(s)|^2]\, ds\} \\ &\geq \inf_{K_{output},\, K_{state}} \sup_{x_0} \{p(x_0) + \sup_{T\geq 0,\, w(\cdot)} (\tfrac{1}{2}\int_0^T [|z(s)|^2 - \gamma^2|w(s)|^2]\, ds \; : \; x_0)\} \\ &\geq \sup_{x_0}\{p(x_0) + V(x_0)\} \end{aligned}$$

as required. □

We next look at the state feedback problem as a special case of the output feedback problem, with $C_2(x) = x$ and $D_{21} = 0$:

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= x. \end{aligned} \tag{5.54}$$

This does not satisfy the standard assumptions, which require D_{21} to be full rank, however, the information state and the value function can still be defined, The key point is that in general, when D_{21} is rank deficient and $C_2(x) = x$, the observation y consists of some state components that are perfectly known, with the remaining ones subject to the influence of measurement disturbances, see [HJM98]. In the case at hand, all state components are perfectly known, and none are subject to measurement disturbances.

Theorem 5.4.10 *In the state feedback case, we take $C_2(x) = x$ and $D_{21} = 0$, and let the information state p_t and value function W be defined for the structure (5.54). Let x_0 be a given plant initial condition with disturbance input $w \in L_{2,loc}$ and controller $u = K_{state}(x)$, giving trajectory $x(\cdot)$ of (2.2) and output measurement $y(\cdot) = x(\cdot)$. Then:*

(i) The information state determined by the signals $u(\cdot)$ and $y(\cdot)$ is given by

$$p_t(x) = \delta_{x(t)} + p_t(x(t)) \tag{5.55}$$

where $p_0 = \delta_{x_0}$ and

$$p_t(x(t)) = \sup_{\tilde{w}(\cdot)}\{\delta_{x_0}(\xi(0)) + \tfrac{1}{2}\int_0^t [|C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2 - \gamma^2|\tilde{w}(s)|^2\, ds\} \tag{5.56}$$

where $\xi(s)$ satisfies

$$\dot{\xi} = A(\xi) + B_2(\xi)u + B_1(\xi)\tilde{w} \tag{2.2)'$$

for $0 \le s \le t$ with $\xi(t) = x(t)$.

(ii) The value functions are related by

$$W(\delta_x) = V(x). \tag{5.57}$$

Proof. Consider the definition (3.7) of the information state, but with $C_2(x) = x$ and $D_{21} = 0$. The constraints on the trajectories $\xi(\cdot)$ of (2.2) read

$$\xi(t) = x(t), \quad y(s) = \xi(s),\ 0 \le s \le t. \tag{5.58}$$

If these constraints are not satisfied, then $p_t(x) = -\infty$.

Let $\tilde{w}(\cdot)$ be given, and consider the corresponding trajectory $\xi(\cdot)$ of (2.2)' with $\xi(t) = x$. To satisfy the constraints, we must have $x = x(t)$, and $\tilde{w}$ must produce a trajectory such that $\xi(\cdot) = x(\cdot)$ (there is always one such $\tilde{w}$, viz. w). Then the definition (3.7) can be interpreted as (5.55), (5.56).

Next, we have

$$W(\delta_{x_0}) = \inf_K \sup_{T\ge 0,\, \tilde{w},\, \xi_0} \{\delta_{x_0}(\xi_0) + \tfrac{1}{2}\int_0^T [|C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2 - \gamma^2|\tilde{w}(s)|^2\, ds\}$$

where $\xi(s)$ satisfies (2.2)' with $\xi(0) = \xi_0$. Then

$$\begin{aligned} W(\delta_{x_0}) &= \inf_K \sup_{T\ge 0,\, \tilde{w}} \{\tfrac{1}{2}\int_0^T [|C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2 - \gamma^2|\tilde{w}(s)|^2\, ds\} \\ &= V(x_0) \end{aligned}$$

where $\xi(s)$ satisfies (2.2)' with $\xi(0) = x_0$. □

5.5 Notes

(i) The material on dissipative systems (except for the nonlinear Strict Bounded Real Lemma) is drawn from [Wil72], [HM76], [vdS96].

(ii) For a thorough treatment of the state feedback H_∞ problem, see [vdS96].

Chapter 6

Storage Functions

So far we have discussed dissipation and stability for the output feedback H_∞ problem in terms of the behavior of the information state system, and we have not explicitly used storage functions for the closed-loop system as in the Hill-Moylan-Willems theory of dissipative systems. Indeed the results of Chapter 4 were expressed in terms of functions $\hat{W}$ of the information state, and in particular the value function W. These functions encode dissipation and stability of the closed loop. The purpose of this section is to express our main results in the Hill-Moylan-Willems language, and in particular, we explicitly identify a storage function.

6.1 Storage Functions for the Information State Closed-Loop

In Chapter 5 we reviewed some of the main elements of the Hill-Moylan-Willems theory. Let us now apply concepts from this theory to the closed loop systems $(G, K_{p_0}^{\mathbf{u}^*_{\hat{W}}})$ produced by information state controllers constructed from solutions $\hat{W}$ to the dissipation PDI (4.26). The most important example of such a closed-loop is built from the central controller via the value function W and control attractor p_e.

The closed-loop system $(G, K_{p_0}^{\mathbf{u}^*_{\hat{W}}})$ has initial conditions $(x_0, p_0) \in \mathbf{R}^n \times \mathcal{X}_e$. A storage function for this system is a function $V(x, p)$ on the state space $\mathbf{R}^n \times \mathcal{X}_e$ and the *dissipation inequality* reads: for any input $w \in L_{2,t}$, for all $t \geq 0$

$$V(x(t), p(t)) + \tfrac{1}{2} \int_0^t |z(s)|^2 \, ds \leq V(x_0, p_0) + \gamma^2 \tfrac{1}{2} \int_0^t |w(s)|^2 \, ds \tag{6.1}$$

along the trajectory in the state space $\mathbf{R}^n \times \mathcal{X}_e$ with initial conditions (x_0, p_0) under the action of the controller $K_{p_0}^{\mathbf{u}^*_{\hat{W}}}$. Since (for certain p_0) $K_{p_0}^{\mathbf{u}^*_{\hat{W}}}$ solves the dissipative control problem for G, it is worthwhile investigating and actually constructing storage functions for the closed loop system. The storage function $V(x, p_0)$ depends on the controller $K_{p_0}^{\mathbf{u}^*_{\hat{W}}}$, which depends in turn both on $\hat{W}$ and p_0, so we write $V^{\hat{W}}(x, p_0)$.

The available storage is defined by

$$V_a^{\hat{W}}(x,p) = \sup_{T\geq 0,\, w\in L_{2,T}} \{-\tfrac{1}{2}\int_0^T [\gamma^2|w(s)|^2 - |z(s)|^2]ds\}, \tag{6.2}$$

where the integral is computed corresponding to $(G, K_{p_0}^{\mathbf{u}^*_{\hat{W}}})$ with system trajectory initialized at $(x(0), p_0) = (x, p)$, and the required supply is defined by

$$V_r^{\hat{W}}(x,p) = -\sup_{T\geq 0,\, w\in L_{2,T}} \{-\tfrac{1}{2}\int_0^T [\gamma^2|w(s)|^2 - |z(s)|^2]ds\}, \tag{6.3}$$

where the integral is computed corresponding to a system trajectory starting at time 0 at $(0, p_0)$ and terminated at time T at $(x(T), p_T) = (x, p)$. The relationship among these storage functions is

$$V_a^{\hat{W}}(x,p) \leq V^{\hat{W}}(x,p) \leq V_r^{\hat{W}}(x,p).$$

A common definition of dissipativity of a system is that it possesses a storage function. Observe that this is *not* precisely the one used in this book, involving β_K, see the definition (2.10). However, if K has a state space realization, β_K can be thought of as a restriction of the available storage for (G, K) to the state space of G. This is illustrated by the next theorem which says this for dissipative information state controllers, as well giving other information about the available storage.

Theorem 6.1.1 *Assume that $K_{p_0}^{\mathbf{u}^*_{\hat{W}}}$ solves the γ-dissipative control problem for G, and that the closed loop system is dissipative in the sense that an available storage function $V_a \geq 0$ exists. Then*

(i) we have

$$\beta_{K_{p_0}^{\mathbf{u}^*_{\hat{W}}}}(x_0) = V_a^{\hat{W}}(x_0, p_0) \ \text{ in } \mathbf{R}^n \times \mathcal{X}_e. \tag{6.4}$$

(ii) The available storage satisfies the upper bound

$$p_0(x_0) + V_a^{\hat{W}}(x_0, p_0) \leq \hat{W}(p_0), \tag{6.5}$$

and so $V_a^{\hat{W}}(x_0, p_0)$ is finite whenever $p_0 \in \mathrm{dom}\hat{W}$ and $p_0(x_0)$ is finite, i.e.

$$\mathrm{dom}\, V_a^{\hat{W}} \subset \mathrm{support}\ p_0 \times \mathrm{dom}\, \hat{W}.$$

(iii) The available storage satisfies the lower bound

$$V_{H_2}(x_0) \leq V_a^{\hat{W}}(x_0, p_0), \tag{6.6}$$

where V is the state feedback value function (defined by (5.39)).

(iv) *The available storage satisfies* $V_a^{\hat{W}}(0, p_0) = 0$ *whenever* $p_0(0) = \hat{W}(p_0)$ *(which means that* $K_{p_0}^{\mathbf{u}^*_{\hat{W}}}$ *is null initialized by Corollary 4.9.10).*

(v) *Under the assumptions of Theorem 4.11.1, we have*

$$V_a^{\hat{W}}(0, p_e) = 0. \tag{6.7}$$

Proof. Equation (6.4) follows directly from the definitions (2.11) and (6.2). To prove (6.5), we have

$$\begin{aligned} V_a^{\hat{W}}(x_0, p_0) + p_0(x_0) &= \sup_{T \geq 0,\, w \in L_{2,T}} \{p_0(x_0) - \tfrac{1}{2} \int_0^T [\gamma^2 |w(s)|^2 - |z(s)|^2] ds\} \\ &\leq \sup_{T \geq 0,\, w \in L_{2,T},\, x_0 \in \mathbf{R}^n} \{p_0(x_0) - \tfrac{1}{2} \int_0^T [\gamma^2 |w(s)|^2 - |z(s)|^2] ds\} \\ &= \sup_{T \geq 0,\, y \in L_{2,T}} \{\langle p_T \rangle\}. \end{aligned}$$

This last equality follows as in the proof of Theorem 3.1.8. The function $\hat{W}$ is dominating, and satisfies (4.48), so that

$$\langle p_t \rangle \leq \hat{W}(p_T) \leq \hat{W}(p_0).$$

Combining these inequalities we get (6.5). This proves item (ii).

Part (i) and (3.5) imply

$$\begin{aligned} V_a^{\hat{W}}(x_0, p_0) &= \beta_{K_{p_0}^{\mathbf{u}^*_{\hat{W}}}}(x_0) \\ &= J_{\delta_{x_0}}(K_{p_0}^{\mathbf{u}^*_{\hat{W}}}) \\ &\geq W(\delta_{x_0}), \end{aligned}$$

and Theorem 5.4.9 and Lemma 5.4.2 then imply

$$\begin{aligned} W(\delta_{x_0}) &\geq \langle \delta_{x_0} + V \rangle \\ &= V(x_0) \\ &\geq V_{H_2}(x_0). \end{aligned}$$

This proves item (iii).

Part (iv) follows for the available storage thanks to (6.5), and part (v) follows as in Theorem 4.11.1. □

Corollary 6.1.2 *If Assumption 5.4.1 holds, then* $V_a^{\hat W}(x_0,p_0) \geq c_{H_2}|x_0|^2$ *is quadratically coercive, and* $\beta_{K_{p_0}^{\mathbf{u}^*_{\hat W}}}(x_0) \geq c_{H_2}|x_0|^2$ *is a coercive bias* ($c_{H_2} > 0$)*. Therefore we can replace the hypothesis that* $(G, K_{p_0}^{\mathbf{u}^*_{\hat W}})$ *is* z*-detectable in item (v) of Theorem 4.10.3 to conclude that the plant state in the closed-loop system is asymptotically stable.*

Proof. The proof follows from Theorem 6.1.1 and the proofs of Theorems 5.4.7 and 5.1.5. □

So far we have dealt with integral inequalities for the storage functions. Since in general the Bounded Real Lemma provides a connection between integral and differential forms of the dissipation inequality, we consider the dissipation PDI for storage functions $V^{\hat W}(x,p)$, viz.

$$\begin{aligned}\sup_w \{ &\nabla_x V^{\hat W}(x,p)\cdot \Big(A(x)+B_1(x)w+B_2(x)\mathbf{u}^*_{\hat W}(p)\Big) \\ &+\nabla_p V^{\hat W}(x,p)[F(p,\mathbf{u}^*_{\hat W}(p), C_2(x)+D_{21}(x)w)] \\ &+\tfrac{1}{2}|C_1(x)+D_{12}(x)\mathbf{u}^*_{\hat W}(p)|^2-\gamma^2\tfrac{1}{2}|w|^2\}\leq 0,\end{aligned} \tag{6.8}$$

and the corresponding PDE

$$\begin{aligned}\sup_w \{ &\nabla_x V_a^{\hat W}(x,p)\cdot \Big(A(x)+B_1(x)w+B_2(x)\mathbf{u}^*_{\hat W}(p)\Big) \\ &+\nabla_p V_a^{\hat W}(x,p)[F(p,\mathbf{u}^*_{\hat W}(p), C_2(x)+D_{21}(x)w)] \\ &+\tfrac{1}{2}|C_1(x)+D_{12}(x)\mathbf{u}^*_{\hat W}(p)|^2-\gamma^2\tfrac{1}{2}|w|^2\}= 0.\end{aligned} \tag{6.9}$$

is satisfied by $V_a^{\hat W}$. Note that this inequality and equality are defined on the infinite dimensional space $\mathbf{R}^n \times \mathcal{X}_e$.

6.2 Explicit Storage Functions

Associated with each function $\hat W$ solving the dissipation PDI (4.41) is the *explicit* storage function

$$e^{\hat W}(x,p) \stackrel{\triangle}{=} -p(x)+\hat W(p). \tag{6.10}$$

Theorem 6.2.1 *The function* $e^{\hat W}(x,p)$ *is non-negative, and satisfies the dissipation inequality (6.1), and so is a storage function for the closed loop system in* $\mathbf{R}^n \times \mathrm{dom}\, W$ *provided* W *and* p *are smooth, and*

$$V_a^{\hat W}(x,p) \leq e^{\hat W}(x,p) \leq V_r^{\hat W}(x,p).$$

Further,

$$\mathrm{dom}\, e^{\hat{W}} \subset \mathrm{support}\ p \times \mathrm{dom}\, \hat{W},$$

and $e(0,p) = 0$ *whenever* $p(0) = \hat{W}(p)$*, and in particular,* $e(0,p_e) = 0$ *(under the assumptions of Theorem 4.11.1). The function* $e^W(x,p)$ *solves the PDE (6.9), and*

$$e^W(x,p) \leq e^{\hat{W}}(x,p) \tag{6.11}$$

for any good solution $\hat{W}$ *of the dissipation PDI (4.41).*

Proof. We use the fact that $\nabla_x e^{\hat{W}}(x,p) = -\nabla_x p(x)$ and $\nabla_p e^{\hat{W}}(x,p) = -E_x + \nabla_p \hat{W}(p)$ (where E_x is the evaluation operator $E_x[f] = f(x)$):

$$\begin{aligned}
0 &\geq \sup_y \{\nabla_p \hat{W}(p)[F(p, \mathbf{u}^*_{\hat{W}}(p), y)]\} \\
&= \sup_y \{\sup_w \{\nabla_x e^{\hat{W}}(x,p) \cdot \Big(A(x) + B_1(x)w + B_2(x)\mathbf{u}^*_{\hat{W}}(p)\Big) \\
&\quad + \tfrac{1}{2}|C_1(x) + D_{12}(x)\mathbf{u}^*_{\hat{W}}(p)|^2 - \gamma^2 \tfrac{1}{2}|w|^2 \ :\ y = C_2(x) + D_{21}(x)w\} \\
&\qquad + \nabla_p e^{\hat{W}}(x,p)[F(p, \mathbf{u}^*_{\hat{W}}(p), y)]\} \\
&= \sup_y \{\sup_w \{\nabla_x e^{\hat{W}}(x,p) \cdot \Big(A(x) + B_1(x)w + B_2(x)\mathbf{u}^*_{\hat{W}}(p)\Big) \\
&\quad + \tfrac{1}{2}|C_1(x) + D_{12}(x)\mathbf{u}^*_{\hat{W}}(p)|^2 - \gamma^2 \tfrac{1}{2}|w|^2 + \nabla_p e(x,p)[F(p, \mathbf{u}^*_{\hat{W}}(p), y)] \\
&\quad :\ y = C_2(x) + D_{21}(x)w\}\} \\
&= \sup_w \{\nabla_x e^{\hat{W}}(x,p) \cdot \Big(A(x) + B_1(x)w + B_2(x)\mathbf{u}^*_{\hat{W}}(p)\Big) \\
&\quad + \tfrac{1}{2}|C_1(x) + D_{12}(x)\mathbf{u}^*_{\hat{W}}(p)|^2 - \gamma^2 \tfrac{1}{2}|w|^2 \\
&\quad + \nabla_p e^{\hat{W}}(x,p)[F(p, \mathbf{u}^*_{\hat{W}}(p), C_2(x) + D_{21}(x)w)]\}
\end{aligned} \tag{6.12}$$

(In these expressions, the optimizing y is $\mathbf{y}^*_{\hat{W}}(p)$.) This implies that $e^{\hat{W}}(x,p)$ satisfies the PDI (6.8). Integrating yields the dissipation inequality (6.1).

The inequality $\hat{W}(p) \geq \langle p \rangle$ implies that $e^{\hat{W}}$ is nonnegative. The remaining assertions follow as in Theorem 6.1.1, and from Theorem 4.5.3, $W \leq \hat{W}$. □

Remark 6.2.2 The above proof used differentials. However, the result could have also been established in integrated form. ▽

Assumption 6.2.3 We now assume that D_{21} is independent of x.

In the PDE (6.12), the optimizing disturbance $\mathbf{w}^*_{e,\hat{W}}$ is given by

$$\mathbf{w}^*_{e,\hat{W}}(x,p) = -\gamma^{-2}[I - D_{21}'E_2^{-1}D_{21}]B_1(x)'\nabla_x p(x)' + D_{21}'E_2^{-1}(\mathbf{y}^*(p) - C_2(x)). \tag{6.13}$$

We use the subscript in $\mathbf{w}^*_{e,\hat{W}}$ to emphasize that this quantity depends on the function e (defined by (6.10)) and $\hat{W}$. A simple computation shows that if $w = \mathbf{w}^*_{e,\hat{W}}(x,p)$, then $y = C_2(x) + D_{21}\mathbf{w}^*_{e,\hat{W}}(x,p) = \mathbf{y}^*_{\hat{W}}(p)$.

Assumption 6.2.4 We now assume $D_{21} = I$ (1 and 2A block cases).

Under Assumption 6.2.4 we have the simplification

$$\mathbf{w}^*_{e,\hat{W}}(x,p) = \mathbf{y}^*_{\hat{W}}(p) - C_2(x).$$

Recall from §4.12.2 the definition of zero dynamics $\mathcal{ZD}^0$ of the system

$$\dot{p} = F(p, \mathbf{u}^*_{\hat{W}}(p), \mathbf{y}^*_{\hat{W}}(p)), \tag{4.93}$$

and let M_s be the stable manifold for the exponentially hyperbolic vector field $A^\times$.

The next theorem describes a set on which $e^{\hat{W}} = V_a^{\hat{W}}$. This set is rather thin in general.

Theorem 6.2.5 *Assume that $\hat{W}$ is a good solution of the dynamic programming PDE (4.16) and*

$$p_0 \in \mathcal{ZD}^0 \cap \mathcal{D}^0_{attr}(p_e). \tag{6.14}$$

Assume $\hat{W}(p_e) = 0$. Then

$$V_a^{\hat{W}}(x_0,p_0) = e^{\hat{W}}(x_0,p_0) \text{ for all } x_0 \in M_s \cap \operatorname{support} p_0. \tag{6.15}$$

Proof. If $w = \mathbf{w}^*_{e,\hat{W}}$ in the closed-loop system, then $y = \mathbf{y}^*_{\hat{W}}$, and so $\hat{W}(p_0) = \hat{W}(p_t)$ for all $t \geq 0$. Since $p_0 \in \mathcal{ZD}^0$, we have $\mathbf{u}^*_{\hat{W}}(p_t) = 0$ and $\mathbf{y}^*_{\hat{W}}(p_t) = 0$ for all $t \geq 0$. Thus the plant dynamics becomes

$$\dot{x} = A^\times(x),$$

and the controller dynamics is

$$\dot{p} = F(p,0,0).$$

If $x_0 \in M_s$, then $x(t) \to 0$ and

$$\begin{aligned} p_t(x(t)) &\to p_0(x_0) + \tfrac{1}{2}\int_0^\infty [|C_1(x(s)) + D_{12}(x(s))u(s)|^2 - \gamma^2|C_2(x(s)) - y(s)|^2]\, ds \\ &= p_e(0) + c, \end{aligned}$$

a finite number, as $t \to \infty$.

