On optimization with competing performance criteria

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1 Introduction

This paper analyzes a problem in which one optimizes objective functions over the space H_N^∞ of vector-valued functions $f = (f_1, \dots f_N)$ defined on the unit circle, $\partial\Delta$, where each coordinate function f_j extends to be analytic on the entire disk Δ. The objective we optimize is described in terms of nonnegative continuous functions Γ defined on $\partial\Delta \times \mathbf{C}^N$.

M-OPT Problem *Given positive functions* $\Gamma_j(e^{i\theta}, z) \in C^1(\mathbf{T}, \mathbf{C}^N)$, $j = 1, \dots, l$ *and a function* $f \in H_N^\infty$ *we define the* l *performances*

$$\gamma_j(f) := \sup_{\theta \in \mathbf{T}} \Gamma_j(e^{i\theta}, f) \quad j = 1, \dots, l$$

Definition *A function* $f^* \in H_N^\infty$ *is called a* Pareto optimum *if*

$$\text{for all} \quad f \in H_N^\infty \quad \exists j \quad \gamma_j(f) \geq \gamma_j(f^*)$$

For $l = 1$ this definition means that f^* minimizes $\sup_\theta \Gamma(e^{i\theta}, f(e^{i\theta}))$.

This type of problem is central to frequency domain system design problems where stability is a key constraint. In particular it is important to the area of H^∞-control [H] and [Fr]. The basic physical idea is simple. Recall that a linear time invariant system has a frequency response function F defined on the imaginary axis and that the system is stable if F has no poles in the closed right half plane (R.H.P.). Behavior of the system when excited with a pure sine wave of frequency ω is determined by $F(i\omega)$. The following often occurs in a design procedure. We are required to build a

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system S but part of the system is given (we are stuck with it), part of the system is designable (denote its frequency response function by f). The performance of the system S at frequency ω is a function $\Gamma(\omega, f(i\omega))$ which depends on ω and on ones choice of the designable subsystem f. Let us take the convention that large Γ is bad while small Γ is good. Then in a worst case "broadband" design we consider the worst performance over all frequencies

$$\sup_{\omega} \Gamma(\omega, f(i\omega))$$

and try to minimize it over all admissible f. If our main constraint is that the designable subsystem f must be stable, then the design problem becomes (OPT), with one Γ, after transforming the R.H.P. to the unit disk. When $N > 1$ these problems usually pose serious difficulties since traditional graphical trial and error methods are inadequate.

A number of authors, Mayne-Polak-Salucidean, Fan-Tits, Streit, Boyd, Daleh, Pearson, Doyle-Glover-Packard, Helton-Merino-Walker and Sideris, have written computer programs to search for an optimal f^* with certain kinds of Γ. See [HM2] for references.

In engineering applications a single performance function Γ can be used to incorporate several performance criteria. In particular, a Pareto optimum f^* for performance functions $\Gamma_1, \ldots, \Gamma_l$ can be viewed as a solution to the optimization problem with only one performance function defined by:

$$\widetilde{\Gamma}(e^{i\theta}, z) := max\left(\frac{\Gamma_1(e^{i\theta}, z)}{\gamma_1(f^*)}, \ldots, \frac{\Gamma_l(e^{i\theta}, z)}{\gamma_l(f^*)}\right) \tag{1}$$

Namely, it is easy to check that f^* is a Pareto optimum for $\Gamma_1, \ldots, \Gamma_l$ if it is an optimum for $\widetilde{\Gamma}$. The main disadvantage of introducing $\widetilde{\Gamma}$ is that it is almost never differentiable even though the Γ_j's are. As a consequence, the results proved for a single performance function optimization, reproduced in the theorem below, can not be applied to $\widetilde{\Gamma}$.

In this article we shall characterize functions f^* which are solutions to M-OPT in the following sense.

Definition *A function $f^* \in H_N^\infty$ is called a* local directional Pareto optimum *(LDP optimum) if, for each $h \in H_N^\infty$, there exists $t_h > 0$ such that*

$$\forall t \in (0, t_h)\ \exists j \quad \gamma_j(f^* + th) \geq \gamma(f^*)$$

For $l = 1$ it simply means that f^* minimizes $\sup_\theta \Gamma(e^{i\theta}, f(e^{i\theta}))$ in the directional sense.

For a single (rather than multiple) performance function earlier result which this paper extends is

Theorem 1.1 [H1] *Let Γ be of class C^3 and $f^* \in H^\infty_N \cap C(\mathbf{T})$ be such that $(\partial\Gamma/\partial z)(e^{i\theta}, f^*(e^{i\theta}))$ never equals 0 on $\mathbf{T}$. If f^* is a local directional optimizer, then*

(I) The function $\Gamma(\cdot, f^*(\cdot))$ is a constant on $\mathbf{T}$,

(II) There exist functions $F \in H^1_N$ and $\lambda : \mathbf{T} \to \mathbf{R}^+$ measurable and positive almost everywhere on $\mathbf{T}$ such that

$$\lambda(e^{i\theta})\frac{\partial\Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta})) = e^{i\theta}F(e^{i\theta}) \quad \text{a.e on } \mathbf{T}$$

When one adds a condition (III) asserting that Γ is convex in some directions , then one obtains a necessary and sufficient condition [HM1]. Indeed [HM1] is the best reference for this result. Theorem 1.1 is extremely useful in that conditions (I) and (II) are a basis for computer diagnostics and software developed by Helton-Merino-Walker, see [HMW].

In this paper we consider several Γ_j's. We show for a special class of Γ_j's ,namely the ones which are the norms of certain rational functions, that the LDP optimum satisfies conditions (I) and (II). Also we give a result which indicates that there is a large class of Γ_j's for which flatness will not hold.

An earlier instance of the flatness condition (I) for Pareto optima was discovered by Nicholas Young (see [PY] for proofs). It applies to jointly minimizing the first, second, third, etc. singular values of matrix valued functions which is quite a different context that the one in this paper.

This paper has the following structure. In section 2 we state and prove the main result of this paper. In section 3 we give two auxiliary results on uniqueness and existence of the Pareto optimum. In section 4 we reproduce the proofs of several lemmas which were proved by J.-M.Trepreau in preprint [T], and which were not published. Section 4 is completely independent of the rest of the paper. In section 5 we discuss the connection between the M-OPT problem which had its origin in engineering mathematics and the analytic disc techniques used in the popular several complex variables problem of extending a function defined on a manifold M in $\mathbf{C}^N$ to a function analytic in a neighborhood of a given point on M.

2 First order conditions

a. Main results

We introduce the notation

$$(\partial\Gamma_1, \ldots, \partial\Gamma_l) = \begin{pmatrix} \frac{\partial}{\partial z_1}\Gamma_1 & \ldots & \frac{\partial}{\partial z_N}\Gamma_1 \\ \ldots & \ldots & \ldots \\ \frac{\partial}{\partial z_1}\Gamma_l & \ldots & \frac{\partial}{\partial z_N}\Gamma_l \end{pmatrix} \tag{2}$$

Denote the unit disc in $\mathbf{C}$ by Δ and its boundary by $\partial\Delta$.

