OPTIMIZATION OVER ANALYTIC FUNCTIONS WHOSE FOURIER COEFFICIENTS ARE CONSTRAINED[1]

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This article gives necessary and sufficient conditions for local solutions to several very general constrained optimization problems over spaces of analytic functions.

The results presented here have many applications, a particular instance of which is the sup-norm approximation of functions continuous on the unit circle in the complex plane by functions continuous on the circle and analytic on the open disk and whose Fourier coefficients satisfy prescribed linear relations.

Also, the results in this article generalize Nevalinna-Pick and Caratheodory-Fejer Interpolation results to allow values of arbitrary derivatives of functions to be assigned or merely bounded. Classically, NP and CF solve only problems with consecutive derivatives specified.

In engineering, constraints on the Fourier coefficients of a frequency response function correspond to constraints on its "time domain" behavior. Indeed the central problems of control theory involve both time and frequency domain constraints. That is precisely what the results in this paper handle.

Key words: H^∞ control with time domain constraints, constrained H^∞ optimization, constrained Nehari problem, analytic function optimization, analytic disks.

1 Introduction

Problem statement. For $N \geq 1$, let $\mathcal{C}_N$ be the space of C^N -valued continuous functions on the unit circle $\partial D \subset C$. Also, let $\mathcal{A}_N$ be the space of functions in $\mathcal{C}_N$ that are the boundary values of functions f analytic on D. The symbol Γ is reserved in this paper for functions $\Gamma : \partial D \times C^N \to R^+$, where R^+ denotes the positive real numbers. These functions will be referred to as "performance functions". The optimization problem considered here is,

Constr − OPT Given $\mathcal{E} \subset A_N$ and $\Gamma : \partial D \times C^N \to R^+$,
find $\gamma^* \in R^+$ such that

$$\gamma^* = \inf_{f \in \mathcal{E}} \sup_{\theta} \Gamma(e^{i\theta}, f(e^{i\theta})) ,$$

and find minimizers f^* in $\mathcal{E}$ if they exist.

The number γ^* is called the *optimal value* and it always exists. A minimizer function f^* is also referred to as a *solution* to Constr-OPT.

In this paper we give necessary and sufficient conditions for a local solution for the case when the set $\mathcal{E}$ is given by a set of equalities or inequalities on linear combinations of Fourier coefficients. More precisely, for C^N -valued functions $G_1, \ldots, G_n$ that are integrable on the unit circle, and for complex numbers $c_1, \ldots, c_n$, let

$$\mathcal{E} = \{f \in \mathcal{A}_N : \textstyle\int_0^{2\pi} f(e^{i\theta})^t \overline{G^{(\ell)}(e^{i\theta})} \frac{d\theta}{2\pi} = c_\ell, \ 1 \leq \ell \leq n\} \tag{1}$$

When $G^{(\ell)}$ is an analytic polynomial $G^{(\ell)}(e^{i\theta}) = \sum_{m=0}^{m_\ell} b_{m,\ell} e^{im\theta}$ for $\ell = 1, \ldots, n$, we have that the set $\mathcal{E}$ is described by equalities in terms of Fourier coefficients of f.

[1] Supported in part by the AFOSR and the NSF

The main result of this paper is a characterization of local solutions to Constr-OPT for functions Γ that satisfy very mild assumptions, when the set $\mathcal{E}$ of admissible functions is a subset of $\mathcal{RA}_N$ of functions in $\mathcal{A}_N$ whose Fourier coefficients are real numbers or vectors. That is, $\mathcal{E}$ has the form

$$\mathcal{E} = \{f \in \mathcal{RA}_N : \int_0^{2\pi} Re(f(e^{i\theta})^t \overline{G^{(\ell)}(e^{i\theta})}) \frac{d\theta}{2\pi} \geq c_\ell,\ 1 \leq \ell \leq n\} \ . \tag{2}$$

It is assumed that in (2) the Fourier coefficients of $G^{(\ell)}$ are real numbers or vectors.