Now since $p_0 \in \mathcal{D}^0_{attr}(p_e)$ we have $p_t \implies p_e + c$ as $t \to \infty$. By continuity, $\hat{W}(p_0) = \hat{W}(p_t) \to \hat{W}(p_e)$ as $t \to \infty$. Therefore

$$e^{\hat{W}}(x(t), p_t) = -p_t(x(t)) + \hat{W}(p_t) \to -(p_e(0) + c) + \hat{W}(p_e + c) = 0 \quad (6.16)$$

as $t \to \infty$.

Next, we know from Theorem 6.2.1 that $e^{\hat{W}}(x_0, p_0) \geq V_a^{\hat{W}}(x_0, p_0$, so we need only verify the opposite inequality. To this end, plug $w = \mathbf{w}^*_{e,\hat{W}}$ into (6.2):

$$V_a^{\hat{W}}(x_0, p_0) \geq -\tfrac{1}{2}\int_0^T [\gamma^2|\mathbf{w}^*_{e,\hat{W}}(s)|^2 - |z(s)|^2]ds.$$

Since $e^{\hat{W}}$ satisfies the PDE (6.9), integration gives

$$-\tfrac{1}{2}\int_0^T [\gamma^2|\mathbf{w}^*_{e,\hat{W}}(s)|^2 - |z(s)|^2]ds = e^{\hat{W}}(x_0, p_0) - e^{\hat{W}}(x(t), p_t),$$

using $w = \mathbf{w}^*_{e,\hat{W}}$. Combining, we get

$$V_a^{\hat{W}}(x_0, p_0) - e^{\hat{W}}(x_0, p_0) \geq -e^{\hat{W}}(x(t), p_t).$$

Now from (6.16) we get

$$V_a^{\hat{W}}(x_0, p_0) - e(x_0, p_0) \geq -\lim_{t\to\infty} e^{\hat{W}}(x(t), p_t) = 0.$$

This implies $V_a^{\hat{W}}(x_0, p_0) \geq e^{\hat{W}}(x_0, p_0)$, completing the proof. □

We next discuss some formulas that connect some of the functions we have been studying. Recall the definition (4.4) which can be written as

$$W(p) = \inf_K \sup_{T\geq 0,\, y(\cdot),\, x_0} \{p(x_0) + \tfrac{1}{2}\int_0^T [|C_1(\xi(s)) + u(s)|^2 - \gamma^2|y(s) - C_2(\xi(s))|^2]\, ds\},$$

where the trajectory $\xi(\cdot)$ is the solution of

$$\dot{\xi} = A^\times(\xi) + B_1(\xi)y + B_2(\xi)u, \quad (2.6)$$

with $u(\cdot) = K(y(\cdot))$, and $\xi(0) = x_0$. Now write

$$W(x, p) \stackrel{\triangle}{=} \sup_{T\geq 0,\, y(\cdot)} \{p(x) + \tfrac{1}{2}\int_0^T [|C_1(\xi(s)) + u(s)|^2 - \gamma^2|y(s) - C_2(\xi(s))|^2]\, ds\}, \quad (6.17)$$

where $u(\cdot) = K_p^{\mathbf{u}^*_{\hat{W}}}$. Then in fact

$$V_a^{\hat{W}}(x, p) = -p(x) + W(x, p). \quad (6.18)$$

Example 6.2.6 (Linear systems) The storage function for linear systems is calculated in §7.2.2. ▽

Chapter 7

Special Cases

In this chapter we discuss some important special cases. In particular, the cases considered have special features due to the problem data which simplifies the construction of the controller. A class of bilinear systems is discussed in §7.1. These systems are linear systems with an additional term in the state dynamics consisting of a product of the state and control. The information state for thus system is finite dimensional, given by a driven Riccati equation, and the dynamic programming PDE which determines the controller is correspondingly defined on a finite dimensional space. Linear systems are special cases of this, and the controller is determined by a Riccati equation. Thus the well-known linear H_∞ controller is obtained from the information state theory as a special case. §7.3 systems which satisfy the certainty equivalence principle are discussed. Here, the controller is constructed from a pair of PDEs defined on a finite dimensional space.

7.1 Bilinear Systems

Consider the following class of bilinear systems:

$$G_{bilinear}: \begin{cases} \dot{x} = A(u)x + B_1 w + B_2 u \\ \\ z = C_1 x + D_{12} u \\ \\ y = C_2 x + D_{21} w. \end{cases} \tag{7.1}$$

Here, we take the control input dimension $m = 1$ for simplicity, and

$$A(u) = A + B_3 u.$$

In our general notation (Chapter 2), $B_2(x)$ would read $B_2 + B_3 x$, in terms of the notation of (7.1). The presence of this B_3 term means that the dynamics have a multiplicative term involving both x and u—hence the term *bilinear*. The matrices are of the appropriate sizes (e.g. B_3 is $n \times n$), and the standard assumptions used earlier apply, except that B_2 is no longer bounded.

Let the initial information state be given by

$$p_0(x) = -\gamma^2 \tfrac{1}{2} x' Y_0^{-1} x, \tag{7.2}$$

with $Y_0 > 0$. Then the information state for $t > 0$ is given explicitly by

$$p_t(x) = -\gamma^2 \tfrac{1}{2} (x - \hat{x}(t))' Y(t)^{-1} (x - \hat{x}(t)) + \phi(t), \tag{7.3}$$

where

$$\begin{aligned} \dot{\hat{x}} = {} & (A(u) - B_1 D_{21}' E_2^{-1} C_2 + Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2))\hat{x} \\ & + (B_1 D_{21}' + Y C_2') E_2^{-1} y + (B_2 + \gamma^{-2} Y C_1' D_{12}) u, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \dot{Y} = {} & (A(u) - B_1 D_{21}' E_2^{-1} C_2) Y + Y (A(u) - B_1 D_{21}' E_2^{-1} C_2)' \\ & + Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2) Y + B_1 [I - D_{21}' E_2^{-1} D_{21}] B_1' \end{aligned} \tag{7.5}$$

$$\dot{\phi} = \tfrac{1}{2} |C_1 \hat{x} + D_{12} u|^2 - \gamma^2 \tfrac{1}{2} (y - C_2 \hat{x})' E_2^{-1} (y - C_2 \hat{x}). \tag{7.6}$$

The initial conditions are

$$\hat{x}(0) = 0, \quad Y(0) = Y_0, \quad \phi(0) = 0. \tag{7.7}$$

This can be readily verified by differentiation and substitution in (3.9).

The significance of this is that when information state p_0 has the form (7.7) then p_t is given by (7.3) for all $t \geq 0$, and can be *computed by solving the finite set of ODEs* (7.4), (7.5), (7.6). This means that there exists an invariant subspace $\mathcal{F} = \mathbf{R}^n \times \mathbf{R}^{n^2} \times \mathbf{R} \subset \mathcal{X} \subset \mathcal{X}_e$. Note in particular that the Y Riccati equation (7.5) is *driven* by the control signal u.

The the minimax cost functional (3.20) can be expressed in terms of these parameters, since

$$\langle p_T \rangle = \phi(T).$$

The problem of computing the optimal control and observation (c.f. (4.23)) is also greatly simplified. This can be done by introducing a finite dimensional "projection" of the value function $W(p)$. Indeed, define

$$\tilde{W}(\hat{x}, Y, \phi) = \inf_{K} \sup_{T \geq 0,\, y \in L_{2,T}} \{\phi(T) \,:\, \hat{x}(0) = \hat{x}, Y(0) = Y, \phi(0) = \phi\}. \tag{7.8}$$

The function $\tilde{W}$ enjoys the following properties: (i) from Theorem 4.3.1, item (ii), we have

$$\nabla_\phi \tilde{W}(\hat{x}, Y, \phi) = 1, \tag{7.9}$$

and

$$\tilde{W}(\hat{x}, Y, \phi) \geq \phi \quad \text{for all } \hat{x}, Y, \phi. \tag{7.10}$$

The PDE (4.16) on the infinite dimensional space $\mathcal{X}$ converts to a PDE on a finite dimensional space (i.e. one defined on a subset of $\mathcal{F} = \mathbf{R}^n \times \mathbf{R}^{n^2} \times \mathbf{R}$), viz.

$$\begin{aligned}
\inf_u \sup_y \{ \nabla_{\hat{x}} \tilde{W}(\hat{x}, Y, \phi) \cdot [(A(u) - B_1 D_{21}' E_2^{-1} C_2 + Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2))\hat{x} \\
+ (B_1 D_{21}' + Y C_2') E_2^{-1} y + (B_2 + \gamma^{-2} Y C_1' D_{12}) u] \\
+ \nabla_Y \tilde{W}(\hat{x}, Y, \phi) \cdot [(A(u) - B_1 D_{21}' E_2^{-1} C_2) Y + Y (A(u) - B_1 D_{21}' E_2^{-1} C_2)' \\
+ Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2) Y + B_1 [I - D_{21}' E_2^{-1} D_{21}] B_1'] \\
+ \nabla_\phi \tilde{W}(\hat{x}, Y, \phi) \cdot [\tfrac{1}{2} |C_1 \hat{x} + D_{12} u|^2 - \gamma^2 \tfrac{1}{2} (y - C_2 \hat{x})' E_2^{-1} (y - C_2 \hat{x})] \} = 0.
\end{aligned} \tag{7.11}$$

In equation (7.11), the gradient $\nabla_{\hat{x}, Y, \phi} \tilde{W}$ has components $\nabla_{\hat{x}} \tilde{W}$, $\nabla_Y \tilde{W}$, $\nabla_\phi \tilde{W}$, and the notation $\nabla_Y \tilde{W} \cdot M$ means $\sum_{i,j=1}^n \frac{\partial \tilde{W}}{\partial Y_{ij}} M_{ij}$ (where M is an $n \times n$ matrix). In general it is not possible to solve the PDE explicitly. However, evaluation of the $\inf$ and $\sup$ in (7.11) gives the following formulas for the optimal control and observation:

$$\begin{aligned}
\mathbf{u}^*(\hat{x}, Y, \phi) &= -E_1^{-1} [D_{12}' C_1 \hat{x} + (B_2' + \gamma^{-2} D_{12}' C_1 Y + \hat{x}' B_3') \nabla_{\hat{x}} \tilde{W}(\hat{x}, Y, \phi)' \\
&\quad + \nabla_Y \tilde{W}(\hat{x}, Y, \phi) \cdot (B_3 Y + Y B_3')] \\
\mathbf{y}^*(\hat{x}, Y, \phi) &= C_2 \hat{x} + (D_{21} B_1' + \gamma^{-2} C_2 Y) \nabla_{\hat{x}} \tilde{W}(\hat{x}, Y, \phi)'.
\end{aligned} \tag{7.12}$$

The finite set of ODEs (7.4), (7.5), (7.6) together with the optimal control specified in (7.12) constitute the optimal information state controller for bilinear systems. This is illustrated in Figure 7.1.

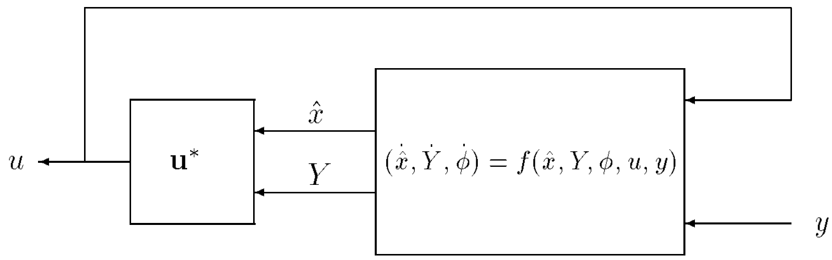


Figure 7.1: Bilinear information state controller. The dynamics f is defined by equations (7.4), (7.5) and (7.6).

7.2 Linear Systems

Consider the following class of linear systems:

$$G_{linear}: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w. \end{cases}$$

This corresponds to a subclass of bilinear systems with $B_3 = 0$, so that multiplicative terms involving both x and u are no longer present. The information state is again given explicitly by (7.3), with the modification $B_3 = 0$. The dynamics are

$$\begin{aligned} \dot{\hat{x}} = (A - B_1 D_{21}' E_2^{-1} C_2 + Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2))\hat{x} \\ +(B_1 D_{21}' + Y C_2') E_2^{-1} y + (B_2 + \gamma^{-2} Y C_1' D_{12}) u, \end{aligned} \tag{7.13}$$

$$\begin{aligned} \dot{Y} = (A - B_1 D_{21}' E_2^{-1} C_2) Y + Y(A - B_1 D_{21}' E_2^{-1} C_2)' \\ +Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2) Y + B_1 [I - D_{21}' E_2^{-1} D_{21}] B_1' \end{aligned} \tag{7.14}$$

$$\dot{\phi} = \tfrac{1}{2}|C_1 \hat{x} + D_{12} u|^2 - \gamma^2 \tfrac{1}{2}(y - C_2 \hat{x})' E_2^{-1} (y - C_2 \hat{x}). \tag{7.15}$$

Remark 7.2.1 In the general bilinear case, the Y-Riccati differential equation (7.5) depends on the control u, while in the linear case, (7.14), it does not. ▽

Consider the function $\hat{W}(\hat{x}, Y, \phi)$ defined by

$$\tilde{W}(\hat{x}, Y, \phi) = \tfrac{1}{2}\hat{x}'(I - \gamma^{-2} Y X)^{-1} X \hat{x} + \phi, \tag{7.16}$$

where $X \geq 0$ is a solution of the X-Riccati equation

$$\begin{aligned} (A - B_2 E_1^{-1} D_{12}' C_1)' X + X(A - B_2 E_1^{-1} D_{12}' C_1) \\ +X(\tfrac{1}{\gamma^2} B_1 B_1' - B_2 E_1^{-1} B_2') X + C_1'(I - D_{12} E_1^{-1} D_{12}') C_1 = 0. \end{aligned} \tag{7.17}$$

Then $\hat{W}$ solves the PDE (7.11) (with $B_3 = 0$) on a certain domain (see below). This can be checked by differentiation and substitution into (7.11).

The optimal control and observation are given by:

$$\begin{aligned} \mathbf{u}^*(\hat{x}, Y, \phi) &= -E_1^{-1}[D_{12}' C_1 \hat{x} + (B_2' + \gamma^{-2} D_{12}' C_1 Y) X (I - \gamma^{-2} Y X)^{-1} \hat{x}] \\ &= -E_1^{-1}[D_{12}' C_1 + B_2' X](I - \gamma^{-2} Y X)^{-1} \hat{x} \\ \mathbf{y}^*(\hat{x}, Y, \phi) &= C_2 \hat{x} + \gamma^{-2}(D_{21} B_1' + C_2 Y) X (I - \gamma^{-2} Y X)^{-1} \hat{x}] \\ &= [C_2 + \gamma^{-2} D_{21} B_1' X](I - \gamma^{-2} Y X)^{-1} \hat{x}. \end{aligned} \tag{7.18}$$

Notice that these formulas are independent of ϕ, but depend explicitly on $\hat{x}, Y$.

The finite set of ODEs (7.4), (7.5), (7.6) (with $B_3 = 0$) together with the optimal control specified in (7.18) constitute the central controller for linear systems. However, the form of this controller can be simplified with the change of variables:

$$\bar{x} = (I - \gamma^{-2}YX)^{-1}\hat{x}. \tag{7.19}$$

Equation (7.4) becomes

$$\begin{aligned}\dot{\bar{x}} &= (A - B_2E_1^{-1}D'_{12}C_1 + (\gamma^{-2}B_1B'_1 - B_2E_1^{-1}B'_1)X)\bar{x} \\ &+(I - \gamma^{-2}YX)^{-1}(B_1D'_{21} + YC'_2)E_2^{-1}(y - [C_2 + D_{21}B'_1X]\bar{x}) \\ &+(I - \gamma^{-2}YX)^{-1}(B_2 + \gamma^{-2}YC'_1D_{12})(u + E_1^{-1}[D'_{12}C_1 + B'_2X]\bar{x}),\end{aligned} \tag{7.20}$$

and the optimal control is simply

$$u = \mathbf{u}^*(\bar{x}, Y, \phi) = \mathbf{u}^*_{state}(\bar{x}) = -E_1^{-1}[D'_{12}C_1 + B'_2X]\bar{x},$$

in complete agreement with the formula (4) for the central controller in [PAJ91] (when one takes $X = X_e$ and $Y = Y_e$ to be the stabilizing solutions). The resulting controller is illustrated in Figure 7.2.

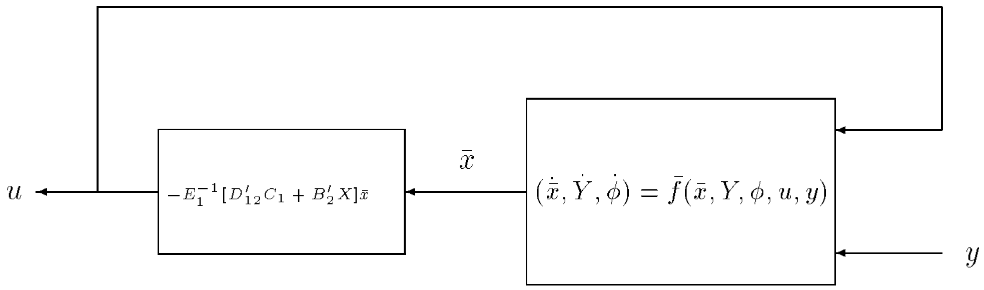


Figure 7.2: Linear optimal information state controller. The dynamics $\bar{f}$ is defined by equations (7.13), (7.14) and (7.15).

The steady state version of the Y equation (7.14) is

$$\begin{aligned}&(A - B_1D'_{21}E_2^{-1}C_2)Y_e + Y_e(A - B_1D'_{21}E_2^{-1}C_2)' \\ &\quad +Y_e(\gamma^{-2}C'_1C_1 - C'_2E_2^{-1}C_2)Y_e + B_1[I - D'_{21}E_2^{-1}D_{21}]B'_1 = 0,\end{aligned} \tag{7.21}$$

where we seek the stabilizing solution Y_e, meaning

$$A - B_1D'_{12}C_2 + Y_e(\gamma^{-2}C_1C'_1 - C_2E_2^{-1}C'_2) \text{ asymptotically stable.} \tag{7.22}$$

The central controller $K^*_{p_e}$ uses $X = X_e$, the stabilizing solution of (7.17), meaning

$$A - B_2 E_1^{-1} D_{12}' C_1 + (\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2') X_e \quad \text{is asymptotically stable.} \tag{7.23}$$

and is initialized at the equilibrium

$$p_e(x) = \tfrac{1}{2}\gamma^2 x' Y_e^{-1} x, \tag{7.24}$$

with state space description

$$\begin{aligned}
\dot{\bar{x}} &= (A - B_2 E_1^{-1} D_{21}' C_1 + (\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_1') X_e)\bar{x} \\
&\quad + (I - \gamma^{-2} Y_e X_e)^{-1} (B_1 D_{21}' + Y_e C_2') E_2^{-1} (y - [C_2 + D_{21} B_1' X_e]\bar{x}) \\
&\quad + (I - \gamma^{-2} Y_e X_e)^{-1} (B_2 + \gamma^{-2} Y_e C_1' D_{12})(u + E_1^{-1}[D_{12}' C_1 + B_2' X_e]\bar{x}), \\
u &= -E_1^{-1}[D_{12}' C_1 + B_2' X_e]\bar{x}.
\end{aligned} \tag{7.25}$$

This controller is illustrated in Figure 7.3.

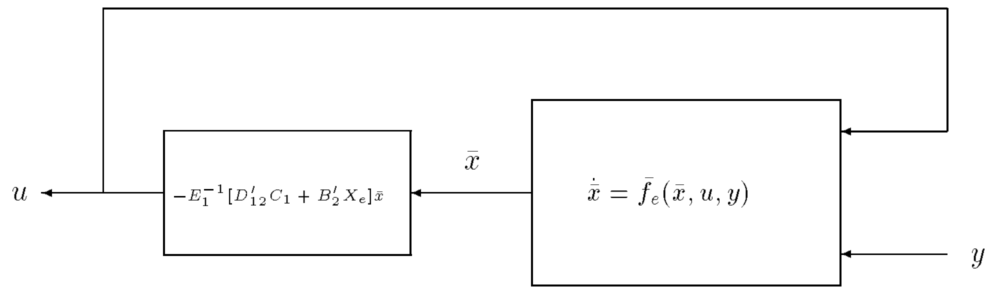


Figure 7.3: Linear central controller. The dynamics $\bar{f}_e$ is defined by equation (7.25).

For detailed results, see [DGKF89], [PAJ91], [GL95], etc. The standard assumptions were given in the introduction, conditions (1.6), (1.7).

7.2.1 Coupling

The coupling condition of [DGKF89] requires that

$$X_e Y_e \ \text{ has spectral radius } \ < \gamma^2. \tag{7.26}$$

Suppose that $\hat{W}(\hat{x}, T, \phi)$ is defined using the stabilizing solution $X = X_e$, so that $\hat{W} = \tilde{W}$, defined by (7.8). Then setting $Y = Y_e$ we get

$$\tilde{W}(\hat{x}, Y_e, \phi) = \tfrac{1}{2}\hat{x}'(1 - \gamma^{-2} Y_e X_e)^{-1} X_e \hat{x} + \phi \quad \text{is finite.}$$

This corresponds to

$$p_e \in q.i.\mathrm{dom}W,$$

recall Theorem 4.11.1. Also note $\hat{W}(0, Y_e, 0) = 0$, corresponding to $W(p_e) = 0$.

