Chapter 12

Time Varying Systems

In this book we treated time invariant systems, although most of the theory goes through for time varying ones. This was primarily to keep the notation from becoming one notch more involved. In this chapter we state results for a time varying plant G.

12.1 Dissipation and the Control Problem

We use notation which is so consistent with that for the time invariant case that there is no point in repeating all the definitions. For example, the plant coefficients are

$$A(t,x), B_1(x,t), B_2(x,t), C_1(x,t), C_2(x,t), D_{ij}(x,t).$$

The closed loop system is γ*-dissipative* provided there exist a gain $\gamma > 0$ and a bias $\beta(x,t)$ such that

$$\begin{cases} \frac{1}{2}\int_{t_1}^{t_2} |z(s)|^2\, ds \ \le\ \gamma^2 \frac{1}{2}\int_{t_1}^{t_2} |w(s)|^2\, ds + \beta(x(t_1), t_1) \\ \text{for all } \ w \in L_2[t_1, t_2] \text{ and all } t_2 \ge t_1, \end{cases} \tag{12.1}$$

where the trajectory of the time varying plant (2.2) is initialized at time t_1 at $x(t_1)$.

12.2 The Two Equations

Again there are two equations, the dynamic programming PDE or PDI and the information state equations.

12.2.1 The Information State

The information state equation remains the same except the coefficients now have t's in them; p_t is defined by

$$p_t(x) = \sup_{\tilde{w}(\cdot)} \{ p_0(\xi(0)) + \tfrac{1}{2} \int_0^t [|\tilde{z}(s)|^2 - \gamma^2 |\tilde{w}(s)|^2] \, ds \tag{12.2}$$

$$: \ \xi(t) = x, y(s) = C_2(\xi(s), s) + D_{21}(\xi(s), s)\tilde{w}(s), \ 0 \le s \le t \}$$

where $\xi(\cdot)$ satisfies

$$\dot{\xi}(s) \ = \ A(\xi(s), s) + B_1(\xi(s), s)\tilde{w}(s) + B_2(\xi(s), s)u, \tag{12.3}$$

which was the state equation in (2.2), and $\tilde{z}(s) = C_1(\xi(s), s) + D_{12}(\xi(s), s)u(s)$.

The dynamics are

$$\dot{p}_t = F(p_t, u(t), y(t), t), \tag{12.4}$$

where $F(p, u, y, t)$ is the nonlinear first-order partial differential operator

$$\begin{aligned} F(p, u, y, t) &= -\nabla_x p \cdot (A(\cdot, t) \\ &\quad + B_1(\cdot, t) D_{21}(\cdot, t)' E_2(\cdot, t)^{-1} (y - C_2(\cdot, t)) + B_2(\cdot, t) u) \\ &\quad + \tfrac{1}{2\gamma^2} \nabla_x p B_1(\cdot, t) [I - D_{21}(\cdot, t)' E_2(\cdot, t)^{-1} D_{21}(\cdot, t)] B_1(\cdot, t)' \nabla_x p' \\ &\quad + \tfrac{1}{2} |C_1(\cdot, t) + D_{12}(\cdot, t) u|^2 - \gamma^2 \tfrac{1}{2} (y - C_2(\cdot, t))' E_2(\cdot, t)^{-1} (y - C_2(\cdot, t)). \end{aligned} \tag{12.5}$$

12.2.2 Dynamic Programming Equation

The time varying dynamic programming PDE is no longer a steady state equation, but is time varying. The value function is defined by

$$W(p, t_1) = \inf_K \sup_{t_2 \ge t_1,\, y \in L_2[t_1, t_2]} \{ \langle p_{t_2} \rangle : p_{t_1} = p \}. \tag{12.6}$$

The corresponding dynamic programming PDE is

$$\frac{\partial}{\partial t} W(p, t) + \inf_{u \in \mathbf{R}^m} \sup_{y \in \mathbf{R}^p} \nabla_p W(p, t)[F(p, u, y, t)] = 0, \tag{12.7}$$

and PDI

$$\frac{\partial}{\partial t} W(p, t) + \inf_{u \in \mathbf{R}^m} \sup_{y \in \mathbf{R}^p} \nabla_p W(p, t)[F(p, u, y, t)] \le 0. \tag{12.8}$$

Let $\mathbf{u}^*(p, t), \mathbf{y}^*(p, t)$ denote optimizers here:

$$\mathbf{u}^*(p, t) = \nabla_p W(p, t)[E_1^{-1}(\cdot, t)(-D_{12}(\cdot, t)' C_1(\cdot, t) + B_2(\cdot, t)' \nabla_x p')]$$

$$\mathbf{y}^*(p, t) = \nabla_p W(p, t)[C_2(\cdot, t) - \gamma^{-2} D_{21}(\cdot, t) B_1(\cdot, t)' \nabla_x p'].$$

The state feedback PDI reads

$$\frac{\partial}{\partial t}V(x,t) + \nabla_x V(x,t)\left(A(x,t) - B_2(x,t)E_1(x,t)^{-1}D_{12}(x,t)C_1(x,t)\right)$$
$$+\tfrac{1}{2}\nabla_x V(x,t)\left(\gamma^{-2}B_1(x,t)B_1(x,t)' - B_2(x,t)E_1(x,t)^{-1}B_2(x,t)'\right)\nabla_x V(x,t)'$$
$$+\tfrac{1}{2}C_1(x,t)'(I - D_{12}(x,t)E_1(x,t)^{-1}D_{12}(x,t)')C_1(x,t) \le 0. \tag{12.9}$$

The optimal state feedback control is

$$\mathbf{u}^*_{state}(x,t) = -E_1(x,t)^{-1}(D_{12}(x,t)'C_1(x,t) + B_2(x,t)'\nabla_x V(x,t)'). \tag{12.10}$$

12.3 The Controller Construction

The information state controller K^*_{p,t_1} initialized at time t_1 at state p is defined by

$$u(t) = K^*_{p,t_1}[y](t) = \mathbf{u}^*(p_t[y],t) \tag{12.11}$$

for $t \ge t_1$, given a solution $W(p,t)$ of (12.8) and have found the corresponding optimizer $\mathbf{u}^*(p,t)$.