We will impose the following assumption on the performance functions:

Assumption 1. *Suppose $N \geq l$ and suppose $\Gamma_j = \left|\frac{P_j(e^{i\theta},z)}{Q_j(e^{i\theta},z)}\right|^2$ where P_j, Q_j are holomorphic polynomials in z with coefficients which are rational functions in $e^{i\theta}$. We assume, in addition, that the coefficients do not have poles on $\partial\Delta$.*

We will consider a candidate for Pareto optimum f^* for which the following assumption holds:

Assumption 2. The function $f^* \in H^\infty_N \cap C^\alpha, \alpha > 1/2$ with performances $\gamma_1^*, \ldots, \gamma_l^*$ satisfies the following condition.

There exists an analytic direction $h_j \in H^\infty_N \cap C^\alpha, \alpha > 1/2$ improving all performances except for γ_j^*. Namely, there exist $C > 0$ and $t_0 > 0$ such that for every $t < t_0$

$$\sup_\theta \Gamma_k(e^{i\theta}, f^*(e^{i\theta}) + th_j(e^{i\theta})) - \sup_\theta \Gamma_k(e^{i\theta}, f^*(e^{i\theta})) < -Ct \text{ for } j = 1, \ldots, l,\ j \neq k$$

Assumption (2) means that all l performances $\Gamma_1, \ldots \Gamma_l$ play an active role. If not all of them matter we have a smaller l.

Consider the following example for $l = 2, N = 2$:

$$\Gamma_1(e^{i\theta}, z) = |\psi_1(e^{i\theta}) - z_1|^2, \quad \Gamma_2(e^{i\theta}, z) = |\psi_2(e^{i\theta}) - z_2|^2$$

Where ψ_j are rational. In this case the problem is completely separable into two independent one-dimensional, single-performance problems. The assumption (1) is satisfied. At the same time (2) does not hold for functions $f = (f_1, f_2)$ such that either f_1 or f_2 is an optimum for Γ_1 alone or Γ_2 alone correspondingly. Therefore there exists no Pareto optimum for Γ_1, Γ_2 which satisfies the assumption (2).

If on the other hand we consider the problem with

$$\Gamma_1'(e^{i\theta}, z) = |\psi_1(e^{i\theta}) - z_1|^2, \quad \Gamma_2'(e^{i\theta}, z) = |\psi_2(e^{i\theta}) - z_1 - z_2|^2$$

then the problem can not be separated into two independent single-performance problems, and for generic ψ_1, ψ_2 almost all Pareto optima satisfy (2).

We also remark that the sublevel sets

$$\mathcal{S}_\theta^j(\gamma) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma\}$$

for j fixed and θ varies need not be "nice", for example they can be unbounded.

Now we state the main result of this paper.

Theorem 2.1 *Suppose the performance functions Γ_j satisfy assumption (1). Suppose that $f^* \in H_N^\infty \cap C^\alpha$ is a local directional optimizer which satisfies assumption (2). Suppose furthermore that $\frac{\partial \Gamma_j}{\partial z}(e^{i\theta}, f^*(e^{i\theta})) \in C^\alpha$ with $\alpha > 1/2$ and that*

$$\operatorname{rank}[\partial\Gamma_1(e^{i\theta}, f^*(e^{i\theta})), \ldots, \partial\Gamma_l(e^{i\theta}, f^*(e^{i\theta}))] = l \tag{3}$$

for every $e^{i\theta} \in \mathbf{T}$.

Then the following holds:

(I) Flatness:

$$\Gamma_j(e^{i\theta}, f^*(e^{i\theta})) = const \quad j = 1, \ldots, l$$

(II) Gradient alignment:

There exists a row vector valued function $\lambda \in C_l^\alpha(\mathbf{T}, \mathbf{R})$, $\lambda \not\equiv 0$, with non-negative entries, such that

$$\lambda(e^{i\theta}) \frac{\partial \Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta})) = e^{i\theta} F, \quad F \in H_N^2$$

Here $\frac{\partial \Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))$ denotes the $l \times N$ derivative matrix (2) evaluated at $z = f(e^{i\theta})$.

Remark We will, in fact, show that for (II) to hold, it is enough to assume only that f^* is an optimum such that $f^* \in C^\alpha$ with $\alpha > 1/2$ and the rank condition (3) holds. In other words, the assumptions (1) and (2) are not needed for (II).

b. Classical Riemann-Hilbert problem

The main step in the proof of the flatness condition is solving the following version of Riemann-Hilbert problem:

Given a $l \times N$ matrix valued function $A \in C^\alpha(\mathbf{T})$ with invertible values, given a closed interval $I \subset \mathbf{T}$, $I \neq \mathbf{T}$, find $h \in H_N^\infty$ such that

$$\begin{array}{c} (\mathrm{Re}(Ah))_1 > 0, \quad \text{for } e^{i\theta} \in I \\ (\mathrm{Re}(Ah))_j > 0, \text{ for all } e^{i\theta}, \ j = 2, \ldots, l \end{array} \tag{4}$$

Here $(\cdot)_j$ *stands for taking the* j*-th entry of a vector.* (later the matrix A will be taken to be equal to $\frac{\partial\Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))$).

The problem (4) is, in fact, a problem about the range of Riemann-Hilbert map

$$w \longrightarrow 2\mathrm{Re}\,(Aw)$$

Questions about the range of the Riemann-Hilbert map arise in several aspects. For example, the gradient alignment condition of Theorem 2.1 will be shown to be equivalent to the fact that the range of the derivative map

$$w \longrightarrow 2\mathrm{Re}\left(\frac{\partial\Gamma(e^{i\theta}, f^*(e^{i\theta}))}{\partial z} w\right)$$

does not contain strictly positive functions, i.e., that all performances can not be improved to first order at the same time.

Questions about the range of the Riemann-Hilbert map arise also in the analytic disc techniques in one theoretical complex variables problem, see section 6 for more details. See the book [Ve] as a standard reference on the theory of Riemann-Hilbert problem.

In this paper we give conditions on A, which insure that problem (4) always has a solution.

To state our main one we need the following definition.

Definition *A function* $u(e^{i\theta}) \in L^\infty(\mathbf{T}, \mathbf{C})$ *has a* pseudomeromorphic continuation *inside* Δ *if there exists a function* $\tilde{u}$*, meromorphic in* Δ *and with finitely many poles in* Δ*, such that*

$$\lim_{r\to 1} \tilde{u}(re^{i\theta}) = u(e^{i\theta}) \quad \text{a.e } \mathbf{T}$$

Theorem 2.2 *Suppose* $l \leq N$*. Suppose that an* $l \times N$ *matrix valued function* $A \in C^\alpha(\mathbf{T}, \mathbf{C})$*,* $\alpha > 1/2$*, takes values of rank* l *on* $\mathbf{T}$*. If*

(i) the entries of A *have pseudomeromorphic continuation inside* Δ

or more generally

(ii) the matrix A *can be written as* $D\tilde{A}$ *where* D *is an invertible diagonal matrix function and* $\tilde{A}$ *has a pseudomeromorphic continuation inside* Δ

then the problem (4) has a solution h*.*

3 Proofs

a. Proof of the flatness condition

First we observe that every Γ_j has a Taylor expansion

$$\Gamma_j(e^{i\theta}, z+tw) = \Gamma_j(e^{i\theta}, z) + t2\mathrm{Re}\left(\frac{\partial \Gamma_j}{\partial z}(e^{i\theta}, z)\cdot w\right) + O(t^2) \tag{5}$$