Numerical algorithms and numerical analysis based on the results in this paper will be the subject of another paper.

When the set $\mathcal{E}$ is equal to $\mathcal{H}_N^\infty$ or $\mathcal{A}_N$ we obtain the unconstrained problem called OPT that has been studied in [HH], [HMar], [HMer1], [HMer2], [HMW]. The characterization of (local) solutions to OPT can be found in [HM1]. There is a rather complete theory of OPT which uses this as a starting point. It gives both qualitative properties of optima and numerical algorithms for computation [HMer1], [HMW].

The reader may find it useful to look at an example of the kind of problems that are encompassed by the general formulation Constr-OPT. Suppose g is a complex-valued continuous function on the unit circle. The distance of g to the set $\mathcal{A}_1$ of continuous functions on the closed disk and analytic on the unit disk is

$$\mathrm{dist}(g, \mathcal{A}_1) = \inf_{f \in \mathcal{A}_1} \sup_\theta |g(e^{i\theta}) - f(e^{i\theta})| \ , \tag{3}$$

and a function f^* for which $\mathrm{dist}(g, \mathcal{A}_1) = ||g - f||_\infty$ is called a best approximation to q from $\mathcal{A}_1$. This is the well known Nehari problem [Y], [FF], [G]. Note that if we set $\mathcal{E} = \mathcal{A}_1$ and

$$\Gamma(e^{i\theta}, z) = |g(e^{i\theta}) - z| \tag{4}$$

then (3) is a problem of the type Constr-OPT.

Now consider the distance from g to a set $\mathcal{E}$ of functions $f \in \mathcal{A}_1$ that satisfy certain relations, such as

$$\begin{array}{rcl} 3\hat{f}_1 - 2\hat{f}_4 + \hat{f}_7 & = & -2 \\ f(\frac{1}{2}) - 3f(-\frac{3}{4}) & = & 1 \\ f''(-\frac{1}{3}) & = & -1 \\ \int_0^{2\pi} \frac{f(e^{i\theta})}{(4e^{i\theta}-1)^2} \frac{d\theta}{2\pi} & = & 0 \\ f(\frac{1}{4}) & = & \frac{1}{8} \\ f'(\frac{1}{4}) & = & \frac{-1}{10} \end{array} \tag{5}$$

It is not difficult to see that all of the constraints in (5) can be written in integral form, that is, $\mathcal{E}$ has the form (1). For example,

$$f(\frac{1}{2}) - 3f(-\frac{3}{4}) = 1$$

can be rewritten via Cauchy's integral formula as

$$\int_0^{2\pi} f(e^{i\theta}) \overline{\left(\frac{1}{(1 - \frac{1}{2}e^{i\theta})} - \frac{3}{(1 - \frac{3}{4}e^{i\theta})} \right)} \frac{d\theta}{2\pi} = 1 \ .$$

The two bottom equalities in (5) are precisely the type of interpolation constraints that appear in Nevalinna-Pick interpolation problems. It should be clear that the remaining equalities are not of the same kind.

One of the results in this paper implies that, provided the set $\mathcal{E}$ is non-empty, there is a unique best approximation to the function g from $\mathcal{E}$, and another result gives optimality conditions that the best approximation has to satisfy.

Engineering Applications. The left-hand-side of each equation in (5) is, for general $f \in \mathcal{A}_1$, complex valued. If the function f has real Fourier coefficients, that is, if f is real on the real axis, then terms such as $\hat{f}_4$, $f''(-1/3)$ are necessarily real numbers, and in this case it makes perfect sense to consider relations similar to (5) but now with inequalities replacing some of the equality relations. This situation arises in engineering, where f may represent the frequency response function associated with a linear time invariant stable system a designer wants to find, and where the (time domain) specifications are given as affine inequalities in terms of the Fourier coefficients of f. Frequency domain specifications are also given and they yield a performance (cost) function. For the sake of this explanation suppose that the performance has the form (4). The worst case performance for a given f is the number

$$\sup_{\theta} |g(e^{i\theta}) - f(e^{i\theta})| \tag{6}$$

In this setting, the best design possible corresponds to the function f that satisfies the time domain specifications and that minimizes (6). That is, we obtained a problem of the type Constr-OPT. Again, results in this paper concern this problem and guarantee existence and uniqueness of solutions when the performance is of the form (4), and also give general conditions that have to be met by solutions, even when the performance function is more general than (4).