12.3.1 Properties of the Controller

The structural properties are again important:

(i) Domination: $W(p,t) \ge \langle p \rangle$ for all $p \in \mathcal{X}_e$, and in particular, for all $p \in \mathrm{dom}W(\cdot,t)$.

(ii) Monotonicity: if p_1 is in $\mathrm{dom}W(\cdot,t)$, and if $p_1 \ge p_2$, then p_2 is in $\mathrm{dom}W(\cdot,t)$ and
$$W(p_1,t) \ge W(p_2,t).$$

(iii) Additive homogeneity: for any constant $c \in \mathbf{R}$, we have
$$W(p \pm c,t) = W(p,t) \pm c. \tag{12.12}$$

Conditions guaranteeing that the construction solves the H_∞ control problem are given in the next theorem.

Theorem 12.3.1 *Assume that a p-smooth function W satisfies the structural conditions (domination, monotonicity, and additive homogeneity) and solves the PDI (12.8). If the information state controller K^*_{p,t_1} constructed from W exists for all $t \ge t_1$, then the closed-loop system (G, K^*_{p,t_1}) is dissipative, i.e., (12.15) holds (with $\beta(x,t) = -p(x) + W(p,t)$).*

Proof. This follows very closely the lines of proofs in Chapter 4. □

The necessity of the conditions in Theorem 12.3.1 is a theory similar to those of Theorem 4.3.1, with similar proofs.

Theorem 12.3.2 *Assume that the γ-dissipative control problem for a time varying plant G is solved by an admissible controller K_0. Then the value function $W(p, t)$ defined on dom$W(\cdot, t) \subset \mathcal{X}_e$ by (12.6) enjoys the following properties:*

(i) $\mathrm{dom} W(\cdot, t)$ *is non-empty, and so* $W(\cdot, t)$ *is finite on a non-empty set* $\mathrm{dom} W(\cdot, t)$.

(ii) W *satisfies the structural properties.*

(iii) $W(-\beta_{K_0}(\cdot, t), t) = 0$.

(iv) The dynamic programming principle holds: for any $t_2 \geq t_1$, $p \in \mathrm{dom} W(\cdot, t_1)$,

$$W(p, t_1) = \inf_{K} \sup_{y \in L_2[t_1, t_2]} \{W(p_{t_2}, t_2) : p_{t_1} = p\}. \tag{12.13}$$

(Identity (12.13) is called the dynamic programming equation.*)*

(v) If for $p \in \mathrm{dom} W(\cdot, t_1)$ *the controller* $\tilde{K}^*_{p,t_1}$ *exists, then* $W(p_t, t)$ *decreases along any optimal closed loop trajectory determined by* $w \in L_{2,loc}$, $x_{t_1} \in \mathbf{R}^n$*:*

$$W(p_{t_1}, t_1) \geq W(p_{t_2}, t_2), \tag{12.14}$$

where $u(t) = \tilde{K}^*_p[y](t)$, $p_{t_1} = p$, *and* $p_{t_2} \in \mathrm{dom} W(\cdot, t_2)$ *for all* $t_2 \geq t_1$.

Moreover, the closed-loop system $(G, \tilde{K}^*_{p,t_1})$ *is dissipative for all* $x_0 \in \mathbf{R}^n$*, i.e.,*

$$\begin{cases} \frac{1}{2} \displaystyle\int_{t_1}^{t_2} |z(s)|^2 \, ds \ \leq \ \gamma^2 \frac{1}{2} \displaystyle\int_{t_1}^{t_2} |w(s)|^2 \, ds - p(x_{t_1}, t_1) + W(p, t_1) \\ \text{for all } \ w \in L_2[t_1, t_2] \text{ and all } t_2 \geq t_1. \end{cases} \tag{12.15}$$

For perspective observe that these theorems can be proved with a modest portion of Chapter 4. There are two reasons for this simplicity. The most profound is that we have not needed to analyse the asymptotics of the information state equation or of the dynamic programming PDI. To complete the time invariant theory one must do this, so the time invariant theory in some respects is harder than the time varying theory. Also domains in which p_t lies, are swept into our assumptions here, while in Chapter 4 they added lots of complication. The second reason for complication is that we have assumed p_0 is everywhere finite and not treated the technically difficult case of singular p's.

Note also that stability can be infered from dissipation provided a suitable detectability condition is satisfied.

12.3.2 Certainty Equivalence

The certainty equivalence control is given by

$$u_{ce}(t) = \mathbf{u}^*_{state}(\bar{x}(t), t), \tag{12.16}$$

where $\mathbf{u}^*_{state}(x, t)$ is the optimal state feedback given by (12.10), and $\bar{x}(t)$ is the unique minimizer of $p_t(x) + V(x, t)$. The certainty equivalence assumptions and theorems of §7.3 generalize in the obvious way.

12.4 Equilibrium Information States

Equilibrium information states do not make sense for time varying systems absent some strong assumptions about the asymptotics of the plant coefficients A, B_1, etc. themselves. We have not pursed this though it is clearly a topic which could provide someone with endless fun.