Suppose now that $f^* \in H_N^\infty \cap C^\alpha$ is a minimizer which produces performances $\gamma_1^*, \ldots, \gamma_l^*$ and that the performance function $\Gamma_1(e^{i\theta}, f^*(e^{i\theta}))$ is not constant. Then we can find $\epsilon_0 > 0$ and $I \subset \mathbf{T}$ so that $\mathbf{T}\setminus I$ is an open, non-empty interval and

$$\Gamma_1(e^{i\theta}, f^*(e^{i\theta}))|_{\mathbf{T}\setminus I} \leq \gamma_1^* - \epsilon_0 = \sup_\theta \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) - \epsilon_0 \tag{6}$$

We first want to find a vector valued function $w \in H_N^\infty \cap C^\alpha$ that satisfies

$$\begin{pmatrix} \frac{\partial \Gamma_1}{\partial z_1}(e^{i\theta}, f^*(e^{i\theta})) & \ldots & \frac{\partial \Gamma_1}{\partial z_N}(e^{i\theta}, f^*(e^{i\theta})) \\ \ldots & \ldots & \ldots \\ \frac{\partial \Gamma_l}{\partial z_1}(e^{i\theta}, f^*(e^{i\theta})) & \ldots & \frac{\partial \Gamma_l}{\partial z_N}(e^{i\theta}, f^*(e^{i\theta})) \end{pmatrix} \begin{pmatrix} w_1 \\ \ldots \\ w_N \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \\ \ldots \\ 0 \end{pmatrix} \tag{7}$$

where φ is an arbitrary function which never equals 0 on I.

Recall that $\Gamma_j(e^{i\theta}, z) = \frac{P_j}{Q_j}\overline{\left(\frac{P_j}{Q_j}\right)}$ where $P_j(e^{i\theta}, z), Q(e^{i\theta}, z)$ are holomorphic polynomials in z with rational coefficients depending on $e^{i\theta}$. The derivative is given by

$$\frac{\partial \Gamma_j}{\partial z_k}(e^{i\theta}, f^*) = \frac{\frac{\partial P_j}{\partial z_k}(e^{i\theta}, f^*)Q_j(e^{i\theta}, f^*) - P_j(e^{i\theta}, f^*)\frac{\partial Q_j}{\partial z_k}(e^{i\theta}, f^*)}{[Q_j(e^{i\theta}, f^*)]^2}\overline{P_j(e^{i\theta}, f^*)Q_j(e^{i\theta}, f^*)}^{-1}$$

We introduce notation

$$\Psi_j(e^{i\theta}) := \overline{P_j(e^{i\theta}, f^*(e^{i\theta}))}^{-1}\overline{Q_j(e^{i\theta}, f^*(e^{i\theta}))}[Q_j(e^{i\theta}, f^*(e^{i\theta}))]^2$$

Note that $P_j \neq 0$ because $\frac{\partial \Gamma}{\partial z}$ has maximal rank. Then $\frac{\partial \Gamma_j}{\partial z_k}(e^{i\theta}, f^*(e^{i\theta}))\Psi_j(e^{i\theta})$ can be extended meromorphically inside the unit disc. By the assumption (1), the meromorphic functions $\frac{\partial \Gamma_j}{\partial z_k}(z, f^*(z))\Psi_j(z)$ do not have poles on the boundary of the unit disc or accumulating to the boundary. Let $\beta(z)$ be the finite Blaschke product such that $\beta(z)\frac{\partial \Gamma_j}{\partial z_k}(z, f^*(z))\Psi_j(z)$ is holomorphic in the unit disc for $j = 1, \ldots, l$ and $k = 1, \ldots N$.

Multiply both sides of (7) on the left by the $l \times l$ diagonal matrix D, which has the diagonal entries $\Psi_1(e^{i\theta}), \ldots, \Psi_l(e^{i\theta})$. Note that each $\Psi_j(e^{i\theta})$ does not vanish anywhere on $\mathbf{T}$.

By the assumptions, the matrix $D\frac{\partial \Gamma}{\partial z}$ has rank l everywhere. Since it is also of class C^α with $\alpha > 1/2$, by Proposition 5.2, there exists the $N \times N$ constant matrix H such that the first l columns of the product $D\frac{\partial \Gamma}{\partial z}H$ are linearly independent everywhere on $\mathbf{T}$.

Denote by B the the first l columns of the holomorphic matrix $D\frac{\partial \Gamma}{\partial z}H\beta$. Let $\widetilde{B}$ be $l \times l$ holomorphic matrix valued function such that

$$B\widetilde{B} = (\det B)I_{l\times l}$$

Then the vector

$$\begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_N \end{pmatrix} = \beta H \begin{pmatrix} \widetilde{B} \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \tag{8}$$

satisfies (7) with $\varphi(e^{i\theta}) \neq 0, \quad \forall \theta$. Here $\begin{pmatrix} \widetilde{B} \\ 0 \end{pmatrix}$ is a $N \times l$ matrix with the last $N - l$ rows equal to zero.

Since φ is non-zero on I, its argument $\arg \varphi|_I$ is well defined. Extend $\varphi|_I$ to the whole $\mathbf{T}$ in such a way that the extension $\widetilde{\varphi}$ has winding number zero. Let h be a holomorphic function such that $\arg h = -\arg \widetilde{\varphi}$. Then $h\varphi|_I$ is real valued and positive and therefore the vector $-hw$ has the property that

$$2\mathrm{Re}\left(\frac{\partial}{\partial z}\Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \cdot [-hw]\right) < -\epsilon_1 \, , e^{i\theta} \in I$$

Choose ϵ_2 small enough so that

$$2\mathrm{Re}\left(\frac{\partial}{\partial z}\Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \cdot [-\epsilon_2 hw]\right) \leq \epsilon_0/2 \, , e^{i\theta} \in \mathbf{T} \setminus I$$

Then there exist positive constants C and t_0 such that for any $t \leq t_0$ the following holds:

$$\sup_\theta \left[\Gamma_1(e^{i\theta}, f^*(e^{i\theta})) - t2\mathrm{Re}\left(\frac{\partial}{\partial z}\Gamma_1(e^{i\theta}, f^*(e^{i\theta}))hw\right)\right] - \sup_\theta \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) < -Ct$$

$$2\mathrm{Re}(\frac{\partial}{\partial z}\Gamma_j(f^*)hw) = 0 \ j = 2, \ldots, l$$

In other words we have produced the analytic direction $-hw$ that "improves" the performance γ_1 and leaves the performances $\gamma_2, \ldots, \gamma_l$ "unchanged" up to the first order.

By assumption (2) , there exists a direction $v \in H^\infty_N \cap C^\alpha$ that "improves" the performances $\gamma_2, \ldots, \gamma_l$:

$$\sup_\theta \Gamma_j(e^{i\theta}, f^*(e^{i\theta}) + tv(e^{i\theta})) - \sup_\theta \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) < -C't, \quad j = 2, \ldots, l$$

Now consider $\tilde{w} = -hw + \varepsilon v$, for ε small enough. Then the analytic direction $\tilde{w}$ "improves" all Γ_j's. Therefore by (5), for small t the function $f^* + t\tilde{w}$ has better performances:

$$\gamma_j(f^* + t\tilde{w}) < \gamma_j(f^*), \quad j = 1, \ldots, l$$

We have reached a contradiction. ♠

Proof of Theorem 2.2 The proof is a line by line repetition of the part of the proof of condition (I). ♠

b.Counterexample Now we give some indications that we need Assumption (1) on Γ's for the flatness to hold. The flatness result, if it holds, would imply that if $\Gamma_2(e^{i\theta}, f(e^{i\theta})), \ldots, \Gamma_l(e^{i\theta}, f(e^{i\theta}))$ are constants and $\Gamma_1(e^{i\theta}, f(e^{i\theta}))$ satisfies (6), then we can find an analytic vector h such that

$$\begin{aligned} \mathrm{Re}\left(\tfrac{\partial \Gamma_1}{\partial z}(e^{i\theta}, f(e^{i\theta})) \cdot h\right) > 0, \quad \text{for } e^{i\theta} \in I \\ \mathrm{Re}\left(\tfrac{\partial \Gamma_j}{\partial z}(e^{i\theta}, f(e^{i\theta})) \cdot h\right) > 0, \text{ for all } e^{i\theta}, \ j = 2, \ldots, l \end{aligned}$$

Considering the question of existence of such h in a little more general setting leads to problem (4)

Proposition 3.2 below shows that there exists an A such that the problem (4) is not solvable. While we have not done so, we suspect that one can construct a simple A for which (4) is not solvable and which can be written as $\frac{\partial \Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))$ for some Γ_j's and f^*.