If we initialize the controller with $Y_0 \neq Y_e$, we require

$$\phi \leq \hat{W}(\hat{x}, Y_0, \phi) < +\infty,$$

and indeed

$$\phi \leq \hat{W}(\hat{x}(t), Y(t), \phi(t)) < +\infty \ \text{ for all } \ t \geq 0,$$

c.f. §4.9.1.

7.2.2 Storage Function

Take $\gamma = 1$. Then the explicit storage function $e^{\hat{W}}(x, p) = -p(x) + \hat{W}(p)$ of Chapter 6 takes the form

$$e^{\hat{W}}(x, p) = \tfrac{1}{2}(x - \hat{x})'Y^{-1}(x - \hat{x}) + \sup_{\bar{x}}[-\tfrac{1}{2}(\bar{x} - \hat{x})'Y^{-1}(\bar{x} - \hat{x}) + \tfrac{1}{2}\bar{x}'X_e\bar{x}].$$

The supremum occurs at

$$Y^{-1}(\bar{x} - \hat{x}) = X_e\bar{x}$$

and we get

$$e^{\hat{W}}(x, p) = \tfrac{1}{2}(x - \hat{x})'Y^{-1}(x - \hat{x}) + \tfrac{1}{2}\hat{x}X_e[I - YX_e]^{-1}\hat{x}. \tag{7.27}$$

7.3 Certainty Equivalence Principle

In this section we write down the key formulas for the central controller in the special case when the certainty equivalence principle of Basar-Bernhard [BB91] holds.

> Assumption 7.3.1 In this section we assume that D_{12} and D_{21} are independent of x.

This assumption simplifies the formulas, and does not sacrifice generality.

The *minimum stress estimate* is defined by

$$\bar{x}(t) = \underset{x \in \mathbf{R}^n}{\mathrm{argmax}}\{p_t(x) + V(x)\}. \tag{7.28}$$

where $V \geq 0$ is the state feedback value function of (5.45). If this exists and is *unique*, the certainty equivalence controller is given by

$$u^{ce}(t) = \mathbf{u}^*_{state}(\bar{x}(t)), \tag{7.29}$$

where $\mathbf{u}^*_{state}(x)$ is the optimal state feedback controller given by (5.46).

Before presenting explicit formulas, we pause to indicate how this principle fits into the general framework used in this book.

Consider the function

$$\hat{W}(p) = \langle p + V \rangle, \tag{7.30}$$

where $V(x)$ is the state feedback value function (which solves the PDE (5.45)). We define a generalization of the *minimum stress estimate*, viz.

$$\bar{x}(p) = \underset{x \in \mathbf{R}^n}{\operatorname{argmax}} \{p(x) + V(x)\}, \tag{7.31}$$

which we *assume* to be unique. Then (see [JB96]) $\hat{W}(p)$ is (Gateaux) differentiable, and

$$\nabla_p \hat{W}(p) = E_{\bar{x}(p)}, \tag{7.32}$$

where E_x is the evaluation map $E_x[h] = h(x)$ for $h \in \mathcal{X}$, so that

$$\nabla_p \hat{W}(p)[h] = h(\bar{x}(p)).$$

Most importantly, the function $\hat{W}(p)$ solves the PDE (4.16), and the optimal control and observation (given by (4.23)) simplify to

$$\begin{aligned} \mathbf{u}^*(p) &= -E_1^{-1}(D_{12}' C_1(\bar{x}(p)) - B_2(\bar{x}(p))' \nabla_x p(\bar{x}(p))'), \\ \mathbf{y}^*(p) &= C_2(\bar{x}(p)) - \gamma^{-2} D_{21} B_1(\bar{x}(p))' \nabla_x p(\bar{x}(p))'. \end{aligned} \tag{7.33}$$

Since, if p is nonsingular and smooth,

$$\nabla_x p(\bar{x}(p)) = -\nabla_x V(\bar{x}(p)),$$

(first-order necessary condition for a maximum) the certainty equivalence control is, *provided the certainty equivalence assumptions are valid*,

$$u^{ce}(t) = \mathbf{u}^*(p_t) = \mathbf{u}^*_{state}(\bar{x}(p_t)). \tag{7.34}$$

This simplifies the general formula (4.16) (4.23) defining the controller, because under the certainty equivalence assumptions we can use the rather simple solution $\hat{W}(p)$ of equation (4.16) which relates directly to the state feedback controller via the minimum stress estimate. This represents a substantial reduction in complexity.

In order to derive a differential equation for $\bar{x}(t)$, we use the following notation

$$\begin{aligned} r_t(x) &= p_t(x) + V(x), \\ \mathbf{y}^*_{state}(x) &= C_2(x) + \gamma^{-2} D_{21} B_1(x)' \nabla_x V(x)', \end{aligned} \tag{7.35}$$

and *assume* that the Hessian matrix satisfies

$$\nabla_x^2 r_t(\bar{x}(t)) > 0, \tag{7.36}$$

meaning that it is positive definite for each $t \geq 0$. Now

$$\nabla_x r_t(\bar{x}(t)) = 0 \text{ for all } t \geq 0,$$

and so differentiating we get

$$\nabla_x \frac{\partial r_t}{\partial t}(\bar{x}(t)) + \dot{\bar{x}}(t)' \nabla_x^2 r_t(\bar{x}(t)) = 0.$$

After differentiating the PDEs (3.9) for p and (5.45) for V, we obtain the following differential equation:

$$\begin{aligned}
\dot{\bar{x}} &= A - B_2 E_1^{-1} D_{21}' C_1 + (\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2') \nabla_x V' \\
&\quad + (B_1 D_{21}' - \gamma^2 (\nabla_x^2 r_t)^{-1} \nabla_x y_s^{*\prime}) E_2^{-1} (y - \mathbf{y}_{state}^*) \\
&\quad + (B_2 + (\nabla_x^2 r_t)^{-1} \nabla_x \mathbf{u}_{state}^{*}{}' E_1)(u - \mathbf{u}_{state}^*),
\end{aligned} \tag{7.37}$$

where the RHS is evaluated at $x = \bar{x}(t)$, and $u \in L_{2,loc}$, $y \in L_{2,loc}$.

It is important to note that the ODE (7.37) depends on the Hessian of $r = p + V$, and so the CE controller dynamics is still infinite dimensional. It is possible to write down a PDE for r:

$$\dot{r} = G(r, u, y), \tag{7.38}$$

where

$$\begin{aligned}
G(r, u, y) &= -\nabla_x r \cdot [A - B_2 E_1^{-1} D_{12}' C_1 + (\gamma^{-2} B_1 B_1' - B_2 E_1^{-1} B_2') \nabla_x V' \\
&\quad + B_1 D_{21}' E_2^{-1} (y - \mathbf{y}_{state}^*) + B_2 (u - \mathbf{u}_{state}^*)] \\
&\quad + \gamma^{-2} \tfrac{1}{2} \nabla_x r B_1 [I - D_{21}' E_2^{-1} D_{21}] B_1' \nabla_x r' \\
&\quad + \tfrac{1}{2} (u - \mathbf{u}_{state}^*)' E_1 (u - \mathbf{u}_{state}^*) \\
&\quad - \gamma^2 \tfrac{1}{2} (y - \mathbf{y}_{state}^*)' E_2^{-1} (y - \mathbf{y}_{state}^*)
\end{aligned} \tag{7.39}$$

This is obtained from the definition $r_t = p_t + V$ and the PDEs (3.9), (5.45) which p_t and V satisfy.

Remark 7.3.2 Except for u and y, the quantities appearing in (7.39) are functions of x, and are *not* evaluated at $\bar{x}$; this includes $\mathbf{u}_{state}^*$ and $\mathbf{y}_{state}^*$. ∇

The structure of the certainty equivalence controller is illustrated in Figure 7.4.

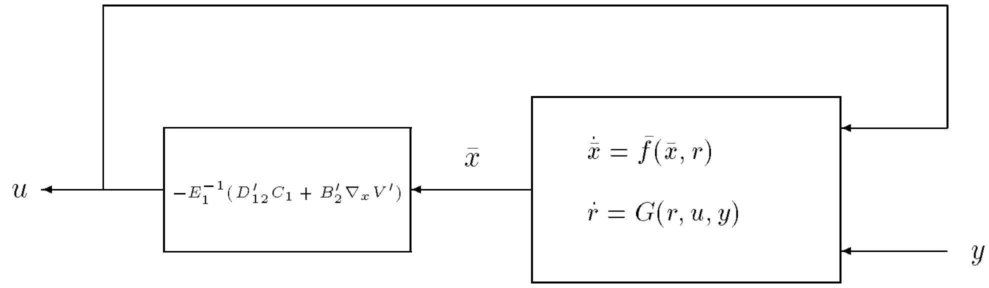


Figure 7.4: Certainty equivalence controller. The dynamics $\bar{f}$ and G are defined by equations (7.37) and (7.38).

Example 7.3.3 *(Bilinear systems)* Connections between certainty equivalence and bilinear systems were studied in [Teo94], [TYJB94]. The state feedback PDE is (5.45) given in Chapter 5. Taking into account the bilinear term, the optimal control and disturbance are given by

$$\begin{aligned} \mathbf{u}^*_{state}(x) &= -E_1^{-1}(D_{12}'C_2 x + (B_2' + x'B_3')\nabla_x V(x)'), \\ \mathbf{w}^*_{state}(x) &= \gamma^{-2} B_1' \nabla_x V(x)'. \end{aligned} \tag{7.40}$$

Assuming the certainty equivalence conditions are satisfied, the function

$$\tilde{\bar{W}}(\hat{x}, Y, \phi) = -\gamma^2 \tfrac{1}{2}(\bar{x} - \hat{x})'Y^{-1}(\bar{x} - \hat{x}) + \phi + V(\bar{x}),$$

solves the PDE (7.11), where $\bar{x} = \bar{x}(\hat{x}, Y, \phi) = \operatorname{argmax}_x\{-\gamma^2 \frac{1}{2}(x - \hat{x})'Y^{-1}(x - \hat{x}) + \phi + V(x)\}$ satisfies

$$-\gamma^2(\bar{x} - \hat{x})'Y^{-1} + \nabla_x V(\bar{x}) = 0.$$

The certainty equivalence controller is therefore given by

$$u^{ce} = -E_1^{-1}(D_{12}'C_2\bar{x} + \gamma^2(B_2' + \bar{x}'B_3')Y^{-1}(\bar{x} - \hat{x})).$$

∇

7.3.1 Breakdown of Certainty Equivalence

Certainty equivalence can break down in two ways. One is the failure of the maximum of $p_t + V$ to be unique. The most dramatic is where the maximum of $p_t + V$ fails to exist. It is this *maximization catastrophe* we discuss in this subsection. Since the functions p_t and V under our assumptions in this book $[p_t + V](x)$ have at most quadratic upper-growth, say

$$[p_t + V](x) \leq c_t|x|^2,$$

for some $c_1 > 0$, we can rule out a maximization catastrophe if $C_t \le 0$ for all $t \ge 0$. But this can fail, so we easily see that the maximum of $p_t + V$ might exist for $t \le t_0$ and then suddenly it will fail to exist (see §11.2 for examples of situations where this can occur).

At a more abstract level this phenomenon corresponds to our condition

$$p_t \in \operatorname{dom} W$$

holding for $t \le t_0$ and then suddenly failing. This was discussed in §4.9.1. Key to this is correct initialisation of the information state, as specified by quadratic upper limiting property: p_0 satisfying (3.37) implies that p_t satisfies (3.38).

Consider the implications of the results in §4.9.1 in the present context of certainty equivalence. Suppose that $p_0 \le p_e$, so that (3.38) implies that p_t is bounded above by

$$p_t(x) \le p_e(x) + k\eta|x|^2 + \frac{c}{\eta} \tag{7.41}$$

for any $\eta > 0$, and any $u, y \in L_2$ (for some constants $k \ge 0$, $c \ge 0$). Assume also that the equilibrium coupling condition holds:

$$p_e(x) + V(x) \le -\varepsilon_e|x|^2 \tag{7.42}$$

for some $\varepsilon_e > 0$. (This is analogous to the DGKF coupling condition §1.2.) Then if we choose $\eta = \varepsilon_e/(2k)$, by combining (7.41) and (7.42), we have

$$\begin{aligned} p_t(x) + V(x) \; &\le p_e(x) + V(x) + k\eta|x|^2 + \frac{c}{\eta} \\ &\le -(\varepsilon_e - k\eta)|x|^2 + \frac{c}{\eta} \\ &\le -\tfrac{1}{2}\varepsilon_2|x|^2 + \frac{2kc}{\varepsilon_e} \end{aligned} \tag{7.43}$$

for all $u, y \in L_2$. This implies that *under these circumstances maximization catastrophe cannot occur*.

Similarly, results parallelling the closed-loop theorems of §4.9.1 can be obtained.

7.4 Notes

(i) From an obvious practical point of view, reducing the computational complexity is of vital importance. In the early 80's, there was considerable interest in the problem of finite dimensional filters (meaning that the filtering computations is reduced from a PDE to a finite set of (stochastic) differential equations), since the discovery by V. Benes of a class of systems admitting finite dimensional filters, [Ben81]. Bensoussan-Elliott [BE96] obtained finite dimensional information states for risk-sensitive control, and Yuliar-James [JY95] obtained similar results for differential games; see also [Teo94], [TYJB94], [Yul96]. Other work on bilinear systems includes [SU96], [HJ96a].

(ii) The certainty equivalence principle for risk-sensitive control and dynamic games is due to Whittle [Whi81] and Basar-Bernhard [BB91] (see also [Ber94], [DBB93], [JB96], [Jam94b]).

(iii) The formulas (7.33), (7.34) for the certainty equivalence controller involves $\nabla_x p$, making it difficult to interpret in singular cases. This problem is overcome in Chapter 10.

(iv) Breakdown of certainty equivalence is discussed in [HJ98], [Jam94b].

Chapter 8

Factorization

8.1 Introduction

There are strong connections between H_∞ control and J-inner outer factorizations of systems, see [BHV91], [Kim97]. In the linear case the H_∞ problem for G is solvable if and only if the reverse arrow system $\widetilde{G}$ admits a J-inner outer factorization. This leads to a parameterization of all H_∞ controllers.

In this chapter we show how to extend this to nonlinear systems. Remarkably the information state constructions adapt perfectly to the factorization problem. They fit so naturally that one again in the nonlinear case might suspect that factoring is the most natural way to deal with H_∞ control. Once we obtain appropriate factors we use our formulas to write down large numbers of solutions to the nonlinear H_∞ control problem. Whether or not this is all solutions remains open.

Our approach to factoring extends the theory for factoring stable systems developed in discrete time by Ball-Helton [BH89], [BH92a] and for continuous time systems by Ball-van der Shaft [BvdS96]. Also there is elegant work due to Baramov and Kimura [BK96]. The key RECIPE presented here was announced and some losslessness properties proved in [HJ94].

8.2 The Problem

8.2.1 Factoring

Typically one says that an operator $\Sigma : \mathbf{S} \to \mathbf{Z}$ factors as

$$\Sigma^I \circ (\Sigma^O)^{-1}$$

if it is the composition

$$\Sigma = \Sigma^I \circ (\Sigma^O)^{-1},$$

where $\Sigma^I : \mathbf{V} \to \mathbf{Z}$ and $\Sigma^O : \mathbf{V} \to \mathbf{S}$. We shall say that Σ has *decomposition* Σ^O, Σ^I provided Σ^I is the composition

$$\Sigma \circ \Sigma^O = \Sigma^I.$$

This is more directly useful for engineering purposes. The system Σ having a decomposition with Σ^O invertible is equivalent to Σ having a factorization

$$\Sigma = \Sigma^I \circ (\Sigma^O)^{-1}.$$

This is important since as we shall see the control problem calls for a decomposition plus a type of invertible factorization. But we emphasize that its importance to control derives from its relationship to decompositon, as will be seen later. Decomposition is illustrated in Figure 8.1.

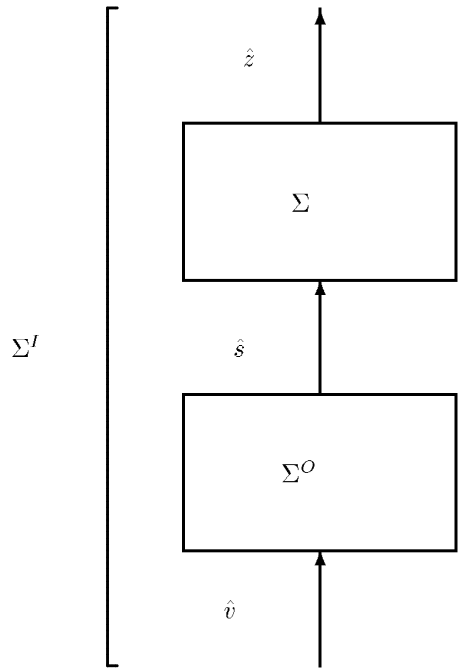


Figure 8.1: The system Σ^I.

There are two ways of thinking about factorizations, depending on whether or not one uses state space realizations for Σ and the two factors Σ^I, Σ^O:

- *Input-output factorization.* In the absence of state space realizations, the operators Σ, Σ^I and Σ^O are regarded simply as mappings defined on the appropriate spaces, and the above decomposition is defined as the usual composition of mappings.

- *State space factorization.* If a state space realization is given, then it is appropriate to find the state space realizations for the two factors. The state space of the composition is defined on the product of the state spaces of the factors.

Of course, if a state space factorization is given, then input-output factorization can be obtained by suitably initializing the state space systems.

Further, an often smaller state space description of the input-output composition can be obtained from the state space representation of the system composition by cutting it down to the set reachable from a fixed initialization, say an equilibrium $(0, 0)$. Thus the state space of the input-output composition often has dimension less than or equal to the dimension of the original system.

We will be concerned with state space factorization in this chapter.

8.2.2 The Set Up

We wish to factor nonlinear systems Σ given in state space form:

$$\Sigma \;:\; \begin{cases} \dot{\hat{x}} \;= \hat{A}(\hat{x}) + \hat{B}(\hat{x})\hat{s}, \quad \hat{x}(0) = \hat{x}_0 \\ \\ \hat{z} \;= \hat{C}(\hat{x}) + \hat{D}(\hat{x})\hat{s} \end{cases} . \tag{8.1}$$

In (8.1) $\mathbf{X}, \mathbf{S}, \mathbf{Z}$ the state, input, and output spaces are vector spaces of finite dimension. These equations define a family of input-output maps $\Sigma_{\hat{x}_0}$ parameterized by the initial condition $\hat{x}_0$. We assume that $\hat{x}_0 = 0$ is an equilibrium $\hat{A}(0) = 0$, $\hat{C}(0) = 0$, with assumptions analogous to Assumption 2.1.1 applying, and write Σ_0 for the input-output map $s \mapsto \hat{z}$ with equilibrium initialization.

> Assumption 8.2.1 Assume $\hat{D}$ is independent of $\hat{x}$ and is invertible, in the sense that
> $$\hat{E} \stackrel{\triangle}{=} \hat{D}' J_{\mathbf{Z}} \hat{D} \text{ invertible.}$$

This assumption will apply throughout this chapter. We will also take $\gamma = 1$.

8.2.3 Dissipation, Losslessness and Being Outer

Now we list definitions of various sensible notions of dissipation and losslessness. In what follows, J will denote a matrix of the form

$$J \stackrel{\triangle}{=} \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}$$

of appropriate size. These are called *signature matrices*. On $\mathbf{S}$ we select a signature matrix $J_{\mathbf{S}}$, as well as one $J_{\mathbf{Z}}$ on $\mathbf{Z}$. If $\hat{s} \in \mathbf{S}$, it is natural to write it as a sum $\hat{s} = \hat{s}_+ + \hat{s}_-$, where $J_{\mathbf{S}}\hat{s}_+ = \hat{s}_+$ and $J_{\mathbf{S}}\hat{s}_- = \hat{s}_-$, in other words,

$$\hat{s} = \begin{pmatrix} \hat{s}_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{s}_- \end{pmatrix} .$$

It is common to use the misleading notation

$$|\hat{s}|^2_{J_{\mathbf{S}}} \stackrel{\triangle}{=} |\hat{s}_+|^2 - |\hat{s}_-|^2 = (J_{\mathbf{S}}\hat{s}, \hat{s}).$$

Sometimes people call $|\cdot|_{J_{\mathbf{S}}}$ the J-norm even though it does not have the main properties of a norm.

(i) Σ is *J-SS-dissipative* if there exists a finite continuous storage function $e(\cdot) \geq 0$, $e(0) = 0$ on $\mathbf{X}$, such that for all $t_1 \leq t_2$, and inputs $\hat{s} \in L_{2,loc}$

$$e(\hat{x}(t_2)) + \int_{t_1}^{t_2} |\hat{z}(t)|^2_{J_{\mathbf{Z}}} dt \leq e(\hat{x}(t_1)) + \int_{t_1}^{t_2} |\hat{s}(t)|^2_{J_{\mathbf{S}}} dt. \tag{8.2}$$

(ii) Σ is *J-SS-lossless* if there exists a finite continuous storage function $e(\cdot) \geq 0$, $e(0) = 0$ on $\mathbf{X}$, such that for all $t_1 \leq t_2$ and inputs $\hat{s} \in L_{2,loc}$

$$e(\hat{x}(t_2)) + \int_{t_1}^{t_2} |\hat{z}(t)|^2_{J_{\mathbf{Z}}} dt = e(\hat{x}(t_1)) + \int_{t_1}^{t_2} |\hat{s}(t)|^2_{J_{\mathbf{S}}} dt. \tag{8.3}$$

(iii) Σ_0 is *J-IO-dissipative* if for $\hat{x}_0 = 0$ and all $T \geq 0$ and inputs $\hat{s}$

$$\int_0^T |\hat{z}(t)|^2_{J_{\mathbf{Z}}} dt \leq \int_0^T |\hat{s}(t)|^2_{J_{\mathbf{S}}} dt. \tag{8.4}$$

(iv) Σ_0 is *J-IO-lossless* if for initial state $\hat{x}_0 = 0$, all inputs $\hat{s}$ in L_2 which produce L_2 outputs $\hat{z}$ satisfy

$$\int_0^\infty |\hat{z}(t)|^2_{J_{\mathbf{Z}}} dt = \int_0^\infty |\hat{s}(t)|^2_{J_{\mathbf{S}}} dt. \tag{8.5}$$

(v) Σ_0 is *L_2- stable* if Σ_0 maps L_2 into L_2.