The main result in this article applies to the area of engineering called H^∞-control, where the main thrust of work is numerical. Several approaches have been developed to solve special cases of Constr-OPT numerically. Work of Rotstein and Sideris [RS] derives state-space algorithms for solving classess of inequality constraints on Fourier coefficients. The traditional Caratheodory-Fejer problem specifies the first k Fourier coefficients of an analytic function while minimizing its sup norm. The [RS] scheme for solving inequality constraints on the f's is to fix a guess at f, solve the CF problem by well known means to get a formula in terms of the $f's$. Then iterate on the f's. Other work using convex programming such as simplex methods, branch and bound, or interior methods is done in [Str], [BB], [EDD].

The results in the present paper can be used to propose numerical methods in the spirit of [HMW], and this will be the subject of a forthcoming paper.

Also addressed by our results is a problem in seismography discussed in [FF]. In a different direction is [HW] which converts problems related to those here to nonlinear eigenvalue problems which for the classical CF problem reduces to ordinary eigenvalues of Hankel matrices.

Notation and definitions. Elements $z \in \mathbf{C}^N$ are represented as column vectors. By z^t we denote the transpose of z, so that $\bar{z}^t z = \|z\|^2_{C^N}$. By $\mathcal{L}^\infty_N$ we represent the set of all $\mathbf{C}^N$ valued measurable functions f on ∂D with finite essential supremum norm $\|f\|_\infty = \sup_\theta \|f(e^{i\theta})\|_{C^N}$, and $\mathcal{H}^\infty_N$ consists of those elements of $\mathcal{L}^\infty_N$ which extend to analytic functions on $\mathbf{D}$. The spaces $\mathcal{L}^p_N$ and $\mathcal{H}^p_N$ $(p \geq 1)$ are defined similarly in terms of the norm $\|f\|_p = \left(\int_0^{2\pi} \|f(e^{i\theta})\|^p_{C^N} \frac{d\theta}{2\pi}\right)^{1/p}$.

Functions $f \in \mathcal{L}^1_N$ have a Fourier series expansion $f(e^{i\theta}) \equiv \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell e^{i\ell\theta}$, with the constants $\hat{f}_\ell \in C^N$ denoting the Fourier coefficients of f. In engineering applications only $\hat{f}_\ell \in R^N$ occur, hence we define the spaces $\mathcal{RL}^p$, $\mathcal{RH}^p$ and $\mathcal{RA}_N$ of functions f in $\mathcal{RL}^p$ resp. $\mathcal{RH}^p$, $\mathcal{RA}_N$, that satisfy $\hat{f}_\ell \in R^N$. The spaces $\mathcal{L}^2_N$ and $\mathcal{H}^2_N$ are Hilbert spaces over C, while $\mathcal{RL}^2_N$ and $\mathcal{RH}^2_N$ are Hilbert spaces over R due to the assumption on the Fourier coefficients. In all cases the inner product is $< f, g >= \int_0^{2\pi} f(e^{i\theta})^t \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$.

If $G \in \mathcal{L}^1_N$ and $c \in R$, a relation of the form $\int f(e^{i\theta})^t \overline{G(e^{i\theta})} \frac{d\theta}{2\pi} = c$ is called *affine equality constraint.* The definition of *affine inequality constraint* is similar with the obvious change of relations from "=" to "$\geq$". The constraint is *homogeneous* if $c = 0$. An inequality constraint is said to be *active at* f if equality holds for f.