We start with the following lemma which gives a characterization of the derivative map (see [BRT],[T]).

Lemma 3.1 *Assume $N \geq l$ and suppose A is a $l \times N$ complex matrix valued function on $\mathbf{T}$, of class C^α with $1/2 < \alpha < 1$, and which has maximal rank ℓ at every point on*

$\mathbf{T}$. *Consider the following Riemann-Hilbert operator:*

$$\mathcal{F} : H^2_N \longrightarrow L^2_l(\mathbf{T}, \mathbf{R}) \quad , \quad \mathcal{F}(w) = 2Re(Aw).$$

Then $(\text{range}\mathcal{F})^\perp \subset L^2(\mathbf{T}, \mathbf{R})$ *consists of all such real valued* ℓ*-vectors* $g \in L^2$ *so that*

$$A^t g \in z H^2_N \tag{9}$$

Proof Suppose $g \in L^2$ belongs to $(\text{range}\mathcal{F})^\perp$. Then

$$\int g \cdot \text{Re}Aw = 0 \; \forall w \in H^2 \tag{10}$$

Since it is true for iw, we have

$$\int g \cdot Aw = 0 \; \forall w \in H^2 \tag{11}$$

Therefore $(\overline{A}^\top g, w) = 0$, where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbf{T}, \mathbf{C})$. Then (9) follows.♠

Remark If we denote the restriction of $\mathcal{F}$ to $H^2_N \cap C^\alpha$ by $\mathcal{F}^\alpha$, then the continuity of $\mathcal{F}$ implies that

$$(\text{range}\mathcal{F}^\alpha)^\perp = (\text{range}\mathcal{F})^\perp \tag{12}$$

The proposition below states that the problem (4) can not be solved for every A.

Proposition 3.2 *Suppose* $N = l = 2$. *Given any closed* $I \subset \mathbf{T}$ *with* $I \neq \mathbf{T}$, *there exists* A *so that for any* $v \in C^\alpha(I)$ *with* $\alpha > 1/2, v > 0$ on I *and any* $\psi \in C^\alpha(\mathbf{T})$ *with* $\psi > 0$, *there exists no solution* $h \in H^\infty_2 \cap C^\alpha$ *to the following Riemann-Hilbert problem:*

$$2Re(Ah) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \; , \; \varphi|_I = v$$

Proof of Proposition 3.2 Take $g_0 = (p(e^{i\theta}), 1)$ with $p = 0$ on $\overline{\mathbf{T} \backslash I}$ and $p > 0$ on $int(I)$. Then $(g_0, (\varphi, \psi))_{L^2} > 0$ for any $\psi > 0$ and any φ with $\varphi|_{int(I)} > 0$.

Now, we take

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ -p(e^{i\theta})e^{i\theta} & e^{i\theta} \end{pmatrix}$$

Obviously, $g_0 A \in zH^2$ and therefore g_0 belongs to $(\text{range}\mathcal{F})^\perp$.♠

c.Proof of condition (II) of Theorem 1

We first need the following theorem

Theorem 3.3 *Suppose a subspace $\mathcal{R} \subset C^\alpha \subset L^2_l(\mathbf{T},\mathbf{R})$ has the property that its complement $(\mathcal{R})^\perp$ is finite dimensional and a subset of C^α_l. Suppose also that its closure in the L^2-topology $\overline{\mathcal{R}}$ satisfies $\overline{\mathcal{R}} \cap C^\alpha = \mathcal{R}$. If $\mathcal{R}$ does not contain vector functions with every component strictly positive then there exists a function $\lambda \not\equiv 0$ in $(\mathcal{R})^\perp$ with every component positive (but not necessarily strictly positive).*

The proof requires the following

Lemma 3.4 *If the vectors $v_1, \dots, v_m$ satisfy*

$$\{x : \exists j, 1 \le j \le m \quad x \cdot v_j > 0\} = \mathbf{R}^n \backslash \{0\}$$

then the set

$$hull(v_1, \dots, v_m) := \{t_1 v_1 + \dots + t_m v_m : \ t_j \ge 0\}$$

is the whole of $\mathbf{R}^n$

Proof Abbreviate $hull(v_1, \dots, v_m)$ by V. Suppose that $V \neq \mathbf{R}^n$. Take any point x_0 which is not in V. Then there exists a plane $\{x : x \cdot \xi = r\}$ which separates x_0 and V, i.e., $x_0 \cdot \xi > r$ and $x \cdot \xi < r$ for every $x \in V$.

Since V is a cone, the latter implies that $x \cdot \xi \le 0$ for every $x \in V$. In particular it means that:

$$v_j \cdot \xi \le 0 \quad j = 1, \dots m$$

We have reached a contradiction.♠

Proof of Theorem 3.3 Suppose that the functions $g_1, \dots g_n$ form a basis of $(\mathcal{R})^\perp$. We want to prove that there exists a linear combination $a_1 g_1(\zeta) + \dots + a_n g_n(\zeta)$ in $(\mathcal{R})^\perp$ that is positive on all of $\mathbf{T}$. Here $a_j \in \mathbf{R}$. Denote the $n \times l$ matrix with rows $g_1(\zeta), \dots g_n(\zeta)$ by $g(\zeta)$. We claim it is enough to show the inequality

$$\inf_{a \in \mathbf{R}^n, ||a||=1} \max_{\zeta \in \mathbf{T}} a^t g(\zeta) \le 0 \tag{13}$$

Here we use the notation

$$\max_{\zeta \in \mathbf{T}} (b_1(\zeta), \dots, b_l(\zeta)) = max(\max_{\zeta \in \mathbf{T}} b_1(\zeta), \dots, \max_{\zeta \in \mathbf{T}} b_l(\zeta))$$

Note that we do not take the absolute values of functions.

The quantity $\max_{\zeta\in\mathbf{T}} a^t g(\zeta)$ depends continuously on a which varies over a compact set in $\mathbf{R}^n$. Therefore if (13) holds then there exists $a_0 \in \mathbf{R}^n$ such that

$$\max_{\zeta\in\mathbf{T}} a_0^t g(\zeta) \leq 0$$

Hence all components of vector function $-a_0^t g(\zeta)$ are positive and the claim follows.