(vi) Σ_0 is *outer* if both Σ_0 and Σ_0^{-1} are L_2 stable.

The terms *dissipative* and *lossless* (without the J prefix) will have the above meanings with J replaced by the matrix $\mathrm{diag}(I, I)$ as appropriate. All of the conditions (i) through (v) imply that dim $\mathbf{Z} \leq$ dim $\mathbf{S}$.

Remark 8.2.2 (i) J-SS-lossless implies J-SS-dissipative implies J-IO-dissipative.

Proof: this follows immediately from $e \geq 0$ and $e(0) = 0$.

(ii) J-SS-lossless does not necessarily imply J-IO-lossless.

The point: if $x(t) \to 0$, then $e(x(t)) \to 0$, which converts the J-SS-lossless equality as $t \to \infty$ to the J-IO-lossless equality. However, J-SS-lossless systems may not be stable.

(iii) A sad nonproperty of J-dissipation is that it does *not imply stability*. Even if one assumes detectability it does not.

The problem: is that $\hat{z}$ can have $\int_0^T |\hat{z}(t)|^2_{J_{\mathbf{Z}}} dt$ uniformly bounded in T but $\hat{z}$ not be in L_2.

∇

8.3 The Information State and Critical Feedback

For fixed $\hat{s} \in L_{2,loc}$, the *information state* $p_t \in \mathcal{X}_e$ *of the system* Σ is defined by

$$p_t(\hat{x}) = p_0(\hat{\xi}(0)) + \tfrac{1}{2} \int_0^t |\hat{C}(\hat{\xi}(s)) + \hat{D}(\hat{\xi}(s))\hat{s}|^2_{J_{\mathbf{Z}}}\, ds, \tag{8.6}$$

where $\hat{\xi}(\cdot)$ is the solution of

$$\dot{\hat{\xi}} = \hat{A}(\hat{\xi}) + \hat{B}(\hat{\xi})\hat{s}, \tag*{(8.1)$'$}$$

with *terminal* condition $\hat{\xi}(t) = \hat{x}$.

The dynamics for p_t, when p_t is a smooth function, is a partial differential equation: for fixed $\hat{s} \in L_{2,loc}$ we have

$$\dot{p}_t = \hat{F}(p_t, \hat{s}(t)), \tag{8.7}$$

where $\hat{F}(p, \hat{s})$ is the differential operator

$$\hat{F}(p, \hat{s}) \overset{\triangle}{=} -\nabla_x p \cdot (\hat{A} + \hat{B}\hat{s}) \; + \; \tfrac{1}{2}\, |\hat{C} + \hat{D}\hat{s}|^2_{J_{\mathbf{Z}}}. \tag{8.8}$$

Recall (§3.3 and Chapter 11) that there is an equilibrium p_e with a basin of attraction $\mathcal{D}_{attr}$. The function p_e is usually singular.

Key to constructing our factors is a function $\hat{\mathtt{s}}^*$ on the information states. Later we write down formulas in various cases for $\hat{\mathtt{s}}^*(p)$, but for now we present our results in terms of an unspecified function $\hat{\mathtt{s}}^*(p)$ which is continuous in some sense with respect to p and which has the properties $\hat{\mathtt{s}}^*(p + c) = \hat{\mathtt{s}}^*(p)$ for any constant $c \in \mathbf{R}$. We shall always assume $\hat{\mathtt{s}}^*(p_e) = 0$.

8.4 RECIPE for the Factors

In this section we present formulas for the factors. We present them as differential equations so they require p_t to be smooth functions in order to make sense (classically). This is just to make the equations seem more conventional and intuitive. However, one can integrate the formulas and present them in integral form, or appeal to weak sense interpretations. This is superior in that this gives inner outer factorization formulas when the functions p_t are singular. Indeed then one can work with an initial state p_0 which is singular as is often the case.

Note that the construction makes sense for any function $\hat{F}$ and $\hat{\mathtt{s}}^*$, not just the information state. Pick $\hat{d}$ so that

$$\hat{d}' \hat{E} \hat{d} = J_{\mathbf{V}}, \tag{8.9}$$

and let $\hat{d}^{-L}$ denote a left inverse of it, that is $\hat{d}^{-L}\hat{d} = I_{\mathbf{V}}$.

(i) Construct Σ^O as follows:

$$\Sigma^O_{p_0} : \begin{cases} \dot{p} = \hat{F}(p, \hat{\mathrm{s}}^*(p) + \hat{d}\hat{v}), \quad p(0) = p_0 \\ \hat{s} = \hat{\mathrm{s}}^*(p) + \hat{d}\hat{v} \\ \mathcal{R}(\Sigma^O_{p_0}) \triangleq \{p \text{ reachable from } p_0 \text{ with inputs } \hat{v} \in L_{2,loc}\} \end{cases} \tag{8.10}$$

(ii) The J-inner factor Σ^I is given by combining (8.1) and (8.10):

$$\Sigma^I_{\hat{\xi}_0,p_0} : \begin{cases} \dot{p} = \hat{F}(p, \hat{\mathrm{s}}^*(p) + \hat{d}\hat{v}), \quad p(0) = p_0 \\ \dot{\hat{\xi}} = \hat{A}(\hat{\xi}) + \hat{B}(\hat{\xi})(\hat{\mathrm{s}}^*(p) + \hat{d}\hat{v}), \quad \hat{\xi}(0) = \hat{\xi}_0 \\ \hat{z} = \hat{C}(\hat{\xi}) + \hat{D}(\hat{\xi})(\hat{\mathrm{s}}^*(p) + \hat{d}\hat{v}) \\ \mathcal{R}(\Sigma^I_{\hat{\xi}_0,p_0}) \triangleq (\mathbf{X}, \mathcal{R}(\Sigma^O_{p_0})). \end{cases} \tag{8.11}$$

(iii) Take the inverse of Σ^O to give:

$$(\Sigma^O_{p_0})^{-1} : \begin{cases} \dot{p} = \hat{F}(p, \hat{s}), \quad p(0) = p_0 \\ \hat{v} = \hat{d}^{-L}(\hat{s} - \hat{\mathrm{s}}^*(p)) \\ \mathcal{R}((\Sigma^O_{p_0})^{-1}) \triangleq \{p \text{ reachable from } p_0 \text{ with inputs } \hat{s} \in L_{2,loc}\} \end{cases} \tag{8.12}$$

Also interesting is the cut down of Σ^I:

$$\Sigma^I_{\hat{\xi}_0,p_0|} \triangleq \Sigma^I_{\hat{\xi}_0,p_0} \text{ restricted to } \mathcal{R}(\Sigma^I_{\hat{\xi}_0,p_0|})$$

where

$$\mathcal{R}(\Sigma^I_{0,p_0|}) \triangleq \{(\hat{\xi}, p) \text{ reachable from } \hat{\xi}_0 = 0, p_0 \text{ and } p_t(\hat{\xi}) \text{ is finite}\}.$$

We often drop the subscripts on the systems if the initial state of the system is obvious from the context.

8.5 Properties of the Factors

In this section we present properties of the factors defined by the formulas given in §8.4. Before doing this we describe necessary properties of the function $\hat{\mathrm{s}}^*$, as well as other assumptions. The function $\hat{\mathrm{s}}^*$ is defined in terms of a PDE (see (8.13) below) which is closely related to the dynamic programming PDE (4.16).

8.5.1 The Factoring PDE

We need a solution $\hat{W}$, $\hat{\mathsf{s}}^*$ to the *factoring PDE*

$$\mathrm{crit}_{\hat{s}\in\mathbf{S}}\ \nabla_p \hat{W}(p)[\hat{F}(p,\hat{s})] = 0 \tag{8.13}$$

defined on a subset of $\mathrm{dom}\hat{W}$ in $\mathcal{X}_e$. Here $\mathrm{crit}_{\hat{s}\in\mathbf{S}}$ stands for a critical point in $\mathbf{S}$ and we assume $\hat{\mathsf{s}}^*(p)$ is such a critical point. It implies that $\hat{W}$ is a function which satisfies

$$\hat{W}(p_t) = \hat{W}(p_0) \tag{8.14}$$

for all $t \geq 0$, where p_t is driven by $\hat{s} = \hat{\mathsf{s}}^*(p_t)$ with an appropriate initialization. The function $\hat{W}$ should satisfy the structural properties (§2.2.2.2). While ultimately understanding (8.13) in weak (e.g. viscosity) senses is important, we now work with smooth $\hat{W}$. Indeed, we will use interpretations as in §4.4.1 to clarify ideas; more general interpretations are left to the reader.

Since $\hat{F}$ is quadratic in $\hat{s}$ there is exactly one critical point which indeed can be computed to be

$$\hat{\mathsf{s}}^*(p) = -\nabla_p \hat{W}(p)[\hat{E}^{-1}(\hat{D}' J_{\mathbf{Z}} \hat{C} - \hat{B}' \nabla_x p')] \tag{8.15}$$

by taking derivatives in (8.13) and setting to 0, where $\hat{E} = \hat{D}' J_{\mathbf{Z}} \hat{D}$. (This formula assumes $\hat{D}$ does not depend on $\hat{x}$.) When we plug $\hat{\mathsf{s}}^*(p)$ into (8.13) we obtain

$$\nabla_p \hat{W}(p)[-\nabla_x p \cdot \hat{A} + \tfrac{1}{2}\hat{C}' J_{\mathbf{Z}} \hat{C}] - \tfrac{1}{2}\hat{\mathsf{s}}^*(p)' \hat{E} \hat{\mathsf{s}}^*(p) = 0. \tag{8.16}$$

See §8.10 for further details.

This differential equation for $\hat{W}$ typically has multiple solutions, and to each one there is an associated $\hat{\mathsf{s}}^*(p)$.

8.5.2 Factoring Assumptions

We make the following assumptions:

> Assumption 8.5.1 We always assume the vector field $\hat{A}$ is incrementally hyperbolic.
>
> Assumption 8.5.2 $\hat{W}$ is smooth on a domain $\mathrm{dom}\,\hat{W}$ and $\hat{W}$ together with critical point $\hat{\mathsf{s}}^*$ solve (8.15). Also $\hat{W}$ should satisfy the structural conditions (recall §2.2.2.2).
>
> Assumption 8.5.3 The function p_0 has the property that for all $\hat{s} \in L_{2,loc}$ and all $t \geq 0$ the function p_t solving the information state equation (8.6) exist and belongs to $\mathrm{dom}\,\hat{W}$.

Assumption 8.5.4 The augmented information state system (8.17)

$$\dot{p} = \hat{F}(p, \hat{\mathsf{s}}^*(p) + \hat{d}\hat{v}) \tag{8.17}$$

when driven by $L_{2,loc}$ input $\hat{v}$ has a solution $p_t \in \operatorname{dom} \hat{W}$ for all $t \geq 0$, for suitable initial states p_0.

Assumption 8.5.5 Assumptions 8.5.3 and 8.5.4 assure that the dynamics of the systems Σ, Σ_0 , Σ^I have solutions for $L_{2,loc}$ inputs. In addition we assume the outputs from these systems are in $L_{2,loc}$.

As we now see these assumptions are sufficient for the RECIPE to produce a J-lossless factor and some of these assumption are necessary. As we shall see Σ^O stable is not necessary for Σ^I to be J-SS-lossless nor is it required by our construction.

8.5.3 The Outer Factor Σ^O

Stability of Σ^O amounts to the *stabilizing* property of $\hat{\mathsf{s}}^*$, p_0 (recall §4.12.1), which in this context means that the following set is non-empty:

$$\begin{aligned} \operatorname{dom}_{cs}\hat{\mathsf{s}}^* = \{p_0 \in \operatorname{dom}\hat{W} \;:\; & \text{the solution } p_t \text{ of (8.10)} \\ & \text{is well-defined with } p_t \in \operatorname{dom}\hat{W} \text{ and} \\ & t \mapsto \hat{\mathsf{s}}^*(p_t) \in L_2[0,\infty) \text{ for all } \hat{v} \in L_2[0,\infty)\} \end{aligned} \tag{8.18}$$

Exactly what assumptions yield this property is not yet understood.

Theorem 8.5.6 *Under the factoring assumptions, the outer factor $\Sigma^O_{p_0}$ is stable provided $p_0 \in \operatorname{dom}_{cs}\hat{\mathsf{s}}^* \cap \mathcal{D}_{attr}(p_e)$, where p_e is a control attractor, in the sense that $\hat{v} \in L_2$ implies $\Sigma^O_{p_0}\hat{v} \in L_2$ and $p_t \Longrightarrow p_e + c(\hat{v})$.*

Proof. If $p_0 \in \operatorname{dom}_{cs}\hat{\mathsf{s}}^*$, and $\hat{v} \in L_2$, then by definition $t \mapsto \hat{\mathsf{s}}^*(p_t) \in L_2$. Thus $t \mapsto \hat{\mathsf{s}}^*(p_t) + \hat{d}\hat{v}(t) = \hat{s}(t) \in L_2$. If also $p_0 \in \mathcal{D}_{attr}(p_e)$, then $p_t \Longrightarrow p_e + c$. □

8.5.4 The Inner Factor Σ^I

The main desirable properties of the inner factor are determined by the factoring PDE. These results are presented in the next theorem.

Theorem 8.5.7 *Let $p_0 \in \operatorname{dom} \hat{W}$ and suppose Assumptions 8.5.1, 8.5.2, 8.5.3 and 8.2.1 hold. Consider the inner factor $\Sigma^I_{\hat{x}_0,p_0|}$ on the state space $\mathcal{R}(\Sigma^I_{\hat{\xi}_0,p_0|})$. Then*

(i) $\Sigma^I_{\hat{\xi}_0,p_0|}$ *is J-SS-lossless,*

(ii) $\Sigma^I_{\hat{\xi}_0,p_0|}$ *is J-SS-dissipative. Now assume in addition that $\hat{W}(p_0) = p_0(0)$. Then*

(iii) $\Sigma^I_{0,p_0|}$ *is J-IO-dissipative, and*

(iv) if $p_0 \in \mathrm{dom}_{cs}\hat{\mathbf{s}}^* \cap \mathcal{D}_{attr}(p_e)$*, where* p_e *is a control attractor with* $\hat{W}(p_e) = p_e(0)$*, and* $\hat{e}$ *is continuous on its domain in* $\mathbf{X} \times \mathcal{X}_e$*, the* $\Sigma^I_{0,p_0|}$ *is J-IO-lossless in the sense that for any input* $\hat{v} \in L_2$ *which produces* $\hat{z} \in L_2$ *and* $p_t \Longrightarrow p_e + c(\hat{v})$*,* $\hat{\xi}(t) \to 0$*, (8.5) holds.*

The proof of this theorem, while not difficult, requires two preliminary lemmas.

Lemma 8.5.8 *Let* $p_0 \in \mathrm{dom}\hat{W}$*. Then for any* $\hat{v}$ *in* $L_{2,loc}$ *such that the information state* p_t *given by (8.6) belongs to* $\mathrm{dom}\hat{W}$ *for all* $t \geq 0$*, we have*

$$\hat{W}(p_t) = \hat{W}(p_0) + \tfrac{1}{2}\int_0^t |\hat{v}(s)|^2_{J_V}\, ds \quad \text{for all } t \geq 0. \tag{8.19}$$

Proof. By definition of $\hat{\mathbf{s}}^*(p)$ as the critical point in the equation (8.13), we have, using also the form of $\hat{F}$,

$$\nabla_p W(p)[\hat{F}(p, \hat{\mathbf{s}}^*(p) + \hat{d}\hat{v})] = \tfrac{1}{2}|\hat{D}\hat{d}\hat{v}|^2_{J_\mathbf{Z}} = \tfrac{1}{2}|\hat{v}|^2_{J_\mathbf{V}}. \tag{8.20}$$

Integrate this to obtain (8.19). □

The next lemma defines a suitable energy (storage) function, c.f. Chapter 6, §6.2.

Lemma 8.5.9 *Suppose that* $\hat{W}$ *is smooth, satisfies the structural properties, and that* $\hat{\mathbf{s}}^*$ *is a critical point for (8.13). Let* $p_0 \in \mathrm{dom}\,\hat{W}$*. Assume that for any* $\hat{v}$ *in* $L_{2,loc}$ *and the augmented information state* p_t *given by the augmented information state dynamics (first line of (8.11)) belongs to* $\mathrm{dom}\hat{W}$ *for all* $t \geq 0$*. Then the storage function*

$$\hat{e}(\hat{x}, p) = -p(\hat{x}) + \hat{W}(p)$$

satisfies the energy balance equation for $\Sigma^I_{p_0}$*:*

$$\hat{e}(\hat{\xi}(t), p_t) + \tfrac{1}{2}\int_0^t |\hat{z}(s)|^2_{J_\mathbf{Z}}\, ds \;=\; \hat{e}(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t |\hat{v}|^2_{J_\mathbf{V}}\, ds, \tag{8.21}$$

for all $\hat{\xi}_0$ *such that* $p_0(\hat{\xi}_0)$ *is finite.*

Proof. First observe, from (8.6), that

$$p_t(\hat{\xi}(t)) = p_0(\hat{\xi}(0)) + \tfrac{1}{2}\int_0^t |\hat{C}(\hat{\xi}(s)) + \hat{D}(\hat{\xi}(s))\hat{s}|^2_{J_\mathbf{Z}}\, ds,$$

and so $p_t(\hat{\xi}(t)$ is finite provided $p_0(\hat{\xi}(0))$ is finite, and hence $\hat{e}(\hat{\xi}(t), p_t)$ is finite. Combining the previous display and (8.19), we get the identity

$$\hat{W}(p_t) - p_t(\hat{\xi}(t)) = \hat{W}(p_0) - p_0(\hat{\xi}(0)) + \tfrac{1}{2}\int_0^t |\hat{v}(s)|^2_{J_V}\, ds - \tfrac{1}{2}\int_0^t |\hat{z}(s)|^2_{J_Z}\, ds.$$

But this is just (8.21). □

Proof of Theorem 8.5.7. We know that $\hat{e}(\hat{x}, p) \geq 0$ by Assumption 8.5.2 and by Assumptions 8.5.3, 8.5.4, 8.5.5 that putting $\hat{v} \in L_{2,loc}$ into $\Sigma^I_{p_0}$ gives output in $L_{2,loc}$, so the J-SS-lossless and therefore J-SS-dissipative properties for $\Sigma^I_{p_0}$ and $\Sigma^I_{p_0|}$ follow directly from (8.21) provided $\hat{e}(\hat{x}, p)$ is finite. When we assume p_0 is everywhere finite we get by the definition of p_t that it is everywhere finite. When p_0 is singular $\hat{e}$ no longer is *everywhere* finite so no longer serves as a storage function for $\Sigma^I_{p_0}$, but at least the J-SS-dissipative properties for $\Sigma^I_{p_0}$ and $\Sigma^I_{p_0|}$ still follow directly from (8.21) provided $\hat{e}(\hat{x}_0, p_0)$ is finite, or equivalently $p_0(\hat{x}_0)$ is finite and $p_0 \in \mathrm{dom}\, \hat{W}$. With $\Sigma^I_{p_0|}$ we are restricting to the part of the state space $\mathbf{X}$ on which p_0 is finite, so J-SS-losslessness, etc, are valid as stated. This proves items (i) and (ii).

If $\hat{W}(p_0) = p_0(0)$, then $\hat{e}(0, p_0) = 0$, and item (iii) follows.

Under the hypotheses of item (iv), we have for such an input $\hat{v}$ that $\hat{e}(\hat{\xi}(t), p_t) \to 0$ as $t \to \infty$, and so J-IO-losslessness follows. □

8.5.5 The Inverse Outer Factor $(\Sigma^O)^{-1}$

The stability of the operator $(\Sigma^O_{p_0})^{-1}$ follows provided the initial condition p_0 is suitably chosen and the function $\hat{\mathsf{s}}^*$ is sufficiently well-behaved.

Theorem 8.5.10 *Under the factoring assumptions, the inverse outer factor $(\Sigma^O_{p_0})^{-1}$ is stable at least to the extent that if $p_0 \in \mathcal{D}_{attr}(p_e)$, where p_e is a control attractor, then $p_t \Longrightarrow p_e + c(\hat{s})$, for any input $\hat{s} \in L_2$. If in addition $t \mapsto \hat{\mathsf{s}}^*(p_t) \in L_2$, then $(\Sigma^O_{p_0})^{-1}\hat{s} \in L_2$.*

Proof. This result follows simply from the definitions. □

8.5.6 Necessity of the RECIPE Formulas

Such results will appear later after describing the application of this theory to control. Ironically the control application greatly facilitates proof and presentation of the necessary results.