If $\phi : \partial D \to C$ is measurable, then the expression $\int \phi$ means $\int_0^{2\pi} \phi(e^{i\theta}) \frac{d\theta}{2\pi}$. If γ is a scalar constant, the relations $\phi = \gamma$ and $\phi > \gamma$ should be interpreted as being valid for almost all $e^{i\theta}$.

We will need the Taylor expansion of $\Gamma(e^{i\theta}, z)$ as a function of z. It is given by

$$\Gamma(e^{i\theta}, z_0 + v) = \Gamma(e^{i\theta}, z_0) + 2\mathrm{Re}(\frac{\partial \Gamma}{\partial z}(e^{i\theta}, z_0)^t v) + \overline{v}^t \frac{\partial^2 \Gamma}{\partial \overline{z} \partial z}(e^{i\theta}, z_0) v + \mathrm{Re}(v^t \frac{\partial^2 \Gamma}{\partial z^2}(e^{i\theta}, z_0) v) + \cdots \tag{7}$$

where $\frac{\partial \Gamma}{\partial z}(e^{i\theta}, z_0)$ is a column vector with entries $\frac{\partial \Gamma}{\partial z_\ell}(e^{i\theta}, z_0)$, $\ell = 1, \ldots, n$. Also, note $\frac{\partial^2 \Gamma}{\partial \overline{z} \partial z}(e^{i\theta}, z_0)$ (resp. $\frac{\partial^2 \Gamma}{\partial z^2}(e^{i\theta}, z_0)$) is an $N \times N$ matrix with entries $\frac{\partial^2 \Gamma}{\partial \overline{z}_\ell \partial z_j}(e^{i\theta}, z_0)$, $\ell, j = 1, \ldots, N$ (resp. $\frac{\partial^2 \Gamma}{\partial z_\ell \partial z_j}(e^{i\theta}, z_0)$).

Theorems 1 through 4 are stated in section 2. The rest of the paper gives the proof but does so in an order that aids exposition. Section 3 states and proves several propositions, which are discussed in the spirit of classical functional analysis. Then the propositions are used in Section 4 to prove the main theorems of Section 2.

2 Main Results

A *local solution* to Constr-OPT over $\mathcal{E}$ is a solution to Constr-OPT over $\mathcal{V} \subset \mathcal{E}$ for some $\mathcal{V}$ a neighborhood of f. It is a *strict local solution* if it is unique in $\mathcal{V}$. A function $h \in \mathcal{E}_0$ is a *descent direction* relative to $f \in \mathcal{E}$ if there exists $\epsilon > 0$ such that for all t between 0 and ϵ,

$$\sup_\theta \Gamma(e^{i\theta}, f(e^{i\theta}) + th(e^{i\theta})) < \sup_\theta \Gamma(e^{i\theta}, f(e^{i\theta})) .$$

The function f is a *directional solution* or *directional minimizer* if there are no descent directions relative to f. Of course, every strict local solution is a directional solution, but the converse statement is false.

Theorem 1 *Let $\{G^{(\ell)} : 1 \leq \ell \leq n\}$ be a given linearly independent subset of $\mathcal{H}^1_N$, and $c_1, \ldots, c_n$ be constants in C. Set*

$$\begin{aligned} \mathcal{E} &= \{f \in \mathcal{A}_N : \textstyle\int f^t \overline{G^{(\ell)}} = c_\ell,\ 1 \leq \ell \leq n\} , \\ \mathcal{E}_0 &= \{f \in \mathcal{A}_N : \textstyle\int f^t \overline{G^{(\ell)}} = 0,\ 1 \leq \ell \leq n\} . \end{aligned} \tag{8}$$