Now we use a dual extension argument. This uses the fact that for any function $\mu \in C_l^\alpha(\mathbf{T})$

$$\max_{\zeta\in\mathbf{T}} \mu(\zeta) = \sup_{u\in B_l^+} \int \mu(\zeta)\cdot u(\zeta)d\zeta$$

where we use the notation $B_l^+ = \{u \in C_l^\alpha(\mathbf{T},\mathbf{R}) : \|u\|_{L^1} = 1, u_j(\zeta) > 0 \ j = 1,\ldots,l\}$. Therefore we need to prove the inequality

$$\inf_{a\in\mathbf{R}^n, \|a\|=1} \sup_{u\in B_l^+} \int (a^t g(\zeta))\cdot u(\zeta)d\zeta \leq 0 \tag{14}$$

Given any $u \in B_l^+$ we define the vector $c(u) \in \mathbf{R}^n$ as follows:

$$c(u) := \left(\int g_1 \cdot u \quad , \ldots , \int g_n \cdot u\right)$$

Lemma 3.5
The set $\{x : \exists u \in B_l^+ \quad x\cdot c(u) > 0\}$ is not the whole of $\mathbf{R}^n \setminus \{0\}$.

Proof Suppose the contrary : $\{x : \exists u \in B_l^+ \quad x\cdot c(u) > 0\} = \mathbf{R}^n \setminus \{0\}$. Since in fact we have an open covering of the unit sphere S^{n-1}, which is compact , there exist $u_1,\ldots,u_m$ such that $\{x : \exists j, 1 \leq j \leq m \quad x\cdot c(u_j) > 0\} = \mathbf{R}^n \setminus \{0\}$. Then, by Lemma 3.4, the convex hull $hull(c(u_1),\ldots,c(u_m)) = \mathbf{R}^n$.

Therefore there exists a positive, non-trivial linear combination of $c(u_j)$'s which is zero:

$$b \in \mathbf{R}^m \ , b \neq 0 \ , b_j \geq 0, \ \sum b_j c(u_j) = 0$$

We consider $\tilde{u}(\zeta) = \sum b_j u_j(\zeta)$. Then $\tilde{u}(\zeta) \in C^\alpha$ would be a vector-function with every component strictly positive . But on the other hand

$$\int g\tilde{u} = \int \sum b_j(gu_j) = \sum b_j c(u_j) = 0$$

which implies that $\tilde{u} \in \overline{\mathcal{R}} \cap C^\alpha = \mathcal{R}$, i.e., we have reached a contradiction.♠

Proof of Theorem 3.3(Continued) Take a point a_0 not in $\{x : \exists u \in B_l^+ \;\; x \cdot c(u) > 0\}$ with $\|a_0\| = 1$. Then

$$\int (a_0^t g(\zeta)) \cdot u(\zeta) d\zeta = a_0^t \int g(\zeta) u(\zeta) d\zeta = a_0^t c(u) \leq 0, \;\; \forall u \in B_l^+$$

and therefore the inequality (14) holds.♠

Proof of condition (II) (Continued) Since f^* is an optimum,there is no strictly positive function in the range of the derivative map

$$\mathcal{F}^\alpha : H_N^2 \cap C^\alpha \to C^\alpha, \;\; \mathcal{F}^\alpha(w)(\zeta) = 2\mathrm{Re}\left(\frac{\partial \Gamma(\zeta, f^*(\zeta))}{\partial z} w(\zeta)\right)$$

We observe that Lemma 5.1 and (12) imply that $(range\mathcal{F}^\alpha)^\perp$ is a finite dimensional subset of C^α. We claim that $\overline{range\mathcal{F}^\alpha} \cap C^\alpha = range\mathcal{F}^\alpha$. To show the claim, we first note that $range\mathcal{F}$, where $\mathcal{F}$ is defined as in Lemma 3.1, is closed in L^2. Therefore $\overline{range\mathcal{F}^\alpha} = range\mathcal{F}$. The C^α-hypoellipticity of $\mathcal{F}$ (see Lemma 5.1) implies that $range\mathcal{F} \cap C^\alpha = range\mathcal{F}^\alpha$ and the claim follows.

The argument above shows that we can apply Theorem 3.3 with $\mathcal{R} = range\mathcal{F}^\alpha$. Therefore, there is a vector valued function λ in $(range\mathcal{F})^\perp$ with each component positive. By Lemma 3.1, λ must satisfy (9), which implies (II). ♠

4 Additional results

In this section we state the results on uniqueness and existence of the M-OPT problem which are, in fact, easy corollaries of the results proved in [HMar] and [HM1].

a.Uniqueness

Consider the sublevel sets

$$\mathcal{S}_\theta^j(\gamma) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma\}$$

Theorem 4.1 *Suppose the performance functions $\Gamma_1, \ldots, \Gamma_l$ satisfy assumption (1). Suppose that the sublevel sets $\mathcal{S}_\theta^j(\gamma_j)$ are strictly convex for every $j = 1, \ldots, l$, every $\theta \in \mathbf{T}$ and every γ_j.*

Assume that real numbers $\gamma_1^*, \ldots, \gamma_l^*$ have the property that for every function f with performances $\gamma_1(f), \ldots, \gamma_l(f)$ satisfying $\gamma_j(f) \leq \gamma_j^*$ the assumption (2) holds and the matrix $\frac{\partial \Gamma(e^{i\theta}, f(e^{i\theta}))}{\partial z}$ has rank l.

Then if $f^* \in H^\infty_N \cap C^\alpha$ with $\alpha > 1/2$ is a Pareto optimum with performances

$$\sup_\theta \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) = \gamma_j(f^*)$$

then this Pareto optimum is unique , namely, there is no other function $f \in H^\infty_N \cap C^\alpha$, $\alpha > 1/2$, with the property $\gamma_j(f) = \gamma_j(f^*)$ for all $j = 1, \ldots, l$.

Proof Suppose that such an f does exist. Then

$$f(e^{i\theta}) \in \partial \mathcal{S}^1_\theta(\gamma^*_1) \cap \ldots \cap \partial \mathcal{S}^l_\theta(\gamma^*_l) \qquad \forall e^{i\theta} \in \mathbf{T} \tag{15}$$

by the flatness result. Here $\gamma^*_j = \gamma_j(f^*)$. Since (15) is true for f^* as well, the function $h = 1/2f + 1/2f^*$ has the property

$$h(e^{i\theta}) \in \mathcal{S}^1_\theta(\gamma^*_1) \cap \ldots \cap \mathcal{S}^l_\theta(\gamma^*_l) \qquad \forall e^{i\theta} \in \mathbf{T}$$

Strict convexity implies that there exists θ_0 satisfying

$$h(e^{i\theta_0}) \in \mathrm{int}\mathcal{S}^1_{\theta_0}(\gamma^*_1) \cap \ldots \cap \mathrm{int}\mathcal{S}^l_{\theta_0}(\gamma^*_l) \tag{16}$$

If the performance functions evaluated at h are flat (i.e., the flatness result (I) of Theorem 1 holds for h) then (16) should hold for every θ and therefore f^* is not an optimum. If the performance functions of h are not flat, then by Theorem 1 they all can be improved simultaneously, and therefore f^* is not an optimum in this case either. ♠

b.Existence

The following theorem was proved in [HMar]

Theorem 4.2 *Suppose that $\Gamma(e^{i\theta}, z)$ is a positive continuous function and suppose that the sublevel sets of Γ satisfy $\partial\mathcal{S}_\theta(\gamma) = \{z \in \mathbf{C}^N : \Gamma(e^{i\theta}, z) = \gamma\}$. Suppose furthermore that the sublevel sets $\mathcal{S}_\theta(\gamma)$ are uniformly bounded and polynomially convex. Suppose that $f^n \in H^\infty_N$ satisfy $lim_n \|\Gamma(e^{i\theta}, f^n(e^{i\theta}))\|_\infty = \gamma$. Let f be a normal limit of f^n. Then $\|\Gamma(e^{i\theta}, f(e^{i\theta}))\|_\infty \leq \gamma$*

As a corollary we can state the following existence result:

Theorem 4.3 *Suppose that $\Gamma_j, j = 1, \ldots, l$ are positive C^1-function. Suppose that the sublevel sets satisfy $\partial\mathcal{S}^j_\theta(\gamma) = \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) = \gamma\}$ for $j = 1, \ldots, l$ and for every γ and θ. Suppose furthermore that the sets $\mathcal{S}^1_\theta(\gamma) \cap \ldots \cap \mathcal{S}^l_\theta(\gamma)$ are uniformly bounded and polynomially convex for every γ.*

Then there exists a Pareto optimum for $\Gamma_1, \ldots, \Gamma_l$.