8.5.7 Singular Cases

The results presented above require $p_0(\hat{\xi}_0)$ finite. More general results for singular cases can be derived along the lines of Chapter 4 using a hyperbolicity assumption; however we do not do so here.

8.6 Examples

8.6.1 Certainty Equivalence

There are conditions (Chapter 7) under which $\hat{W}$ given by the formula

$$\hat{W}(p) \stackrel{\triangle}{=} \langle p + \hat{V} \rangle \tag{8.22}$$

solves the factoring PDE (8.13). Here $\hat{V}$ denotes the stabilizing solution to *the state feedback factoring PDE*

$$\begin{aligned}&\nabla_{\hat{x}}\hat{V} \cdot (\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}}\hat{C}) - \tfrac{1}{2}\nabla_{\hat{x}}\hat{V}\hat{B}\hat{E}^{-1}\hat{B}'\nabla_{\hat{x}}\hat{V}' \\ &+\tfrac{1}{2}\hat{C}' J_{\mathbf{Z}}(J_{\mathbf{Z}} - \hat{D}\hat{E}^{-1}\hat{D}')J_{\mathbf{Z}}C = 0\end{aligned} \tag{8.23}$$

which is non-negative, $\hat{V}(0) = 0$, and stabilizing, to wit the vector field

$$\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}}\hat{C} - \hat{B}\hat{E}^{-1}\hat{B}'\nabla_{\hat{x}}\hat{V}'$$

is incrementally L_2 exponentially stable.

The assumptions which make (8.22) true are:

(i) $\hat{V}$ is finite, smooth, $\hat{V}(0) = 0$ and $\hat{V} \geq 0$.

(ii) The supremum in (8.22) exists and occurs at a unique $\hat{x}$ denoted $\bar{\hat{x}}(p)$.

Then

$$\hat{\mathbf{s}}^*(p) = \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p))$$

where $\hat{\mathbf{s}}^*_{state}$ is given by

$$\hat{\mathbf{s}}^*_{state}(\hat{x}) \stackrel{\triangle}{=} -E^{-1}[\hat{D}' J_{\mathbf{Z}}\hat{C} + \hat{B}'\nabla_x\hat{V}']_{|\hat{x}} \tag{8.24}$$

which incidentally solves the “state feedback” factorization problem treated in §8.9. Also note that $\hat{\mathbf{s}}^*_{state}(\hat{x}) = \hat{\mathbf{s}}^*(\delta_{\hat{x}})$ where $\hat{\mathbf{s}}^*$ is given by (8.15). Here we use the fact $\nabla_p\hat{W}(\delta_{\hat{x}})[h] = h(\hat{x})$ for all suitable functions h (see Chapter 10).

Observe that the “stabilizing” property of $\hat{V}$ means that

$$\begin{aligned}\dot{\hat{x}} &= [\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}'\hat{C}](\hat{x}) - \hat{B}E^{-1}\hat{B}'\nabla_x\hat{V}'(\hat{x}) \\ &= \hat{A}(\hat{x}) + \hat{B}\hat{\mathbf{s}}^*_{state}(\hat{x})\end{aligned}$$

is asymptotically stable.

Assumption (ii) above is what one would call the *certainty equivalence assumption* in this setting. Note that $\hat{V} \geq 0$ implies $\hat{W}(p) = \langle p + \hat{V} \rangle \geq \langle p \rangle$.

The RECIPE in the certainty equivalence case is:

(i) Outer factor:

$$\Sigma^O_{p_0}: \begin{cases} \dot{p} \ = \ \hat{F}(p, \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p)) + \hat{d}\hat{v}), \quad p(0) = p_0 \\ \hat{s} \ = \ \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p)) + \hat{d}\hat{v} \\ \mathcal{R}(\Sigma^O_{p_0}) \ \stackrel{\triangle}{=} \ \{p \text{ reachable from } p_0 \text{ with inputs } \hat{v} \in L_{2,loc}\} \end{cases} \tag{8.25}$$

(ii) The J-inner factor:

$$\Sigma^I_{\hat{\xi}_0, p_0}: \begin{cases} \dot{p} \ = \ \hat{F}(p, \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p)) + \hat{d}\hat{v}), \quad p(0) = p_0 \\ \dot{\hat{\xi}} \ = \hat{A}(\hat{\xi}) + \hat{B}(\hat{\xi})(\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p)) + \hat{d}\hat{v}), \quad \hat{\xi}(0) = \hat{\xi}_0 \\ \hat{z} \ = \ \hat{C}(\hat{\xi}) + \hat{D}(\hat{\xi})(\hat{\mathbf{s}}^*(\bar{\hat{x}}(p)) + \hat{d}\hat{v}) \\ \mathcal{R}(\Sigma^I_{\hat{\xi}_0, p_0}) \stackrel{\triangle}{=} (\mathbf{X}, \mathcal{R}(\Sigma^O_{p_0})). \end{cases} \tag{8.26}$$

(iii) Inverse outer factor:

$$(\Sigma^O_{p_0})^{-1}: \begin{cases} \dot{p} \ = \ \hat{F}(p, \hat{s}), \quad p(0) = p_0 \\ \hat{v} \ = \ \hat{d}^{-L}(\hat{s} - \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}(p))) \\ \mathcal{R}((\Sigma^O_{p_0})^{-1}) \ \stackrel{\triangle}{=} \ \{p \text{ reachable from } p_0 \text{ with inputs } \hat{s} \in L_{2,loc}\} \end{cases} \tag{8.27}$$

8.6.2 $\hat{A}$ Stable

The stable equilibrium p_e for the information state equation (8.6) is the singular information state δ_0. The natural initialization in this case is $p_0 = \delta_0$. Then Σ^O has state $p_t = \delta_{\bar{\hat{x}}(t)}$ where $\bar{\hat{x}}(\cdot)$ satisfies

$$\Sigma^O_{\delta_{\bar{\hat{x}}_0}} \ : \begin{cases} \dot{\bar{\hat{x}}} \ = \hat{A}(\bar{\hat{x}}) + \hat{B}(\bar{\hat{x}})[\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}] \quad \bar{\hat{x}}(0) = \bar{\hat{x}}_0, \\ \hat{s} \ = \hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}, \end{cases} \tag{8.28}$$

where $\hat{\mathbf{s}}^*_{state}$ is given by (8.24), which is determined by a solution $\hat{V}$ of the state feedback factoring PDE (8.13). If $\hat{V}$ is stabilizing, then Σ^O is certainly L_2 stable. Note that certainty equivalence holds in this case, since with $p = \delta_{\hat{\xi}}$, we have

$$\bar{\hat{x}}(\delta_{\hat{\xi}}) = \hat{\xi}.$$

Of course $(\Sigma^O)^{-1}$ and Σ^I are derived easily from (8.28). Indeed,

$$(\Sigma^O_{\delta_{\bar{\hat{x}}_0}})^{-1} \ : \ \begin{cases} \dot{\hat{x}} \ = \hat{A}(\hat{x}) + \hat{B}(\hat{x})\hat{v}, \quad \hat{x}(0) = \hat{x}_0, \\ \\ \hat{v} \ = \hat{d}^{-L}(\hat{s} - \hat{\mathbf{s}}^*_{state}(\hat{x})). \end{cases} \tag{8.29}$$

If $\hat{A}$ is incrementally L_2 exponentially stable, then $(\Sigma^O_{\delta_{\bar{\hat{x}}_0}})^{-1}$ is L_2 stable. So Σ^O is outer.

The J-inner factor is given by

$$\Sigma^I_{\delta_{\bar{\hat{x}}_0},\hat{\xi}_0} \ : \ \begin{cases} \dot{\bar{\hat{x}}} \ = \hat{A}(\bar{\hat{x}}) + \hat{B}(\bar{\hat{x}})[\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}] \quad \bar{\hat{x}}(0) = \bar{\hat{x}}_0, \\ \\ \dot{\hat{\xi}} \ = \hat{A}(\hat{\xi}) + \hat{B}(\hat{\xi})(\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}), \quad \hat{\xi}(0) = \hat{\xi}_0, \\ \\ \hat{z} \ = \hat{C}(\hat{\xi}) + \hat{D}(\hat{\xi})(\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}). \end{cases} \tag{8.30}$$

In the case $\bar{\hat{x}}_0 = \hat{\xi}_0$, this reduces to

$$\Sigma^I_{\delta_{\bar{\hat{x}}_0},\bar{\hat{x}}_0} \ : \ \begin{cases} \dot{\bar{\hat{x}}} \ = \hat{A}(\bar{\hat{x}}) + \hat{B}(\bar{\hat{x}})[\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}] \quad \bar{\hat{x}}(0) = \bar{\hat{x}}_0, \\ \\ \hat{z} \ = \hat{C}(\bar{\hat{x}}) + \hat{D}(\bar{\hat{x}})(\hat{\mathbf{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}). \end{cases} \tag{8.31}$$

8.6.3 $\hat{A}$ Strictly Antistable

The results in the preceeding subsection §8.6.2 show how existing results in the literature can be obtained from the information state perspective. The stability of $A^\times$ assumed there results in an information state which projects to a finite dimensional ODE (when correctly initialized). In contrast to this, we mention that the information state framework provides a complete theory when $A^\times$ is not stable, for instance if $A^\times$ is antistable. When $A^\times$ is antistable, the information state is nonsingular, and does not in general project to a finite dimensional system (except in linear and bilinear cases). This case is genuinely infinite dimensional. The energy function $e(x,p)$ is finite for all p's occuring, since they are nonsingular. Further, hyperbolic cases can also be treated using the information state framework, as our results show.

8.6.4 Bilinear Systems

Consider the bilinear system

$$\Sigma \ : \ \begin{cases} \dot{\hat{x}} \ = \hat{A}(\hat{s})\hat{x} + \hat{B}\hat{s}, \quad \hat{x}(0) = \hat{x}_0 \\ \\ \hat{z} \ = \hat{C}\hat{x} + \hat{D}\hat{s} \end{cases} . \tag{8.32}$$

where $\hat{B}$, $\hat{C}$ and $\hat{D}$ are matrices of appropriate sizes, and $\hat{A}(\hat{s})$ is an affine matrix function of $\hat{s}$. As in Chapter 7, §7.1, the information state is given by

$$p_t(\hat{x}) = -\tfrac{1}{2}(\hat{x} - \hat{\bar{x}}(t))'\hat{Y}(t)^{-1}(\hat{x} - \hat{\bar{x}}(t)) + \phi(t), \tag{8.33}$$

where

$$\begin{aligned} \dot{\hat{\bar{x}}} &= (\hat{A}(\hat{s}) + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{C})\hat{\bar{x}} + (\hat{B} + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{D})\hat{s}, \\ \dot{\hat{Y}} &= \hat{A}(\hat{s})\hat{Y} + \hat{Y}\hat{A}(\hat{s})' + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{Y}, \\ \dot{\phi} &= \tfrac{1}{2}|\hat{C}\hat{\bar{x}} + \hat{D}\hat{s}|^2. \end{aligned} \tag{8.34}$$

provided p_0 is such a quadratic form.

The critical feedback $\hat{\mathsf{s}}^*(\hat{\bar{x}}, \hat{Y})$ is obtained from the bilinear factoring PDE

$$\begin{aligned} &\mathrm{crit}_{\hat{s}}\{\nabla_{\hat{\bar{x}}}\hat{W}(\hat{\bar{x}}, \hat{Y}, \phi)\cdot[(\hat{A}(\hat{s}) + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{C})\hat{\bar{x}} + (\hat{B} + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{D})\hat{s}] \\ &\quad +\nabla_{\hat{Y}}\hat{W}\cdot[\hat{A}(\hat{s})\hat{Y} + \hat{Y}\hat{A}(\hat{s})' + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{Y}] + \nabla_{\phi}\hat{W}\cdot[\tfrac{1}{2}|\hat{C}\hat{\bar{x}} + \hat{D}\hat{s}|^2]\} = 0 \end{aligned} \tag{8.35}$$

This gives the RECIPE for the bilinear case:

(i) Outer factor:

$$\Sigma^O : \begin{cases} \dot{\bar{x}} = (\hat{A}(\hat{s}) + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{C})\bar{x} + (\hat{B} + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{D})(\hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}) \\ \dot{\hat{Y}} = \hat{A}(\hat{\mathsf{s}}^*(\hat{x}, \hat{Y}) + \hat{d}\hat{v})\hat{Y} + \hat{Y}\hat{A}(\hat{\mathsf{s}}^*(\hat{x}, \hat{Y}) + \hat{d}\hat{v})' + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{Y}, \\ \hat{s} = \hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}. \end{cases} \tag{8.36}$$

(ii) Inverse outer:

$$(\Sigma^O)^{-1} : \begin{cases} \dot{\hat{\bar{x}}} = (\hat{A}(\hat{s}) + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{C})\hat{\bar{x}} + (\hat{B} + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{D})\hat{s}, \\ \dot{\hat{Y}} = \hat{A}(\hat{s})\hat{Y} + \hat{Y}\hat{A}(\hat{s})' + \hat{Y}\hat{C}'J_{\mathbf{Z}}\hat{Y}, \\ \hat{v} = \hat{d}^{-L}(\hat{s} - \hat{\mathsf{s}}^*(\hat{\bar{x}}, \hat{Y})). \end{cases} \tag{8.37}$$

(iii) J-inner:

$$\Sigma^O : \begin{cases} \dot{\bar{x}} = (\hat{A}(\hat{s}) + \hat{Y}\hat{C}' J_{\mathbf{Z}} \hat{C})\bar{x} + (\hat{B} + \hat{Y}\hat{C}' J_{\mathbf{Z}} \hat{D})(\hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}) \\ \dot{\hat{Y}} = \hat{A}(\hat{\mathsf{s}}^*(\hat{x}, \hat{Y}) + \hat{d}\hat{v})\hat{Y} + \hat{Y}\hat{A}(\hat{\mathsf{s}}^*(\hat{x}, \hat{Y}) + \hat{d}\hat{v})' + \hat{Y}\hat{C}' J_{\mathbf{Z}} \hat{Y}, \\ \dot{\hat{\xi}} = \hat{A}(\hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}) + \hat{B}(\hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}), \\ \hat{z} = \hat{C}\hat{\xi} + \hat{D}(\hat{\mathsf{s}}^*(\bar{x}, \hat{Y}) + \hat{d}\hat{v}). \end{cases} \tag{8.38}$$

8.6.5 Linear Systems

Typically one initializes p_0 at p_e, the control attractor. If not p_t will probably converge to p_e anyway. For linear systems, this corresponds to using a solution $\hat{Y}_e \geq 0$ of the Riccati equation

$$\hat{A}\hat{Y}_e + \hat{Y}_e\hat{A}' + \hat{Y}_e\hat{C}' J_{\mathbf{Z}} \hat{Y}_e = 0, \tag{8.39}$$

which satisfies the stabilizing condition

$$A + \hat{Y}_e\hat{C}' J_{\mathbf{Z}} \hat{C} \quad \text{asymptotically stable.} \tag{8.40}$$

In this case $(\Sigma^O)^{-1}$ is stable, and is given by

$$(\Sigma^O)^{-1} : \begin{cases} \dot{\hat{x}} = (\hat{A} + \hat{Y}_e\hat{C}' J_{\mathbf{Z}} \hat{C})\hat{x} + (\hat{B} + \hat{Y}_e\hat{C}' J_{\mathbf{Z}} \hat{D})\hat{s} \\ \hat{v} = \hat{d}^{-L}(\hat{s} + \hat{E}^{-1}(\hat{D}' J_{\mathbf{Z}} \hat{C} + \hat{B}'\hat{X}_e)\hat{x}). \end{cases} \tag{8.41}$$

On the other hand Σ^O is is determined from a solution $\hat{X}_e \geq 0$ of the Riccati equation

$$\begin{aligned} &(\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}} \hat{C})'\hat{X}_e + \hat{X}_e(\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}} \hat{C}) \\ &-\hat{X}_e\hat{B}\hat{B}'\hat{X}_e + \hat{C}' J_{\mathbf{Z}}(J_{\mathbf{Z}} - \hat{D}\hat{E}^{-1}\hat{D}')J_{\mathbf{Z}}\hat{C} = 0, \end{aligned} \tag{8.42}$$

and is given by

$$\Sigma^O_{p_e} : \begin{cases} \dot{\bar{x}} = (\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}} \hat{C} - \hat{B}\hat{E}^{-1}\hat{B}'\hat{X}_e)\bar{x} \\ \qquad +(I - \hat{Y}_e\hat{X}_e)^{-1}(\hat{B} + \hat{Y}_e\hat{C}' J_{\mathbf{Z}} \hat{D})(s + \hat{E}^{-1}(\hat{D}' J_{\mathbf{Z}} \hat{C} + \hat{B}'\hat{X}_e)\bar{x}) \\ \hat{s} = -\hat{E}^{-1}(\hat{D}' J_{\mathbf{Z}} \hat{C} + \hat{B}'\hat{X}_e)\bar{x} + \hat{d}\hat{v} \end{cases} \tag{8.43}$$

This system is stable if and only if $\hat{X}_e$ satisfies the stabilizing condition

$$\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}' J_{\mathbf{Z}} \hat{C} - \hat{B}\hat{E}^{-1}\hat{B}'\hat{X}_e \quad \text{asymptotically stable.} \tag{8.44}$$

However, our constructions require $\hat{X}_e$ to be nonnegative or positive only, but not necessarily stabilizing. Thus the RECIPE is more flexible than traditional J-lossless, outer factorization, which would require that both $\hat{Y}_e$ and $\hat{X}_e$ be stabilizing. Of course, we require the spectral radius of $\hat{Y}_e\hat{X}_e$ to be strictly less than one.

The J-inner factor is given by

$$\Sigma^I_{\hat{\xi}_0,\bar{x}_0} : \begin{cases} \dot{\bar{x}} = (\hat{A} - \hat{B}\hat{E}^{-1}\hat{D}'J_{\mathbf{Z}}\hat{C} - \hat{B}E^{-1}\hat{B}'\hat{X}_e)\bar{x} \\ \qquad +(I - \hat{Y}_e\hat{X}_e)^{-1}(\hat{B} + \hat{Y}_e\hat{C}'J_{\mathbf{Z}}\hat{D})(s + \hat{E}^{-1}(\hat{D}'J_{\mathbf{Z}}\hat{C} + \hat{B}'\hat{X}_e)\bar{x}), \\ \qquad \bar{x}(0) = \bar{x}_0, \\ \dot{\hat{\xi}} = \hat{A}\hat{\xi} + \hat{B}(-\hat{E}^{-1}(\hat{D}'J_{\mathbf{Z}}\hat{C} + \hat{B}'\hat{X}_e)\bar{x} + \hat{d}\hat{v}), \quad \hat{\xi}(0) = \hat{\xi}_0 \\ \hat{z} = \hat{C}(\hat{\xi}) + \hat{D}(\hat{\xi})(-\hat{E}^{-1}(\hat{D}'J_{\mathbf{Z}}\hat{C} + \hat{B}'\hat{X}_e)\bar{x} + \hat{d}\hat{v}). \end{cases} \tag{8.45}$$

8.7 Factoring and Control

Now we apply factoring to the standard problem of H_∞ control. For simplicity we set $\gamma = 1$. This is the following standard generalized regulator arrangement with nonlinear plant G and nonlinear controller K shown in Figure 8.2.

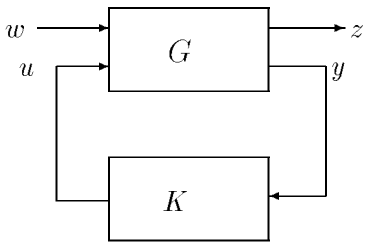


Figure 8.2: Closed-loop system (G, K) for H_∞ control.

We shall soon see how one solves this by taking Σ to equal the reversed arrow system $\tilde{G}$ (2.6) of G (2.2) and by factoring Σ. Before pursuing the control solution heavily we first study factoring for its own sake in new coordinates which turn out to be natural to the control problem.

To any signature matrix $J_{\mathbf{V}}$ on a vector space $\mathbf{V}$ there is a natural decomposition of $\mathbf{V}$ into $\mathbf{V}_+$ and $\mathbf{V}_-$, subspaces on which J is I and $-I$ respectively. Thus it is natural to conformably partition the input and output spaces of the system Σ we have been studying:

$$\hat{s} = \begin{pmatrix} u \\ y \end{pmatrix}, \quad \hat{z} = \begin{pmatrix} z \\ w \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix},$$

so for definiteness

$$u \in \mathbf{S}_+, \;\; y \in \mathbf{S}_-, \;\; v_2 \in \mathbf{V}_+, \;\; \text{and} \;\; v_1 \in \mathbf{V}_-,$$

in which case one gets system equations:

$$\Sigma : \begin{cases} \dot{\xi} \;=\; A^{\times}(\xi) + B_1(\xi)y + B_2(\xi)u \\ z \;=\; C_1(\xi) + D_{12}u \\ w \;=\; -C_2(\xi) + y. \end{cases} \tag{8.46}$$

Here the notation $A^{\times}$ and $\hat{A}$ are both consistent with previous usage, $A^{\times}$ if we think in terms of $\tilde{G}$ and $\hat{A}$ if we think in terms of Σ. Note that

$$\hat{A} = A^{\times}, \quad \hat{B} = (B_2, B_1), \quad \text{and} \quad \hat{C} = \begin{pmatrix} C_1 \\ -C_2 \end{pmatrix}.$$

This structure assumed of G is the nonlinear analogue of the "2A block problem" (recall Chapter 2) in that we assume the $\hat{D}$ term (as in (8.47) for Σ is

$$\hat{D} = \begin{pmatrix} D_{12} & 0 \\ 0 & I \end{pmatrix}. \tag{8.47}$$

We shall also assume that $\hat{d}$ in these coordinates is block diagonal, to wit

$$\hat{d} = \begin{pmatrix} \hat{d}_{11} & 0 \\ 0 & I \end{pmatrix}, \tag{8.48}$$

with $\hat{d}_{11} : \mathbf{V}_+ \to \mathbf{S}_+$, e.g. we could take $\hat{d}_{11} = E_1^{-1/2}$.

Then in pictures the J-inner outer factoring in Figure 8.1 becomes Figure 8.3.

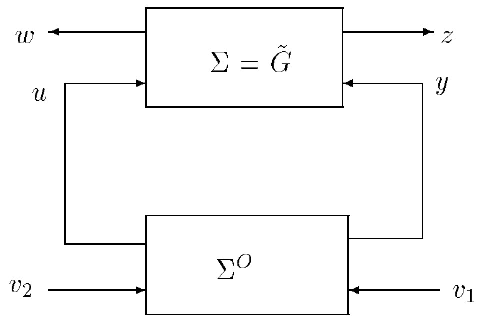


Figure 8.3: Inner factor $\Sigma^I = \Sigma \circ \Sigma^O$.

Here, $z \in \mathbf{Z}_+$, $w \in \mathbf{Z}_-$, and $v_2 \in \mathbf{V}_+$ and $v_1 \in \mathbf{V}_-$. Thus

$$| \begin{pmatrix} z \\ w \end{pmatrix} |^2_{J_{\mathbf{Z}}} = \begin{pmatrix} z' & w' \end{pmatrix} J_{\mathbf{Z}} \begin{pmatrix} z \\ w \end{pmatrix} = |z|^2 - |w|^2,$$

$$| \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} |^2_{J_{\mathbf{V}}} = \begin{pmatrix} v_2' & v_1' \end{pmatrix} J_{\mathbf{V}} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = |v_2|^2 - |v_1|^2.$$

The formulas for the factorization of $\Sigma = \tilde{G}$ can be read from the formulas given by the RECIPE in §8.4 with the substitutions given above.

There are strong statements possible about the $+$ and $-$ components of signals centering on the fact that a bound on $|v|_{J_{\mathbf{V}}} = |\hat{v}_+|^2 - |\hat{v}_-|^2$ has little strength because of cancellation, while a bound on $|\hat{v}_2|^2 + |\hat{v}_1|^2$ is a highly substantive statement. The schematic way of looking at this which is most helpful in exploiting its strengths is what we think of as *reversing arrows*.

One can simply reverse the w and y arrows in Figure 8.3. The systems Σ, Σ^O, and system $(\tilde{\Sigma}, \tilde{\Sigma}^I)$ obtained by reversing arrows on $G = \tilde{\Sigma}$ and $\hat{K} = \tilde{\Sigma}^O$ in the figures are uniquely determined one from the other; see Figures 8.3 and 8.4. Note that a controller $K : y \mapsto u$ is obtained by setting $v_2 = 0$ and ignoring v_1, so that $\hat{K} = \tilde{\Sigma}^O$ is in fact an *augmented controller*.

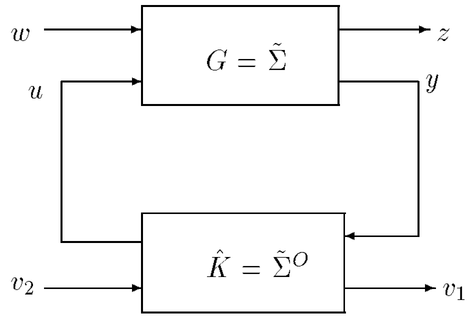


Figure 8.4: Reversed-arrow system $(\tilde{\Sigma}, \tilde{\Sigma}^O) = (G, \hat{K})$.

Also note

$$F(p, u, y) = \hat{F}(p, \hat{s}) = \hat{F}(p, (u, y)),$$

so the system $\hat{K} = \tilde{\Sigma}^O : (v_2, y) \mapsto (u, v_1)$ is given by

$$\hat{K}_{p_0} = \tilde{\Sigma}^O_{p_0} : \begin{cases} \dot{p} = F(p, \mathbf{u}^*(p) + \hat{d}_{11} v_2, y), \quad p(0) = p_0 \\ u = \mathbf{u}^*(p) + \hat{d}_{11} v_2 \\ v_1 = -\mathbf{y}^*(p) + y. \\ \mathcal{R}(\hat{K}_{p_0}) = \{p \text{ reachable from } p_0 \text{ with } v_2, y \in L_{2,loc}\} \end{cases} \tag{8.49}$$

The composition $(G, \hat{K}) = (\tilde{\Sigma}, \tilde{\Sigma}^O)$ is defined in the obvious way on the state space

$$\mathcal{R}(G_{x_0}, \hat{K}_{p_0}) \stackrel{\triangle}{=} \mathbf{R}^n \times \mathcal{R}(\hat{K}_{p_0}).$$

It naturally cuts down to the state space

$$\mathcal{R}(G_{x_0}, \hat{K}_{p_0})_| \stackrel{\triangle}{=} \{(x,p) : \text{ reachable from } (x_0, p_0) \text{ with } p_0(x_0) \text{ finite}\}.$$

The closed loop system $(G, \hat{K}) = (\tilde{\Sigma}, \tilde{\Sigma}^O)$ is given by

$$(G_{x_0}, \hat{K}_{p_0}) \, : \, \begin{cases} \dot{x} \, = A(x) + B_2(x)\mathbf{u}^*(p) + B_1(x)w, \quad x(0) = x_0 \\ \dot{p} \, = F(p, \mathbf{u}^*(p) + \hat{d}_{11}v_2, C_2(x) + w), \quad p(0) = p_0 \\ z \, = C_1(x) + D_{12}\mathbf{u}^*(p) \\ v_1 \, = -\mathbf{y}^*(p) + C_2(x) + w \end{cases} \tag{8.50}$$

We shall require the following well posedness type of assumption.

> Assumption 8.7.1 *(Reversibility.)* When the system $\Sigma^I_{\hat{\xi}_0, p_0}$ initialized at any state $(\hat{\xi}_0, p_0)$ is driven by any v_1, v_2 in $L_{2,loc}$, then the w and z components of its output is in $L_{2,loc}$ and all $w \in L_2$ occur (as v_1, v_2 sweep all of $L_{2,loc}$.
>
> When the system $(G_{x_0}, \hat{K}_{p_0})$ initialized at any state (z_0, p_0) is driven by any w, v_2 in $L_{2,loc}$, then the v_1 and z components of its output is in $L_{2,loc}$ and all $v_1 \in L_2$ occur (as w, v_2 sweep all of $L_{2,loc}$).

Dissipation properties behave as follows under the reversing arrows transformation.

Theorem 8.7.2 *Suppose Σ^I is reversible.* The following statements are true.*

(i) Σ^I is J-SS-dissipative if and only if $(G, \hat{K})$ is I-SS-dissipative. Likewise for $\Sigma^I_|$ and $(G, \hat{K})_|$.

(ii) Σ^I is J-SS-lossless if and only if $(G, \hat{K})$ is I-SS-lossless. Likewise for $\Sigma^I_|$ and $(G, \hat{K})_|$.

(iii) If $(G, \hat{K})$ (constructed above) is I-SS-dissipative with energy function e, then it is L_2 stable, and if also (z, v_1)-detectable, then it is weakly internally stable. Now assume $p_0 \in \mathcal{D}_{attr}(p_e)$, $e(0, p_0) = 0$, $e(0, p_e) = 0$, where p_e is a control attractor. Then if in addition $(G_{x_0}, \hat{K}_{p_0})$ is I-SS-lossless with continuous storage function e (satisfying $e(x, p + c) = e(x, p)$) then it is I-IO-lossless.

*Here no specific construction of Σ^I or of $e(x,p)$ is needed; this argument works for any J-SS-lossless vs. I-SS-lossless system.

Proof. The energy balance equation (8.21) for the $\Sigma^I = (\Sigma, \Sigma^O)$ system Figure 8.3 in our current notation is

$$\begin{aligned} e(\hat{\xi}(t), p_t) &+ \tfrac{1}{2}\int_0^t [|z(s)|^2 - |w(s)|^2]\, ds \\ &= e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|v_2(s)|^2 - |v_1(s)|^2]\, ds. \end{aligned} \tag{8.51}$$

This holds for all v_1, v_2 in $L_{2,loc}$. Reversing arrows does not alter this equality, and we write

$$\begin{aligned} e(\hat{\xi}(t), p_t) &+ \tfrac{1}{2}\int_0^t [|z(s)|^2 + |v_1(s)|^2]\, ds \\ &= e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|w(s)|^2 + |v_2(s)|^2]\, ds. \end{aligned} \tag{8.52}$$

however for $(G, \hat{K})$, the inputs are w and v_2 and we need to know that the inequality is true whenever w and v_2 sweep all of $L_{2,loc}$. This is not implied by Σ^I being J-SS-lossless alone but is precisely what is implied when we add the reversibility assumption. The I-SS-lossless and J-SS-lossless equivalences for $(G, \hat{K})$ and Σ^I follow. I-SS-dissipation follows from (8.51), (8.52) with inequality:

$$\begin{aligned} e(\hat{\xi}(t), p_t) &+ \tfrac{1}{2}\int_0^t [|z(s)|^2 - |w(s)|^2]\, ds \\ &\le e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|v_2(s)|^2 - |v_1(s)|^2]\, ds \end{aligned} \tag{8.53}$$

for all v_1, v_2 in $L_{2,loc}$, and reversing arrows

$$\begin{aligned} e(\hat{\xi}(t), p_t) &+ \tfrac{1}{2}\int_0^t [|z(s)|^2 + |v_1(s)|^2]\, ds \\ &\le e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|w(s)|^2 + |v_2(s)|^2]\, ds \end{aligned} \tag{8.54}$$

for all $w, v_2 \in L_{2,loc}$. This proves conclusions (i) and (ii).

Now suppose that $(G, \hat{K})$ is I-SS-dissipative, so that (8.54) holds for all $w, v_2 \in L_{2,loc}$. If $w, v_2 \in L_2$, we see that $z, v_1 \in L_2$, and so $(G, \hat{K})$ is L_2 stable, and moreover, if $(G, \hat{K})$ is (z, v_1)-detectable, then it is weakly internally stable by Theorem 2.1.3. Using the remaining hypotheses, we have

$$e(x(t), p_t) + \tfrac{1}{2}\int_0^t [|z(s)|^2 + |v_1(s)|^2]\, ds = e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|w(s)|^2 + |v_2(s)|^2]\, ds$$

for all $w, v_2 \in L_2$ and all $t \ge 0$. Sending $t \to \infty$ gives I-IO-losslessness. This proves part (iii). $\square$

8.7.1 RECIPE for Solving the Control Problem

Recall the H_∞ control problem which opened this book, see Chapter 2. Now we show how factoring produces solutions to it. Recall that we take $\gamma = 1$.

Given the plant G, the recipe goes as follows:

(i) Reverse arrows on G to get $\tilde{G}$. Denote $\tilde{G}$ by Σ.

(ii) Find if possible a decomposition of Σ:

$$\Sigma \circ \Sigma^O = \Sigma^I$$

with Σ^I a J-SS-dissipative reversible system and Σ^O outer. This decomposition could, for example, be obtained using the RECIPE, but need not be.

(iii) Re-reverse arrows to get $\hat{K}$ from Σ^O and $(G, \hat{K})$ from Σ^I.

(iv) Connect any strict dissipative load L to $\hat{K}$ as in Figure 8.5 to produce a closed loop system, call it

$$K \stackrel{\triangle}{=} (\hat{K}, L).$$

We assume that this system is well-defined in that $L_{2,loc}$ inputs y produce $L_{2,loc}$ signals u, v_1, v_2. This is the desired controller for the H_∞ control problem (illustrated in Figure 8.5).

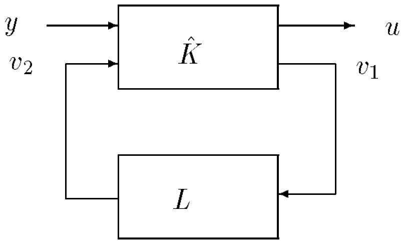


Figure 8.5: The controller $K = (\hat{K}, L)$.

Theorem 8.7.3 *Consider any factorization $\Sigma^I = \Sigma \circ \Sigma^O$. Suppose Σ^I is J-SS-dissipative. Make the reversibility Assumption 8.7.1, and assume that $K = (\hat{K}, L)$ is well defined, where L is a IO-dissipative system. Then $K = (\hat{K}, L)$ is a controller which solves the H_∞ control problem, provided the closed loop system (G, K) is z-detectable. That is, the closed loop system is dissipative and weakly internally stable (meaning that $w \in L_2$ implies $u, y, z, x \in L_2$ and $x(t) \to 0$).*

Proof. $(G, \hat{K})$ is I-SS-dissipative by Theorem 8.7.2, so that

$$e(\hat{\xi}(t), p_t) + \tfrac{1}{2}\int_0^t [|z(s)|^2 + |v_1(s)|^2]\, ds$$

$$\leq e(\hat{\xi}_0, p_0) + \tfrac{1}{2}\int_0^t [|w(s)|^2 + |v_2(s)|^2]\, ds.$$

for all $w, v_2 \in L_{2,loc}$. Now L is I-IO-dissipative by hypothesis, so

$$\tfrac{1}{2}\int_0^t |v_2|^2 ds \le \tfrac{1}{2}\int_0^t |v_1|^2 ds$$

for all $v_1 \in L_{2,loc}$. Combining these two inequalities we see that (G, K) is dissipative. Now z-detectibility provides the weak internal stability, by Theorem 2.1.3. This means in particular that the signals u, y, z are in L_2 whenever $w \in L_2$, but we cannot conclude that v_1, v_2 are in L_2 without further hypothesis (they are in $L_{2,loc}$). □

Note that the stabilizing property of $\hat{\mathbf{s}}^*, p_0$ is not used here, nor is J-SS-losslessness. Still we are able to produce many solutions to the H_∞ control problem. This however, will not (at least for linear systems) be all solutions unless we invoke these constraints.

8.7.2 Parameterizing All Solutions

In linear H_∞ control Theorem 8.7.3 parameterizes all solutions to the H_∞ control problem with norm less than one provided that both Σ^O and $(\Sigma^O)^{-1}$ are L_2 stable. For nonlinear systems clearly Theorem 8.7.3 parameterizes many solutions, but whether or not it parameterizes all is an open question.

8.8 Necessity of the RECIPE

The results given in §8.5 we have obtained so far for the general factorization problem specified in §8.2 have been of the sufficiency type, in that the factors were constructed from a solution $\hat{W}$ of the factoring PDE (8.13) and properties of the factors were obtained. In this section, as foreshadowed in §8.5.6, we consider the main problem of necessity, i.e., finding a solution $\hat{W}$ with the required critical feedback function $\hat{\mathbf{s}}^*$. To do this, we make use of the connections between control and factorization explored in §8.7. This will require a slight restriction in generality, but with the benefit of efficiency.

Theorem 8.8.1 *Suppose $\hat{D}$ has the block diagonal form:*

$$\hat{D} = \begin{pmatrix} D_{12} & 0 \\ 0 & I \end{pmatrix}. \tag{8.47}$$

Suppose that Σ has a decomposition into $\Sigma^I = \Sigma \circ \Sigma^O$ which is J-SS-dissipative and the the reversiblility Assumption 8.7.1 holds. Then by Theorem 8.7.3 a solution K to the H_∞ control problem exists, thus the value function W given by equation 4.4 in Chapter 4 exists. Suppose this W is smooth. Then the critical feedback function is given by

$$\hat{\mathbf{s}}^*(p) = \begin{pmatrix} \mathbf{u}^*(p) \\ \mathbf{y}^*(p) \end{pmatrix},$$

where the functions in the RHS are given by (4.23), and the RECIPE given in §8.4 produces a decomposition Σ^I and Σ^O with Σ^I which is J-SS-lossless, and indeed Assumption 8.5.2 holds.

Proof. Since we assume W is smooth our main necessity Theorem 4.3.1 tell us that W satisfies the dynamic programming principle. Under appropriate smoothness assumptions discussed in Chapter 4 W solves the dynamic programming PDE

$$\inf_u \sup_y \nabla_p W(p)[F(p,u,y)] = 0, \tag{4.16}$$

and gives formulas (4.23) for the optimizers $\mathbf{u}^*, \mathbf{y}^*$. The special form of $\hat{D}$ implies the quadratic part of F has decoupled u, y appearing in a way which makes

$$\inf_u \sup_y \nabla_p W(p)[F(p,u,y)] \;=\; \mathrm{crit}_s \nabla_p W(p)[\hat{F}(p,s)]$$

with $s = (u, y)$. Indeed $\hat{\mathtt{s}}^*(p) = \mathrm{crit}_s \nabla_p W(p)[\hat{F}(p,s)]$ is unique and equals

$$\operatorname*{argmin}_u \operatorname*{argmax}_y \nabla_p W(p)[F(p,u,y)].$$

Thus we produce $\hat{\mathtt{s}}^*$ and W satisfying the formulas in §8.5.1, and Assumption 8.5.2 holds. $\square$

The remaining Assumptions listed in §8.5.2 should be checked. This can be approached using the more complicated framework and results given in Chapter 4. A significant point is that our necessity theory (as presented in Chapter 4) does not imply Assumptions 8.5.4 and 8.5.5 which say that solutions p_t exists for the augmented system (8.17) and that the reversibility Assumption 8.7.1 holds. These are significant open questions.

8.9 State Reading Factors

A natural notion is that of factors being able to read the state rather than just a measurement. In this section we give formulas for a state reading J-lossless outer factorization. We associate with this the picture shown in Figure 8.6.

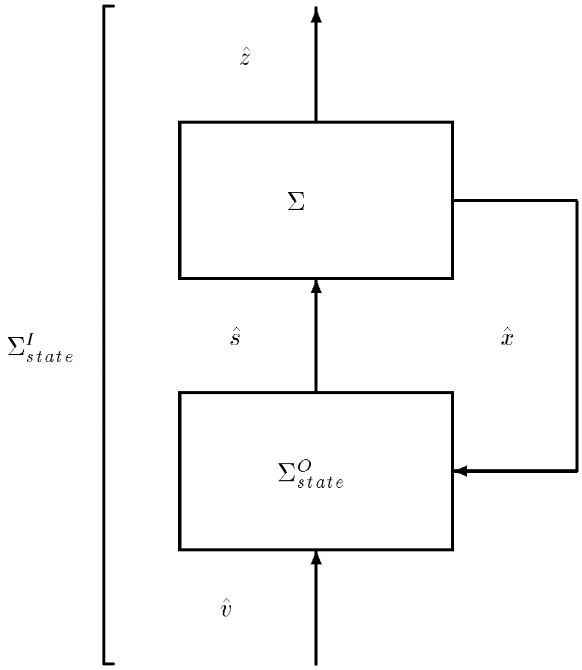


Figure 8.6: State reading inner factor Σ^I.

8.9.1 RECIPE for State Reading Factors

The following RECIPE for the state reading factors uses the critical state feedback function $\hat{\mathbf{s}}^*_{state}(x)$ defined by (8.24), which is defined in terms of a stabilizing solution $\hat{V}(\hat{x})$ to the state feedback factoring PDE (8.23).

(i) Construct the factor Σ^O_{state} by

$$\Sigma^O_{state} : \begin{cases} \dot{\bar{x}} \;=\; \hat{A}(\bar{x}) + \hat{B}(\bar{x})(\hat{\mathbf{s}}^*_{state}(\bar{x}) + \hat{d}\hat{v}) \\ \\ \hat{s} \;=\; \hat{\mathbf{s}}^*_{state}(\bar{x}) + \hat{d}\hat{v} \end{cases} \tag{8.55}$$

(ii) Take the inverse of Σ^O_{state} to give:

$$(\Sigma^O_{state})^{-1} : \begin{cases} \dot{\hat{x}} \;=\; \hat{A}(\hat{x}) + \hat{B}(\hat{x})\hat{s} \\ \\ \hat{v} \;=\; \hat{d}^{-1}(\hat{s} - \hat{\mathbf{s}}^*_{state}(\hat{x})) \end{cases} \tag{8.56}$$

(iii) The J-inner factor Σ^I_{state} is given by composing (8.1) and (8.55):

$$\Sigma^I_{state,\bar{\hat{x}}_0,\hat{\xi}_0} : \begin{cases} \dot{\bar{\hat{x}}} \;=\; \hat{A}(\bar{\hat{x}}) + \hat{B}(\bar{\hat{x}})(\hat{\mathtt{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}), \quad \bar{\hat{x}}(0) = \bar{\hat{x}}_0 \\ \dot{\hat{\xi}} \;=\; \hat{A}(\hat{\xi}) + \hat{B}(\hat{\xi})(\hat{\mathtt{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}), \quad \hat{\xi}(0) = \hat{\xi}_0 \\ \hat{z} \;=\; \hat{C}(\hat{\xi}) + \hat{D}(\hat{\xi})(\hat{\mathtt{s}}^*_{state}(\bar{\hat{x}}) + \hat{d}\hat{v}). \end{cases} \tag{8.57}$$

As mentioned above, the critical state feedback $\hat{\mathtt{s}}^*_{state}$ is given by the formula (8.24) which is exactly the same as

$$\hat{\mathtt{s}}^*_{state}(\hat{x}) = \hat{\mathtt{s}}^*(\delta_{\hat{x}})$$

with $\hat{\mathtt{s}}^*$ given by (8.15).