Remark Note that we do not make assumption (1) in this theorem. In particular, the flatness property of the optimum need not hold.

Proof First, we reduce the problem to the case of one performance function by introducing

$$\widetilde{\Gamma}(e^{i\theta}, z) := max\left(\Gamma_1(e^{i\theta}, z), \ldots, \Gamma_l(e^{i\theta}, z)\right)$$

We start with initial guess $f^1 \equiv 1$. We then make further guesses f^n's, improving $\widetilde{\Gamma}$ if possible. Since $\widetilde{\mathcal{S}}_\theta(\gamma)$'s are uniformly bounded, the set $\{f^n\}$ is uniformly bounded in H_N^∞ and therefore has a subsequence, converging locally uniformly in the open unit disc to some limit f. Applying Theorem 4.2, we conclude that f is an optimum for $\widetilde{\Gamma}$ with $\sup\limits_\theta \widetilde{\Gamma}(e^{i\theta}, f(e^{i\theta})) = \gamma$. ♠

5 Technical lemmas

In this section we reproduce the proofs of several lemmas that were proved by J.-M.Trepreau, and which were not published. They can all be found in the preprint [T].Lemma 5.1 is very close to the results proved in [BG].

Lemma 5.1 *Let the map*

$$\mathcal{F} : H_N^2 \to L_l^2(\mathbf{T}, \mathbf{R})$$

be defined by $\mathcal{F}(w) = 2\text{Re}(Aw)$*, where the* $l \times N$ *matrix* $A \in C^\alpha$ *has rank* l *everywhere on* $\mathbf{T}$*. Then* $(\text{range}\mathcal{F})^\perp$ *is, in fact, a subset of* $C^\alpha(\mathbf{T}, \mathbf{R})$

Proof First we prove the claim for the case $N = l$. Consider the real-linear map

$$\mathcal{G} : q \in L^2(\mathbf{T}, \mathbf{R}) \to q + i\mathcal{H}(q) = w \to Re(Aw) \in L^2(\mathbf{T}, \mathbf{R}) \tag{17}$$

Here $\mathcal{H}$ is the Hilbert transform. Then $\mathcal{G}$ is a pseudodifferential operator with the invertible symbol which equals $Re(A) - i \cdot sign\xi \cdot Im(A)$ and therefore it is elliptic. Then the C^α-hypoellipticity (see [Tay]) implies that the kernel of $\mathcal{G}$ is spanned by C^α-functions. The same is true for its transpose $\mathcal{G}^t$, and therefore $(\text{range}\mathcal{G})^\perp$ is spanned by C^α-functions as well.

For the case $N > l$, we multiply the $l \times N$ matrix A by the $N \times N$ holomorphic matrix valued function H from Proposition 5.2. Then we consider the operator $\mathcal{F}_l$ defined by the first l columns of AH. Thus

$$(\text{range}\mathcal{F})^\perp \subset (\text{range}\mathcal{F}_l)^\perp \subset C^\alpha$$

♠

Proposition 5.2 *Suppose $A \in C^\alpha(\mathbf{T})$, with $\alpha > 1/2$, is a $l \times N$-matrix valued function of maximum rank l at every point on $\mathbf{T}$. Then there exists a $N \times N$ constant complex matrix H, such that the first l columns of AH are linearly independent at every point on $\mathbf{T}$.*

To prove this proposition we will need the following lemma, in which the role of the assumption $\alpha > 1/2$ becomes clear.

Lemma 5.3 *Given $u(x) \in C^\alpha([0,1], \mathbf{C})$, with $\alpha > 1/2$, the range of u has zero Lebesgue measure in $\mathbf{C}$.*

Proof Divide the interval [0,1] into n equal parts by points j/n, $j = 0, \ldots, n$. Then the range of u can be covered by the n discs of radius $C(1/n)^\alpha$ centered at $u(j/n)$,$j = 0, 1, \ldots, n$, where C is the Holder constant of u. The total measure of these discs does not exceed

$$n\pi(C(1/n)^\alpha)^2 = C^2 \pi n^{1-2\alpha}$$

Since $\alpha > 1/2$, this quantity tends to zero as n approaches infinity. ♠

To prove Proposition 5.2 we begin by proving a special case.

Lemma 5.4 *Let $a \in C^\alpha(\mathbf{T}, \mathbf{C})$, $\alpha > 1/2$, be the N-vector such that $a \neq 0$ anywhere on $\mathbf{T}$. Then there exists $\beta \in \mathbf{C}^N$ such that $a(e^{i\theta}) \cdot \beta$ vanishes nowhere on $\mathbf{T}$.*

Proof
We will prove the statement for any open subset I of $\mathbf{T}$ by induction on N (we will assume that a is of class C^α *uniformly* on I). Since the case $N = 1$ is trivial, we assume that the lemma holds for $N - 1$. Let $Z \subset I$ be the zero set of a_N and $Z_0 \subset I$ be an open neighborhood of Z on which $(a_1, \ldots, a_{N-1}) \neq 0$. By the induction hypothesis, there exists $(\beta_1, \ldots, \beta_{N-1})$ such that $a_1\beta_1 + \ldots + a_{N-1}\beta_{N-1}$ does not vanish on Z_0. Also the function $(a_1\beta_1 + \ldots + a_{N-1}\beta_{N-1})/a_N \in C^\alpha(I \setminus Z)$ can not be onto $\mathbf{C}$ by the Lemma 5.5 below (note that σ/a_N is bounded on $I \setminus Z_0$).

Therefore, we can find a number $-\beta_N$ that is not in the range of σ/a_N, and so $a_1\beta_1 + \ldots + a_{N-1}\beta_{N-1} + a_N\beta_N$ vanishes nowhere on I.♠

Lemma 5.5 *Let $\sigma := a_1\beta_1 + \ldots + a_{N-1}\beta_{N-1}$. The range of the function $\sigma/a_N \in C^\alpha(Z_0 \setminus Z)$ is not the whole of $\mathbf{C}$.*

Proof Note that $Z_0 \setminus Z$ is a countable union of open intervals (t_j, t_{j+1}). We will show that the range of each of the restrictions $\sigma/a_N \in C^\alpha((t_j, t_{j+1}))$ is of measure zero in $\mathbf{C}$. Note that $\sigma \neq 0$ on Z_0 and a_N can vanish only at the end points t_j. If it vanishes at t_j, then it satisfies

$$|a_N(t) - 0| \leq c|t - t_j|^\alpha \tag{18}$$

which implies

$$|\sigma(t)/a_N(t)| \geq c'|t-t_j|^{-\alpha} \tag{19}$$

Divide $\mathbf{C}$ into the annuluses $L_k = \{z : k \leq |z| \leq k+1\}$.