In the context of the state feedback control problem (recall Chapter 5), the state reading version $(G, \hat{K}_{state})$ of the corresponding re-reversed arrow system described in (8.49) is still of interest. Here the the system $\hat{K}_{state} : (v_2, y) \mapsto (u, v_1)$ is given by

$$\hat{K}_{state} : \begin{cases} \dot{x} \;=\; A^{\times}(x) + B_2(x)(\mathbf{u}^*_{state}(x) + \hat{d}_{11}v_2) + B_1(x)y \\ u \;=\; \mathbf{u}^*_{state}(x) + \hat{d}_{11}v_2 \\ v_1 \;=\; -\mathbf{y}^*(x) + y. \end{cases} \tag{8.58}$$

Of course, this can be related to the state feedback control problem, as in §8.7.

8.9.2 Properties of State Reading Factors

We now briefly present some basic results about the state reading factors.

Theorem 8.9.1 *The factor Σ^O_{state} is L_2 stable provided $\hat{V}$ is taken to be the stabilizing solution of the state feedback factoring PDE (8.23).*

Proof. If $\hat{V}$ is stabilizing, then the system $(\hat{A} + \hat{B}\hat{\mathtt{s}}^*_{state}, \hat{B})$ is incrementally exponentially L_2 stable, hence $\Sigma^O_{state}\hat{v} \in L_2$ whenever $\hat{v} \in L_2$. □

Remark 8.9.2 The inverse $(\Sigma^O)^{-1}_{state}$ is often not stable, since $\hat{A}$ is often not a stable vector field.

Theorem 8.9.3 *Σ^I_{state} is J-SS-lossless.*

Proof. Since $\hat{V}$ is a solution of the state feedback factoring PDE (8.23), we have

$$\nabla_{\hat{x}}\hat{V}(\hat{x}) \cdot (\hat{A}(\hat{x}) + \hat{B}(\hat{x})(\hat{\mathtt{s}}^*_{state}(\hat{x}) + \hat{d}\hat{v})) = -\tfrac{1}{2}|\hat{C} + \hat{D}(\hat{\mathtt{s}}^*_{state} + \hat{d}\hat{v})|^2_{J_{\mathbf{Z}}} + \tfrac{1}{2}|\hat{v}|^2_{J_{\hat{v}}}.$$

Integrating this gives the energy balance equation (8.21). □

8.9.3 Separation Principle

Now we observe a most elegant fact. When certainty equivalence holds, the factoring formulas in the RECIPE of §8.6.1 are actually the composition of a state reading factor and an information state estimator.

Note when the initial conditions for Σ^I_{state} satisfy $x_0 = \xi_0$, the ξ and x dynamics for Σ^I_S are the same. Thus the cutdown $\Sigma^I_{state|}$ given by

$$\Sigma^I_{state|} := \begin{cases} \dot{x} \ = \hat{A}(x) + \hat{B}(x)(\hat{\mathsf{s}}^*_{state}(x) + \hat{d}\hat{v}), \qquad x(0) = x_0 \\ \\ \hat{z} \ = \hat{C}(x) + \hat{D}(x)(\hat{\mathsf{s}}^*_{state}(x) + \hat{d}\hat{v}) \end{cases} \tag{8.59}$$

has the same input-output behavior as Σ^I_{state}. Our separation theorem uses $\Sigma_{state|}$ and says:

Theorem 8.9.4 *Under the the hypotheses of §8.6.1, The general J-SS-lossless system Σ^I given by the RECIPE of §8.6.1 can be expressed as the composition*

$$\Sigma^I_{x_0,p_0} = \Sigma^I_{state|} \circ \Sigma^O_{p_0}$$

where $\Sigma^O_{p_0}$ is the information state type of estimator given by the RECIPE in §8.4.

Proof. Read the formulas (in the RECIPE of §8.6.1). □

8.10 Non-square Factors and the Factoring PDE

In linear control the J-inner outer factorizations are used to give a formula for all solutions to the H_∞ control problem (recall §8.7.2). Thus they go beyond what is needed merely to find one solution. In nonlinear control one often pays a great price in terms of implementability for achieving more than is absolutely required. As we shall see obtaining one solution corresponds to factoring with Σ_0 which are non-square. So we discuss non-square factoring in this section.

8.10.1 Non-square Factoring PDE

First notice the construction in the RECIPE has some freedom in it, for example the choice of $\hat{d}$, and the RECIPE makes perfect sense if dimension $\mathbf{V}$ is less than or equal to that of $\mathbf{Z}$. This freedom is thus reflected in the choice of the space $\mathbf{V}$, and hence the signal $\hat{v}$. In fact, the main assumption we used in the RECIPE was

$$\hat{D}'\hat{d}'J_{\mathbf{Z}}\hat{d}\hat{D} = J_{\mathbf{V}}, \tag{8.9}$$

and in connection with the control problem we needed reversibility.

The main points we wish to make are that the reversibility (recall §8.7) does not use that $\hat{d}_{11}$ is invertible, and therefore Theorem 8.7.2 holds as is. Secondly, Theorem 8.5.7 on losslessness holds with modifications in the formula (8.15) for $\hat{\mathsf{s}}^*$ and the factoring PDE (8.16) for $\hat{W}$ in §8.5.1. Thirdly, we would like a derivation of these relations which applies to the directional derivative $\nabla_p^0 \hat{W}$ which need not be a linear functional (but is convex).

Theorem 8.10.1 *Consider the factor Σ^I determined by the RECIPE, and define*

$$\Lambda \stackrel{\triangle}{=} \text{ range } \hat{d} \ \subset \mathbf{S},$$

with Λ not in general equal to $\mathbf{S}$, but with $\hat{d}$ satisfying (8.9). Assume the factoring PDE (equations (8.16) and (8.63) below) can be interpreted in the sense of directional derivative, with $\nabla_p^0 \hat{W}(p)[\cdot]$ continuous. Then Σ^I is J-SS-lossless if and only if

$$\nabla_p^0 \hat{W}(p)[\hat{F}(p, \hat{\mathsf{s}}^*(p) + \hat{d}\hat{v}) = \tfrac{1}{2}|\hat{v}|^2_{J_{\mathbf{V}}} \quad \text{for all } \ \hat{v} \in \mathbf{V}. \tag{8.60}$$

Assume $\nabla_p^0 \hat{W}(p)$ is linear on the space of functions

$$\mathbf{S}_{\hat{F}P_\Lambda} = \text{ span}\,\{\frac{d\hat{F}}{d\hat{s}}(p, \hat{\mathsf{s}}^*(p))P_\Lambda]\hat{s} \ : \ \hat{s} \in \mathbf{S}\}. \tag{8.61}$$

Then identity (8.60) is equivalent to the relations

$$P_\Lambda \hat{E}\hat{\mathsf{s}}^* = -\nabla_p^0 \hat{W}(p)[P_\Lambda(\hat{D}' J_{\mathbf{Z}} \hat{C} - \hat{B}' \nabla_x p')], \tag{8.62}$$

and

$$\begin{aligned} &\nabla_p^0 \hat{W}(p)[-\nabla_x p\hat{A} + \tfrac{1}{2}\hat{C}' J_{\mathbf{Z}} \hat{C} + (\hat{C}' J_{\mathbf{Z}} \hat{D} - \nabla_x p\hat{B})(I - P_\Lambda)\hat{\mathsf{s}}^*(p)] \\ &\qquad + \hat{\mathsf{s}}^{*\prime}(p)\hat{E}(\tfrac{1}{2}I - P_\Lambda)\hat{\mathsf{s}}^*(p) = 0. \end{aligned} \tag{8.63}$$

Here, $P_\Lambda : \mathbf{S} \to \Lambda \subset \mathbf{S}$ denotes projection.

Of course, when we take $\hat{d}$ invertible, with $\mathbf{V} = \Lambda = \mathbf{S}$, we have $P_\Lambda = I$, and so (8.62) and (8.63) reduce to (8.15) and (8.16), respectively.

Linearity of $\nabla_p^0 \hat{W}(p)[\cdot]$ on the subspace Λ amounts to true differentiability of $\hat{W}$ in the directions along the subspace. The following corollary treats this linearity issue.

Corollary 8.10.2 *Under the assumptions of Theorem 8.10.1, the identity (8.60) is equivalent to*

$$\nabla_p^0 \hat{W}(p)[\hat{F}(p, \hat{\mathsf{s}}^*(p)) + \frac{d\hat{F}}{d\hat{s}}(p, \hat{\mathsf{s}}^*(p))\tilde{\hat{s}}] = 0 \quad \text{for all } \ \tilde{\hat{s}} \in \Lambda. \tag{8.64}$$

This identity is also equivalent to

$$\begin{aligned} &\nabla_p^0 \hat{W}(p)[\hat{F}(p, \hat{\mathsf{s}}^*(p))] = 0, \ \text{ and} \\ &\nabla_p^0 \hat{W}(p)[\frac{d\hat{F}}{d\hat{s}}(p, \hat{\mathsf{s}}^*(p))P_\Lambda] = 0. \end{aligned} \tag{8.65}$$

The proofs of these results are deferred to §8.10.3.

8.10.2 Reversing Arrows on One Port

The diagrams in Figure 8.7 concern descriptions of the H_∞ controller, in particular parameterizations using the signal $\hat{v}$. The standard problem of H_∞ control as in Figure 8.7 diagram (a) converts to diagrams (b) and (c) where signals flow in one direction. The operator $\tilde{K}$ in diagram (c) is a non-square outer factor.

This is just an instance of the basic principle which we now describe. Consider a system K with a single input y and output u. If we were studying this system in an experiment we would observe input-output pairs, and the set of all input-output pairs we would observe is the set

$$\{(y, u)\} = \{(y, Ky)\}.$$

Parameterizing this set in terms of y is just one possible construction. Quite possibly we could also parameterize this set as $\left\{ \left(A(\hat{v}), B(\hat{v}) \right) \right\}$, where A and B are suitably chosen nonlinear operators. This verbal discussion corresponds to Figure 8.8 where $\tilde{K}$ is the map $\tilde{K}(\hat{v}) \stackrel{\triangle}{=} (A(\hat{v}), B(\hat{v}))$.

Clearly, for given K many pairs A, B exist. The converse is true only under some assumptions on A to the effect that its range is onto.

This discussion tells us immediately that if we have the solution to the standard problem of H_∞ control in Figure 8.7 (a), then we also have a non-square factorization of $\tilde{G}$ as indicated in Figure 8.7 (c). This can be seen also in terms of Figure 8.7 (b) by setting v_2 to 0; we obtain $\tilde{K}v_1 = \tilde{\Sigma}^O(0, v_1)$. Conversely, if we have a factorization whose range meets certain conditions, then we have a solution to the control problem. The next example illustrates this.

Example 8.10.3 The famous central controller of H_∞ control (defined in Chapter 4) is an example. Recall that it is given by the top two lines of

$$\tilde{\tilde{K}} : \begin{cases} \dot{p} \;=\; F(p, \mathbf{u}^*(p), y), \qquad p(0) = p_0, \\ u \;=\; \mathbf{u}^*(p), \\ v_1 \;=\; -\mathbf{y}^*(p) + y. \end{cases} \tag{8.66}$$

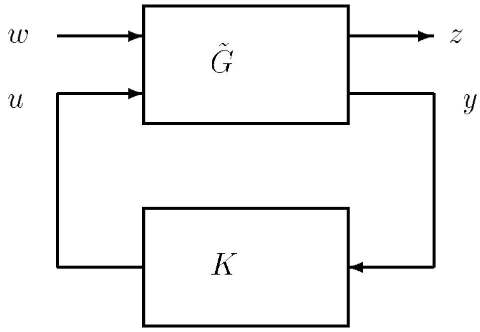
This is (8.49) with v_2 equals 0 and $\hat{d}_{11} = 0$. The system is reversible in that $v_1 = -\mathbf{y}^*(p) + y$ can be solved for y in terms of v_1. The appropriate $J_{\mathbf{V}}$ is $J_{\mathbf{V}} v_1 = -v_1$ which is negative definite. Now we write out formulas (8.62) and (8.63).

The ingredients of these formulas are

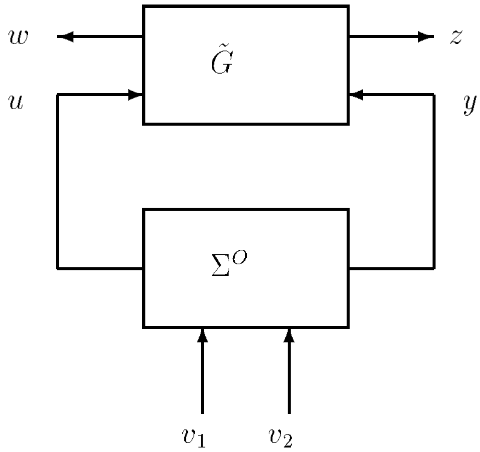
$$P_\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad P_\Lambda \hat{D} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad P_\Lambda \hat{E} = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and

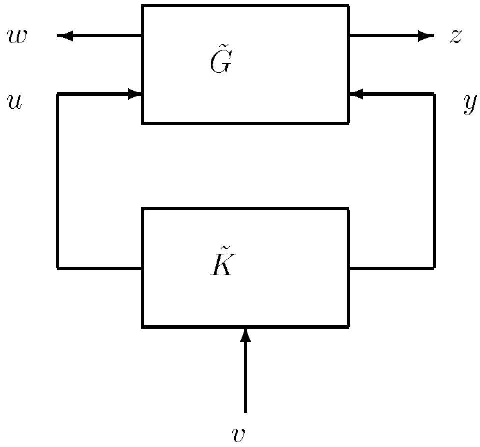
$$P_\Lambda(\hat{D}' J_{\mathbf{Z}} \hat{C} - \hat{B}' \nabla_x p') = \begin{pmatrix} 0 \\ C_2 - B_1' \nabla_x p' \end{pmatrix}.$$



(a) Closed-loop system (G, K) for H_∞ control.

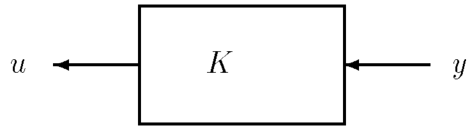


(b) Square inner factor Σ^I showing parameter $\hat{v} = (v_2, v_1)$ entering square Σ^0 (Figure 8.3 repeated).

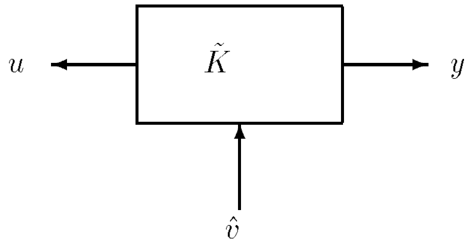


(c) Non-square inner factor Σ^I showing parameter $\hat{v} = v$ entering non-square Σ^0

Figure 8.7: Square and non-square H_∞ controller parameterizations.



(a) Operator K and input-output pairs $(y, u) = (y, Ky)$.



(b) Operator $\tilde{K}$ and parameter $\hat{v}$.

Figure 8.8: Operator parameterizations.

Plug this into (8.62) to get

$$\mathbf{y}^*(p) = \nabla_p^o \hat{W}(p)[C_2 - B_1' \nabla_x p'].$$

Turn to the factoring PDE (8.63). We calculate

$$\hat{C}' J_{\mathbf{Z}} \hat{C} = C_1' C_1 - C_2' C_2,$$

$$(\hat{C}' J_{\mathbf{Z}} \hat{D} - \nabla_x p \hat{B})(I - P_\Lambda)\hat{\mathbf{s}}^*(p) = (C_1' D_{12} - \nabla_x p B_2)\mathbf{u}^*(p),$$

and

$$\hat{\mathbf{s}}^{*\prime}(p)\hat{E}(\tfrac{1}{2}I - P_\Lambda)\hat{\mathbf{s}}^*(p) = \tfrac{1}{2}\mathbf{u}^{*\prime}(p)E_1\mathbf{u}^*(p) - \tfrac{1}{2}\mathbf{y}^{*\prime}(p)\mathbf{y}^*(p).$$

Plug these into (8.63) and get

$$\begin{aligned} &\nabla_p^0 \hat{W}(p)[-\nabla_x p A^\times + \tfrac{1}{2}(C_1' C_1 - C_2' C_2) + (C_1' D_{12} - \nabla_x p B_2)\mathbf{u}^*(p)] \\ &\quad + \tfrac{1}{2}(\mathbf{u}^{*\prime}(p)E_1\mathbf{u}^*(p) - \mathbf{y}^{*\prime}(p)\mathbf{y}^*(p)) = 0. \end{aligned} \tag{8.67}$$

Note that while $\mathbf{y}^*$ is uniquely determined, $\mathbf{u}^*$ is relatively undetermined. Note that any reversible controller of this form satisfying the previous equation with inequality ≤ 0 is a solution to the dissipative control problem. The standard choice (4.23) for $\mathbf{u}^*$ is obtained by doing the most natural thing, namely, minimizing the left side (Hamiltonian) of (8.67) in $\mathbf{u}^*$. Assuming that $\nabla_p^0 \hat{W}(p)[\cdot]$ is linear we get that the minimizer is $\mathbf{u}^* = \nabla_p^0 \hat{W}(p)[E_1^{-1}(-D_{12}' C_1 + B_2' \nabla_x p']$ which is indeed (4.23). Thus once $\hat{d}_{11} = 0$ has been specified our RECIPE determines $\mathbf{y}^*$ uniquely from $\mathbf{V}$, but there may be some freedom in the choice of $\mathbf{u}^*$. $\triangledown$

8.10.3 Proof of Factoring PDE

The core of the proof of Theorem 8.10.1 is the next lemma.

Lemma 8.10.4 *Let ν be a function $\nu : \mathcal{X}_e \to \mathbf{R}$ satisfying*

(i) ν *is convex,*

(ii) $\nu[p + c] = \nu[p] + c$ *if* $c \in \mathbf{R}$ *is constant,* $p \in \mathcal{X}_e$, *and*

(iii) $\nu[\lambda p] = \lambda \nu[p]$ *if* $\lambda \in \mathbf{R}$, $\lambda \geq 0$, *(positive homogeneous).*

(iv) ν *is continuous at zero (relative to the topology of* $\mathcal{X}_e$*).*

Let

$$f(\hat{s}) \stackrel{\triangle}{=} a + b\hat{s} + \tfrac{1}{2}\hat{s}' c \hat{s}$$

be a $\mathcal{X}_e$ *valued quadratic function of* $\hat{s} \in \mathbf{S}$, *where* c *is a constant selfadjoint invertible matrix. Let* $\Lambda \subset \mathbf{S}$ *be a subspace. Then a function* $\hat{\mathtt{s}}^*(p)$ *of* $p \in \mathcal{X}_e$ *has the property*

$$\nu[f(\hat{\mathtt{s}}^* + \tilde{\hat{s}})] = \tfrac{1}{2}\tilde{\hat{s}}' c \tilde{\hat{s}} \ \text{ for all } \ \tilde{\hat{s}} \in \Lambda \tag{8.68}$$

if and only if

$$\nu[f(\hat{\mathtt{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)\tilde{\hat{s}}] = 0 \quad \text{for all} \quad \tilde{\hat{s}} \in \Lambda. \tag{8.69}$$

Then identity holds if and only if

$$\begin{aligned} &\nu[f(\hat{\mathtt{s}}^*)] = 0, \ \text{ and} \\ &\nu[\tfrac{df}{d\hat{s}}(\hat{s}^*)P_\Lambda \hat{s}] = 0 \ \text{ for all } \ \hat{s} \in \mathbf{S}. \end{aligned} \tag{8.70}$$

Now assume that ν *is linear on the subspace of functions*

$$\mathbf{S}_{f,P_\Lambda} \stackrel{\triangle}{=} \text{span}\,\{\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda \hat{s} \ : \ \hat{s} \in \mathbf{S}\}.$$

Then (8.70) holds if and only if

$$\begin{aligned} &\nu[f(\hat{\mathtt{s}}^*)] = 0, \ \text{ and} \\ &\nu[\tfrac{df}{d\hat{s}}(\hat{s}^*)P_\Lambda] = 0, \end{aligned} \tag{8.71}$$

which holds if and only if $\hat{\mathtt{s}}^*$ *and* ν *satisfy*

$$P_\Lambda c \hat{\mathtt{s}}^* = -\nu[P_\Lambda b'], \tag{8.72}$$

and

$$\nu[a + b(1 - P_\Lambda)\hat{\mathtt{s}}^*] + \hat{\mathtt{s}}^{*\prime} c(\tfrac{1}{2}I - P_\Lambda)\hat{\mathtt{s}}^* = 0. \tag{8.73}$$

Since c *is invertible we may select* $\hat{\mathtt{s}}^*$ *in* Λ *satisfying (8.70) in which case (8.72) and (8.73) become*

$$\begin{aligned} &\hat{\mathtt{s}}^* = -(P_\Lambda c P_\Lambda)^{-1}\nu[P_\Lambda b'] \\ &\nu[a] - \tfrac{1}{2}\hat{\mathtt{s}}^{*\prime} c \hat{\mathtt{s}}^* = 0. \end{aligned} \tag{8.74}$$

Proof. We begin the proof with some basic facts. First,

$$f(\hat{\mathbf{s}}^*) = a + b\hat{\mathbf{s}}^* + \tfrac{1}{2}\hat{\mathbf{s}}^{*,\prime} c\hat{\mathbf{s}}^*, \quad \text{and} \quad \frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*) = b + \hat{\mathbf{s}}^{*\prime} c, \tag{8.75}$$

and secondly, by convexity and positive homogeneity,

$$\nu[\alpha + \beta] = 2\nu[\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta] \leq 2(\tfrac{1}{2}\nu[\alpha] + \tfrac{1}{2}\nu[\beta]) = \nu[\alpha] + \nu[\beta]. \tag{8.76}$$

This means that ν is sublinear.

Continuing with the proof, since $\tilde{\hat{s}}$ and c are constant, (8.68) is equivalent to

$$\nu[f(\hat{\mathbf{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] + \tfrac{1}{2}\tilde{\hat{s}}' c\tilde{\hat{s}} = \tfrac{1}{2}\tilde{\hat{s}}' c\tilde{\hat{s}} \quad \text{for all} \quad \tilde{\hat{s}} \in \Lambda,$$

and (8.68) is true if and only if (8.69) holds.

Now assume that (8.69) holds. Set $\tilde{\hat{s}} = 0$ in (8.69) to get $\nu[f(\hat{\mathbf{s}}^*)] = 0$. Fix $\tilde{\hat{s}} \in \Lambda$, and let $r > 0$ be a positive constant. Then $r\hat{\mathbf{s}} \in \Lambda$, so by (8.69) we have

$$\nu[f(\hat{\mathbf{s}}^*) + r\frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] = 0$$

which holds if and only if

$$r\nu[\frac{1}{r}f(\hat{\mathbf{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] = 0$$

which in turn is equivalent to

$$\nu[\frac{1}{r}f(\hat{\mathbf{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] = 0$$

since $r > 0$. Sending $r \to +\infty$, and making use of the assumed continuity, we get

$$\nu[\frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] = 0$$

for all $\tilde{\hat{s}} \in \Lambda$. Since the projection $P_\Lambda : \mathbf{S} \to \Lambda$ is onto, this is equivalent to

$$\nu[\frac{df}{d\hat{s}}(\hat{s}^*)P_\Lambda\hat{s}] = 0 \ \text{ for all } \ \hat{s} \in \mathbf{S}.$$

This proves that (8.69) implies (8.70).

For the converse, assume (8.70) holds. Let $\tilde{\hat{s}} \in \Lambda$. Then by (8.76), we have

$$\nu[f(\hat{\mathbf{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] \leq \nu[f(\hat{\mathbf{s}}^*)] + \nu[\frac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] = 0.$$

Also, using (8.70),

$$\begin{aligned}
0 &= \nu[f(\hat{\mathbf{s}}^*(p))] \\
&= \nu[(f(\hat{\mathbf{s}}^*(p)) + \tfrac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}) + \tfrac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)(-\tilde{\hat{s}})] \\
&\leq \nu[f(\hat{\mathbf{s}}^*(p)) + \tfrac{df}{d\hat{s}}(\hat{\mathbf{s}}^*)\tilde{\hat{s}}] + 0
\end{aligned}$$

since $-\hat{\mathtt{s}} \in \Lambda$. These inequalities prove that (8.70) implies (8.69).

Assume (8.70). By letting $\hat{s}$ equal basis vectors in $\mathbf{S}$, we see that all components of $\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda$ must satisfy

$$\nu[(\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda)_i] = 0.$$

This means that all components of the row vector $\nu[\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda]$ must be zero; but this is just the second line of (8.71).

Now assume that ν is linear on the stated subspace. Then the second line of (8.71) implies

$$\nu[\pm(\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda)_i] = 0$$

for each component $g_i = (\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)P_\Lambda)_i$ of $\frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)$. Then as shown at the end of the proof

$$\nu[g\hat{s}] = 0 \tag{8.77}$$

for all $\hat{s} \in \mathbf{S}$. This proves that (8.71) implies (8.70), under the linearity assumption.

We continue making the linearity assumption. Next, we wish to show that (8.71) implies (8.72) and (8.73). By the proven equivalences, set $\tilde{\hat{s}} = -P_\Lambda \hat{\mathtt{s}}^*$ in (8.69) to obtain

$$\nu[f(\hat{\mathtt{s}}^*) + \frac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)(-P_\Lambda \hat{\mathtt{s}}^*)] = 0, \tag{8.78}$$

and hence (8.73). Using (8.75) and the second line of (8.70) we get (8.72).

Next we assume (8.72), (8.73) and verify (8.71). Equation (8.72) is clearly equivalent to the second line of (8.71). Now

$$\begin{aligned}
&\nu[f(\hat{\mathtt{s}}^*)] = \nu[(f(\hat{\mathtt{s}}^*) + \tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)(-P_\Lambda \hat{\mathtt{s}}^*)) - \tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)(-P_\Lambda \hat{\mathtt{s}}^*)] \\
&\le \nu[f(\hat{\mathtt{s}}^*) + \tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)(-P_\Lambda \hat{\mathtt{s}}^*))] + \nu[\tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*)(P_\Lambda \hat{\mathtt{s}}^*)] \\
&= 0.
\end{aligned}$$

Now (8.73) is equivalent to (8.78), and so

$$\begin{aligned}
0 &= \nu[f(\hat{\mathtt{s}}^*(p) + \tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*(p))(-P_\Lambda \hat{\mathtt{s}}^*(p))] \\
&\le \nu[f(\hat{\mathtt{s}}^*(p))] + \nu[\tfrac{df}{d\hat{s}}(\hat{\mathtt{s}}^*(p))(-P_\Lambda \hat{\mathtt{s}}^*(p))] \\
&= \nu[f(\hat{\mathtt{s}}^*(p))].
\end{aligned}$$

Thus $\nu[f(\hat{\mathtt{s}}^*(p))] = 0$. This completes the verification of (8.71), assuming (8.72), (8.73).

The identities (8.74) now follow by direct substitution.

Proof of (8.77). The hypothesis is

$$\nu[\pm g_i] = 0$$

for all components of the vector g. We do not use the linearity assumption here. Let $\hat{s} \in$ S. Then

$$\begin{aligned} \nu[g\hat{s}] &= \nu[\textstyle\sum_i g_i \hat{s}_i] \\ &= \nu[\textstyle\sum_i g_i \,\mathrm{sign}\,(\hat{s}_i)|\hat{s}_i|] \\ &\leq \textstyle\sum_i \nu[g_i \,\mathrm{sign}\,(\hat{s}_i)]|\hat{s}_i| \\ &= 0, \end{aligned}$$

so $\nu[g\hat{s}] \leq 0$. Also,

$$0 = \nu[0] \leq \nu[a] + \nu[-a]$$

for $a = g\hat{s} = \sum_i g_i \,\mathrm{sign}\,(\hat{s}_i)|\hat{s}_i|$, and

$$\nu[-a] \leq \sum_i \nu[g_i\,(-1)\mathrm{sign}(\hat{s}_i)|\hat{s}_i| = 0,$$

so $\nu[g\hat{s}] \geq 0$. □

Proof of Theorem 8.10.1 and Corollary 8.10.2. To prove this note that energy conservation (8.21) is equivalent to (8.19), which is just the integrated form of (8.20), or (8.60).

The remainder of the theorem and the corollary are proven by applying Lemma 8.10.4, with

$$f(\hat{s}) = \hat{F}(p, \hat{s}), \quad \text{and} \quad \nu[\cdot] = \nabla_p^0 \hat{W}(p)[\cdot].$$

Here,

$$a = -\nabla_x p\hat{A} + \tfrac{1}{2}\hat{C}' J_{\mathbf{Z}} \hat{C}, \;\; b = -\nabla_x p\hat{B} + \hat{C}' J_{\mathbf{Z}} \hat{D}, \;\; \text{and} \;\; c = \hat{E} = \hat{D}' J_{\mathbf{Z}} \hat{D}.$$

□

8.11 Notes

(i) J inner-outer factoring was seen to be related to classical linear H_∞ control at an early stage. The connection came through Nevanlinna-Pick interpolation in that the original solution [ZF81], [ZF83] to the H_∞ control problem relied on such interpolation or generalizations of it. The paper [BH83] showed (in what would now be called a behavioral setting) how Nevannlina-Pick and commutant lifting problems were solved by J-inner-outer factorization. The first (see also Chang-Pearson [CJ84]) paper [FHZ84] to solve the MIMO H_∞ problem actually used such factoring as did the famous proof of [BR87]. Kimura and associates took factoring to great depth in an excellent series of publications, c.f. [Kim92], [Kim95], [BK96], [Kim97].

(ii) The first work on extending H_∞ control to nonlinear systems took the factoring approach we have described here. These layed out solutions of varying effectiveness to the discrete time factoring problem when $\hat{A}$ is stable, i.e. when the plant is stable and weights are carefully chosen: the paper [BFHT87b], [BFHT87a] used Volterra expansions, and [BH88c] did it behaviorally. Then came state space. The formulas of §8.6.2 were first described in discrete time in [BH88a], [BH89] with a thorough treatment in [BH92a]. See the book [vdS96] for an excellent treatment of the continuous time case by Ball and van der Schaft. Also there is the elegant work of Baramov and Kimura [BK96]. All of these approaches pertained to the $\hat{A}$ stable situation and used only the HJBI equation. See also [Bar98a], [Bar98b].

(iii) As we saw in this chapter to solve the factoring problem when $\hat{A}$ is not stable requires the information state equation. The approach taken here, the RECIPE and some of its J lossless properties, was announced in [HJ94], and the bilinear case was studied in [HJ96a].

Chapter 9

The Mixed Sensitivity Problem

9.1 Introduction

In the mixed sensitivity problem one is given a plant P and wishes to find a controller K achieving a given H_∞ performance, see Figure 9.1.

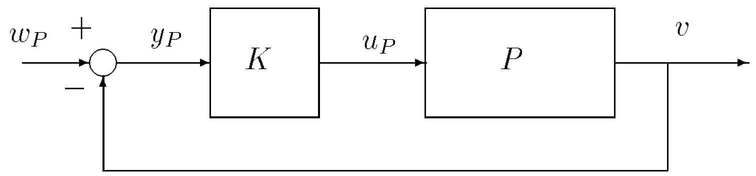


Figure 9.1: Mixed sensitivity setup

If the the weights are chosen correctly for a mixed sensitivity problem then one gets the standard 2A block problem of H_∞ control. We shall show under strong hypotheses on our setup:

> Result 9.1.1 *If the plant P is strongly hyperbolic and has a $k = k_{as}$ dimensional antistable manifold and if the weights are well behaved (defined below), then the key dynamics of the two block problem are strongly hyperbolic and has a k dimensional antistable manifold. This allows a construction for an information state controller requiring on line computation having complexity of the same order as numerical solution of a PDE driven (by inputs) on $k < n$ dimensional space.*

Now it often happens in practice that plants P are stable or at worst have a low dimensional instability. Thus the numerics of constructing the central controller solving a measurement feedback problem might not be much more complex than that of constructing a state feedback controller. We caution the reader that the off line computation in designing an *uncompromised* H_∞ controller would still be huge (by this we mean a full implementation without effort to reduce complexity).

9.2 Notation and Other Details

Now we apply the solution given in the previous sections to the mixed sensitivity problem. The G in standard problem of H_∞ control which we treat in this paper is typically not the plant in classical control problems, such as mixed sensitivity (see Figure 9.1). Rather, G is built from the plant P together with weights.

We treat the mixed sensitivity problem for a nonlinear plant P having state space representation

$$P: \begin{cases} \dot{x}_P = A_P(x_P) + B_P(x_P)u_P \\ \\ v = C_P(x_P). \end{cases} \tag{9.1}$$

The mixed sensitivity problem is to design the compensator K so that (i) the following dissipation inequality holds

$$\begin{aligned} &\tfrac{1}{2}\int_0^T [|\mathcal{W}_1(y_P)(s)|^2 + |\mathcal{W}_2(v)(s)|^2]\,ds \\ &\leq \gamma^2 \tfrac{1}{2}\int_0^T |w_P(s)|^2\,ds + \beta_P(x_P(0)), \end{aligned} \tag{9.2}$$

for all $w_P \in L_{2,T}$, $T \geq 0$, for some finite $\beta_P(x_P(0)) \geq 0$, and, (ii) the closed loop system is internally stable (see Figure 9.1). Here $\mathcal{W}_1$ and $\mathcal{W}_2$ are systems which weight the signals in the closed loop system. The relevant system of equations is: $u_P = y_P + P(u_P)$, $K(y_P) = u_P$.

To set the problem in the framework of the standard H_∞-problem for G as in (2.2), we should make the following identifications (Figure 9.1) $w = w_P$, $u = u_P$, $y = y_P$, $z = (z_1', z_2')'$, with $z_1 = \mathcal{W}_1(y_P)$, $z_2 = \mathcal{W}_2(v)$. To write down the state space equations for G, we need to specify the form of the state space realizations of the weights. A key issue is the proper choice of weights.

9.3 Choosing the Weights

In the linear case, physically correct choices of the frequency domain weights weights dictate that for complex numbers s we have

$$\mathcal{W}_1(s) \to 0 \text{ and } \mathcal{W}_2(s)P(s)K(s) \to \text{ constant } \neq 0$$

as $s \to \infty$, [HM98a]. Some perspective on this is given in §1.8, 1.9. Here we give another. To pick weights correctly, one must make an *a priori* decision on the rate at which the compensator $K(s)$ must go to 0 at ∞. For simplicity of exposition we make the choice that $K(s) \to c \neq 0$ as $s \to \infty$. One can select other asymptotics by extracting them from the K and placing them in P. Reasonable nonlinear analogues of this with a "proper" plant are given by the following.

We treat the nonlinear analog of the case where $P(s)$ has a first order zero at $s = \infty$ and we desire a compensator satisfying $K(s) \to c \neq 0$. Then $\mathcal{W}_2(s)$ must be asymptotic to s as $s \to \infty$. This motivates the state space realizations

$$\mathcal{W}_1 : \begin{cases} \dot{x}_1 &= A_{\mathcal{W}_1}(x_1) + B_{\mathcal{W}_1}(x_1)y \\ \\ z_1 &= C_{\mathcal{W}_1}(x_1), \end{cases} \tag{9.3}$$

$$\mathcal{W}_2 : \begin{cases} \dot{x}_2 &= A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)v \\ \\ z_2 &= C_{\mathcal{W}_2}(x_2) + D_{\mathcal{W}_2}(x_2)\frac{\partial v}{\partial t}. \end{cases} \tag{9.4}$$

Write $n_P = \dim x_P$, $n_1 = \dim x_1$, $n_2 = \dim x_2$, $n = n_P + n_1 + n_2$, and $x = (x_P, x_1, x_2)$. We say that the weights *well behaved* provided that $\mathcal{W}_1$ and

$$\dot{x}_2 = A^{\mathcal{W}_2}(x_2) + B^{\mathcal{W}_2}(x_2)v, \; z_2 = C^{\mathcal{W}_2}(x_2) + D^{\mathcal{W}_2}(x_2)v$$

are asymptotically stable in a very strong sense. (This is not the same as saying $\mathcal{W}_1$ and $\mathcal{W}_2$ stable since the $\frac{d}{dt}v$ destabilizes $\mathcal{W}_2$.)

9.4 Standard Form

Using such weights the extended system G (2.2) (of Chapter 2) with state variables $x = (x_P, x_1, x_2)'$ corresponding to (9.1), (9.3), (9.4) is derived from Figure 9.2.

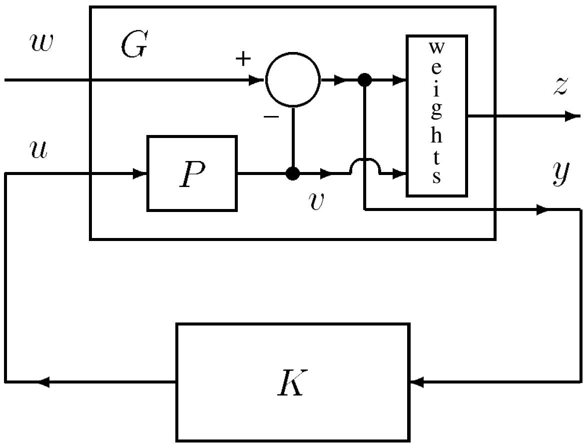


Figure 9.2: Mixed sensitivity embedded in the standard H_∞ problem arrangement (recall Figure 1.1).

The equations for G are:

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_P \\ \\ \dot{x}_2 \\ \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} A_P(x_P) + B_P(x_P)u \\ \\ (A_{\mathcal{W}_1}(x_1) \\ \quad +B_{\mathcal{W}_1}(x_1)(-C_P(x_P) + w)) \\ \\ A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)C_P(x_P) \end{pmatrix} \\
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} C_{\mathcal{W}_1}(x_1) \\ \\ (C_{\mathcal{W}_2}(x_2) + D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)) \\ \\ \quad .(A_P(x_P) + B_P(x_P)u)) \end{pmatrix} \\
y &= -C_P(x_P) + w.
\end{aligned}
\tag{9.5}
$$

Of course, $u = K(y)$. From (9.5) it is evident that

$$
A^{\times}(x_P, x_1, x_2) = \begin{pmatrix} A_P(x_P) \\ \\ A_{\mathcal{W}_1}(x_1) \\ \\ A_{\mathcal{W}_2}(x_2) + B_{\mathcal{W}_2}(x_2)C_P(x_P) \end{pmatrix}, \tag{9.6}
$$

and the "D" terms of G satisfy

$$
D_{11}(x) = 0, \;\; D_{21}(x) = I, \;\; D_{22}(x) = 0,
$$

$$
D_{12}(x) = (0, [D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)]B_P(x_P)]')'.
$$

Thus $D_{21}(x) = I_p$, and if

$$
D_{\mathcal{W}_2}(x_2)\nabla_{x_P}[C_P(x_P)]B_P(x_P) \;\; \text{is full rank,}
$$

then

$$
E_1(x) = D_{12}(x)'D_{12}(x) > 0
$$

is invertible, and indeed we have produced the *2A block problem*.

The stability of the weights insure that the instabilities of this system are in perfect correspondence with the instabilities of P.

Theorem 9.4.1 *Assume that the vector field A_P is incrementally hyperbolic,* $\dim M_{as}^P =$ *k_{as}, and that the vector fields $A_{\mathcal{W}_1}$, $A_{\mathcal{W}_2}$ are incrementally L_2 exponentially stable. Then the vector field $A^{\times}$ defined by (9.6) is incrementally hyperbolic with antistable manifold M_{as} of dimension k_{as}.*

Proof. By the definition of (global) hyperbolicity, we assume that after a coordinate transformation that $\mathbf{R}^{n_P} = M_a^P \times M_{as}^P$, and $x_P = (x_{Ps}, x_{Pas})$. Define

$$M_s = M_s^P \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2},$$

and

$$M_{as} = M_{as}^P \times \{0_{n_1}\} \times \{0_{n_2}\}.$$

The hypotheses imply that if $x(0) \in M_s$, then $x(t) \to 0$ as $t \to \infty$, where $x(\cdot)$ is the solution to the $A^\times$ dynamics. Similarly, if $x(0) \in M_{as}$, then $x(t) \to 0$ as $t \to -\infty$. By construction, $\dim M_{as} = k_{as} = \dim M_{as}^P$, as required. □

9.5 Formula for the Controller

To get an explicit formula for the controller one merely combines Corollary 10.2.2 and formulas (10.38) describing the controller for the two block problem with formulas (9.6) and (9.5) give the two block problem in terms of the mixed sensitivity problem. We reiterate that the dynamics of the controller has the numerical complexity of a PDE on a $k = \dim M_{as}^P << n$ dimensional space. Writing them out in $A_{\mathcal{W}}$ notation is not particularly informative, so we shall not. However, we will be somewhat more explicit in the special case that $A^\times$ is strongly stable. Indeed, see [BH92b]:

Corollary 9.5.1 *Assume*

(i) $B_1(x_P, x_1, x_2) = (0, B_{\mathcal{W}_1}(x_1)', 0)'$ *and* $B_2(x_P, x_1, x_2) = (B_P(x_P)', 0, 0)'$ *are independent of* x_P, x_1.

(ii) $A_P(x_P)$ *is incrementally* L_2 *exponentially stable.*

(iii) There exists a (smooth) solution $V(x_P, x_1, x_2)$ *solving the PDE (5.45) with*

$$A^*(x_P, x_1, x_2) = A - B_2C_1 + (\gamma^{-2}B_1B_1' - B_2B_2')\nabla_x V'$$

incrementally L_2 *exponentially stable.*

Then the controller $K_{\delta_0}^*$ *defined by (10.39) solves the mixed sensitivity control problem for* P *with internal stability. This controller is finite dimensional, or from the information state PDE viewpoint, it is given by a PDE on 0 dimensional space.*

9.6 Notes

(i) Result 9.1.1 was given for stable discrete time systems in [BH89], [BH92b]. Coupled with the discrete time version of singular controller [BH92b], this converts solution of the mixed sensitivity problem for stable plants to solving the usual HJBI difference analog of (5.45). Corollary 9.5.1 in this chapter gives these formulas in continuous time for the special class of systems we study.