Away from points where a_N vanishes, we use the fact that both σ and a_N are of class C^α uniformly on $Z \setminus Z_0$ to conclude that the restriction of the range of $(\sigma/a_N)\mid_{(t_j,t_{j+1})}$ onto L_k is a curve which is *uniformly* C^α. In the neighborhood of points were a_N vanishes, we use the estimate (19) to reach the same conclusion. Therefore the range of $(\sigma/a_N)\mid_{(t_j,t_{j+1})}$ has zero measure in $\mathbf{C}$ by Lemma 5.3.♠

Proof of Proposition 5.2 We write $A = (A_1, \ldots, A_N)$, where A_j's are the columns of A. For $p = 1, \ldots, l$ we denote the vector formed by the first p components of A_j by A_j^p. By induction on p , we will show that there exists an invertible constant matrix H such that the matrix $((AH)_1^p, \ldots, (AH)_p^p)$ has rank p at every point.

Since A has rank l at every point, $(A_1^1, \ldots, A_N^1)$ has rank 1 and by Lemma 5.4 there exists $< \beta_1, \ldots, \beta_N >$ such that $\beta_1 A_1^1 + \ldots + \beta_N A_N^1$ does not vanish. Let H_1 be an invertible matrix with the first column $< \beta_1, \ldots, \beta_N >^t$. Then $(AH_1)_1^1$ has rank 1 everywhere.

Now we assume that the claim is true for p. Replacing A by AH_p we can assume that $(A_1^p, \ldots, A_p^p)$ has rank p everywhere. This implies that, at every point on $\mathbf{T}$, one of the determinants

$$d_j = det(A_1^{p+1}, \ldots, A_p^{p+1}, A_j^{p+1}), \quad j = p+1, \ldots, N \tag{20}$$

is non zero.By Lemma 5.4, we can find $\beta_j, j = p+1, \ldots, N$ such that $d_{p+1}\beta_{p+1} + \ldots + d_N\beta_N$ is non zero on $\mathbf{T}$. Let $\widetilde{H}$ be a $(N-p) \times (N-p)$ invertible matrix whose first column is $\beta_{p+1}, \ldots, \beta_N$ and set

$$H_{p+1} = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & \widetilde{H} \end{pmatrix} \tag{21}$$

to obtain an N-dimensional matrix. Then the matrix whose columns are $(AH_{p+1})_1^{p+1}, \ldots, (AH_{p+1})_{p+1}^{p+1}$ has the determinant equal to

$$det(A_1^{p+1}, \ldots, A_p^{p+1}, \sum_{j=p+1}^{N} \beta_j A_j^{p+1}) = d_{p+1}\beta_{p+1} + \ldots + d_N\beta_N$$

and therefore it has rank $p+1$ everywhere. ♠

6 Connection with the analytic disc technique used in the problem of extending CR-functions

We discuss some connections between the M-OPT problem and the analytic disc techniques used for studying the theoretical several complex variables problem of extending a function defined on a manifold M in $\mathbf{C}^N$ to a function analytic in a neighborhood of a given point on M.

a. Attached analytic discs correspond to the flatness condition of M-OPT

First we give some definitions. A smooth manifold $M = \{z \in \mathbf{C}^N : \rho_1(z) = \ldots = \rho_l(z) = 1\}$ is called *generic* if its defining functions $\rho_1, \ldots, \rho_l$ satisfy $\partial\rho_1 \wedge \ldots \wedge \partial\rho_l \neq 0$, where $\partial\rho_j$ are defined as in (2). We say that a continuous function u on M is a *CR-function* if it is annihilated by all tangential antiholomorphic vector fields:

$$\sum_j a_j \frac{\partial}{\partial \overline{z}_j} u = 0, \quad \text{if} \quad \sum_j a_j \frac{\partial}{\partial \overline{z}_j} \rho_k = 0, \quad \forall k$$

We study the question of local extendibility of CR functions on M holomorphically to some neighborhood of a given point p on M. The basic theorem for the analytic disc method is the approximation theorem by Baouendi-Treves.

Theorem 6.1 (See [BT]) *Suppose M is a generic manifold, p is a point on M and u is a CR function on $M \cap \Omega$, where Ω is a neighborhood of p.Then there exists $\Omega' \subset \Omega$, a smaller neighborhood of p, such that u is a (uniform on $M \cap \Omega'$) limit of holomorphic polynomials.*

It follows from the maximum modulus theorem that if we can fill some neighborhood of p with the interiors of analytic discs f attached to M (i.e. the boundary $f(\partial\Delta) \subset M$), then the function u can be extended holomorphically to (a part of) that neighborhood.

Construction of analytic discs attached to a manifold was the main tool used by many authors to prove the extendability results, see [Tu1],[Tu2],[BRT], the book [Bog] and references in it.

The main result of this paper Theorem 2.1(I) on flatness, when stated geometrically, is an assertion about attached analytic discs. Namely, the flatness condition proved in [HM1] means that the optimal analytic disc f^* is attached to a loop of hypersurfaces $M_\theta = \{\Gamma(e^{i\theta}, z) = c\}$ in $\mathbf{C}^N$. The flatness condition proved in this paper says that f^* is attached to a loop of generic manifolds in $\mathbf{C}^N$, namely, $M_\theta = \{\Gamma_j(e^{i\theta}, z) = c_j, \ j = 1, \ldots, l\}$.

b. The notion of defect versus the gradient alignment condition

Since CR functions analytically continue to the set which is the union of 'small' analytic discs attached to M, it is clear that this set is important. With this in mind we let $\mathcal{R}_M$ denote the subset of $\mathbf{C}^N$ which is swept out by analytic discs attached to M. Beware that not every CR function on M extends to $\mathcal{R}_M$ since the 'small' disc requirement has been dropped. Basic information about it is provided by the notion of defect.

Definition *Given an analytic disc f attached to a manifold $M = \{z \in \mathbf{C}^N : \rho_1(z) = \ldots = \rho_l(z) = 0\}$ we define the defect* $\operatorname{def} f$ *of the disc f as the real dimension of the space*

$$\left\{ \lambda \in C_l^\alpha(\mathbf{T}, \mathbf{R}) : \lambda(e^{i\theta}) \frac{\partial \rho(f(e^{i\theta}))}{\partial z} \in H_N^2 \right\}$$

It can be shown (see e.g. [BRT]) that for any p on M there exists $\epsilon(p)$ such that if $\|f\|_{C^\alpha} < \epsilon(p)$ then the defect is an integer between 1 and l.

The notion of defect was introduced for small discs by Tumanov in [Tu1]. The definition we give was first introduced in [BRT]. The advantage of the latter is that it makes sense for any disc (not just small discs).

It was proved by Tumanov that if for any $\epsilon > 0$ there exists an analytic disc $f \in H_N^\infty \cap C^{1,\alpha}$ of defect zero and such that $\|f(e^{i\theta}) - p\|_{C^{1,\alpha}} < \epsilon$, then the interiors of small analytic discs attached to M sweep out a wedge with edge M near p, and therefore any CR-function on M extends to that wedge.

To a given disc f attached to M with $f(0) = q$ we associate a M-OPT problem

Find $h \in H_N^\infty$ which is a Pareto optimum for l performance functions $\Gamma_{q1}, \ldots, \Gamma_{ql}$ defined by

$$\Gamma_{qj}(e^{i\theta}, z) = \rho_j(e^{i\theta} z + q) \quad j = 1, \ldots, l \tag{22}$$

Note that

$$\frac{\partial \Gamma_q(e^{i\theta}, h(e^{i\theta}))}{\partial z} = e^{i\theta} \frac{\partial \rho(f(e^{i\theta}))}{\partial z}$$

Then the condition that f has defect zero is equivalent to

$$\lambda(e^{i\theta}) \frac{\partial \Gamma(e^{i\theta}, f(e^{i\theta}))}{\partial z} \notin e^{i\theta} H_N^2, \quad \forall \lambda \in C_l^\alpha(\mathbf{T}, \mathbf{R}) \tag{23}$$

which means that the condition (II) of Theorem 2.1 is violated.

In other words, if disc f has defect zero, then it can not be a Pareto optimum for $\Gamma_{q1}, \ldots, \Gamma_{ql}$ where $q = f(0)$.

We use this observation to prove the following proposition which shows that we can fill out an open set in $\mathbf{C}^N$ by discs attached to a hypersurface and close to a disc f of defect zero. Similar resuts of Tumanov require f to be small.

Proposition 6.2 *Suppose the disc $f \in H_N^\infty \cap C^\alpha$ is attached to a smooth hypersurface $M = \{z \in \mathbf{C}^N : \rho(z) = 1\}$. Suppose the defect of f is equal to zero. Then the set $\{g(0) \in H_N^\infty : g(\partial\Delta) \subset M\}$ contains a neighborhood of $f(0)$ in $\mathbf{C}^N$*

Note that we use H^∞ discs, rather than $H^\infty \cap C^\alpha$ discs, to fill a neighborhood in $\mathbf{C}^N$.

A similar result, for the special case when M is a generic manifold of real codimension N, can be found in [G]. The theorem in [G] is stated in terms of factorization indices of the matrix $\frac{\partial\rho(f)}{\partial z}$, but it can be restated as above.

Proof of Proposition 6.2 We can assume that $\rho(f(0)) > 1$. Consider the performance functions $\Gamma_{q1}, \ldots, \Gamma_{ql}$ given by (22). Then (23) implies that condition (II) of theorem 2.1 is violated and the quantity

$$\gamma(q) = \inf_{h \in H^\infty} \sup_\theta \Gamma_q(e^{i\theta}, h(e^{i\theta}))$$

is less than one.

We need the following lemma.

Lemma 6.3 *The function $q \to \gamma(q)$ is continuous on $\mathbf{C}^N$.*

Proof of Lemma 6.3 For any given ϵ, take $\delta > 0$ such that

$$\|h_1 - h_2\|_\infty \leq \delta \implies \left\|\rho(h_1(e^{i\theta})) - \rho(h_2(e^{i\theta}))\right\|_\infty \leq \epsilon$$

Then by the definition of $\gamma(q)$, the inequality $\|q - \tilde{q}\| < \delta$ implies $\gamma(\tilde{q}) \geq \gamma(q) - \epsilon$. At the same time it implies $\gamma(q) \geq \gamma(\tilde{q}) - \epsilon$. Therefore $\|\gamma(\tilde{q}) - \gamma(q)\| \leq \epsilon$. ♠

Note that the function $q \to f_q^*$ where f_q^* is the optimum for Γ_q need not be continuous.

Proof of Proposition 6.2 (continued) By Lemma 6.3 , there exists U, an open neighborhood of q in $\mathbf{C}^N$, such that $\gamma(\tilde{q}) < 1$ for all $\tilde{q} \in U$. Take a point $\tilde{q} \in U$. Let $g \in H_N^\infty$ be a solution to OPT problem with $\Gamma_{\tilde{q}}$. Then $\rho(e^{i\theta} g(e^{i\theta}) + \tilde{q}) < 1$ for every θ. Denote by Ω the connected component of the set

$$\{z \in \Delta : \rho(g(z)) > 1\}$$

containing zero. Construct Ω^*, the set containing Ω, according to the following procedure. Take a point $\xi \in \Delta \setminus \Omega$. If there exists a closed Jordan curve in Ω which encircles ξ, then we set $\xi \in \Omega^*$. If there is no such curve, we set $\xi \notin \Omega^*$. Then Ω^* is an open simply connected region such that $\partial\Omega^* \subset \partial\Omega$ and therefore $g|_{\partial\Omega^*} \equiv 0$.

Let $\Psi : \Omega \to \Delta$ be the Riemann map such that $\Psi(0) = 0$. Then $g \circ \Psi^{-1}$ is the analytic disc which is attached to M and which passes through $\tilde{q}$. ♠

c. Manifolds of analytic discs

Motivated by the successful application mentioned above, the set of analytic discs attached to a manifold was studied in general. It was shown in [BRT] that the set of analytic discs, infinitely close to a point, forms an infinite dimensional manifold. Namely, it was shown that the derivative map

$$\mathcal{F} : H_N^\infty \cap C^\alpha \to C^\alpha(\mathbf{T}, \mathbf{R}), \quad \mathcal{F}(w) = 2\mathrm{Re}\left(\frac{\partial\rho(f(e^{i\theta}))}{\partial z} w\right) \tag{24}$$

is onto $C^\alpha(\mathbf{T}, \mathbf{R})$ for f close to a constant disc. Then application of the local submersion theorem shows that we have a manifold.

In papers [F],[T],[O] the restriction on the size of the disc f was removed and it was shown that under certain conditions the map (24) is onto.

In the engineering set up, if the set of discs close to f^* forms a manifold then f^* is not an optimum. More precisely, suppose we are given disc f^* attached to a loop of manifolds $M_\theta = \{\Gamma(e^{i\theta}, z) = c\}$. Then if the map (24) is onto, then there exists a holomorphic direction h such that

$$2\mathrm{Re}\left(\frac{\partial\Gamma}{\partial z} h\right) < 0$$

and therefore $\gamma(f^* + th) < \gamma(f^*)$ for small t which means that f^* is not an optimum.

The following proposition shows how the defect is connected with the "manifold question" discussed above.

Proposition 6.4 (See [BRT],[T]) *Suppose the disc $f \in H_N^\infty \cap C^\alpha$ is attached to a smooth manifold $M = \{z \in \mathbf{C}^N : \rho_1(z) = \ldots = \rho_l(z) = 0\}$. Let E be the set of discs passing through point $f(0)$, i.e., $E = \{g \in H_N^\infty \cap C^\alpha : g(0) = f(0)\}$. If the defect of f is equal to zero, then the derivative map*

$$\mathcal{F}_0 : T_f E \to C^\alpha(\mathbf{T}, \mathbf{R}), \quad \mathcal{F}_0(h) = 2\mathrm{Re}\left(\frac{\partial\rho(f)}{\partial z} h\right)$$

is onto. Here T_fE denotes the tangent space to E at f, i.e., $T_fE = \{h \in H_N^\infty \cap C^\alpha : h(0) = 0\}$

The proof of Proposition 6.4 is a line by line repetition of the proof of Lemma 3.1. One can easily verify that Proposition 6.4 implies that the set of discs attached to M and passing through $f(0)$ forms a Banach manifold near f.

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