Let $\Gamma(e^{i\theta}, z)$ be a 3 times continuously differentiable positive valued function on $\partial D \times C^N$. Let $f^* \in \mathcal{E}$ be such that $\frac{\partial \Gamma}{\partial z}(\cdot, f^*(\cdot))$ is never 0 on ∂D. Denote by χ the function $\chi(e^{i\theta}) = e^{i\theta}$. If the function f^* is a directional solution to Constr-OPT over $\mathcal{E}$ then

a. *There exist a constant $\gamma^* \in R$ such that $\Gamma(\cdot, f^*) = \gamma^*$.*
b. *There exists a positive valued measurable function λ^*, a function $F^* \in \mathcal{H}^1_N$ and constants $\kappa^*_\ell \in R$ such that*

$$\frac{\partial \Gamma}{\partial z}(\cdot, f^*) = \lambda^* \ (\ \chi\ F^* + \sum_\ell \kappa^*_\ell \overline{G^{(\ell)}}\ ) . \tag{9}$$

c. *If $A = \frac{\partial^2 \Gamma}{\partial \overline{z} \partial z}(\cdot, f^*(\cdot))$, $B = \frac{\partial^2 \Gamma}{\partial z^2}(\cdot, f^*(\cdot))$, and $\mathcal{N} := \{h \in \mathcal{E}_0 : \mathrm{Re}(\frac{\partial \Gamma}{\partial z}(\cdot, f^*)^t h) = 0\ \}$, then*

$$\sup_\theta \ \left(\overline{h}^t A h + Re(h^t B h) \right) \ \geq\ 0 \qquad (\forall h \in \mathcal{N} \setminus \{0\}) . \tag{10}$$

Conversely, If a., b., and c. with strict inequality hold, then f^* is a directional solution to Constr-OPT.

The hypothesis of continuity of f^* guarantees the existence of dual extremal functions G^* in the proof of necessity of condition b.

In the case of scalar valued analytic functions ($N = 1$) we have that the following theorem applies.

Theorem 2 *With the notation and assumptions of Theorem 1, if $N = 1$ then conditions a. and b. are sufficient for f^* to be a directional solution.*

When the set of constraints on the Fourier coefficients is given by a set of inequalities, for the problem to be stated correctly we need to define the problem in spaces of functions whose Fourier coefficients are real (when $N = 1$), or in general elements of R^N. In this context, performance functions Γ are required to be *real symmetric*. By this we mean that

$$\Gamma(e^{i\theta}, z) = \Gamma(e^{-i\theta}, \overline{z}) \qquad \forall\theta \ \forall z \ .$$

The following theorem combines both equality and inequality constraints.

Theorem 3 *Let $\{G^{(\ell)} : 1 \le \ell \le n\}$ be a given linearly independent subset of $\mathcal{RH}^1_N$, and let $c_1, \ldots, c_n$ be real numbers. Set*

$$\mathcal{E} = \left\{ f \in \mathcal{RA}_N : \begin{array}{cccccc} \int f^t \overline{G^{(\ell)}} & = & c_\ell, & 1 & \le \ell \le & n_1 \\ \int f^t \overline{G^{(\ell)}} & \ge & c_\ell, & n_1 + 1 & \le \ell \le & n_1 + n_2 \end{array} \right\}, \tag{11}$$

and let $\mathcal{E}_0$ be the subset of $\mathcal{RA}_N$ defined by affine relations similar to those from the definition of $\mathcal{E}$ but only now they are homogeneous. Let $\Gamma(e^{i\theta}, z)$ be a 3 times continuously differentiable real symmetric positive valued function on $\partial D \times C^N$. Denote by χ the function $\chi(e^{i\theta}) = e^{i\theta}$. Let $f^* \in \mathcal{E}$ be such that $\frac{\partial\Gamma}{\partial z}(\cdot, f^*)$ is never 0. Suppose that all of the inequality constraints are active at f^*.

If the function f^* is a local directional solution to Constr-OPT over $\mathcal{E}$ then

a. There exist a constant $\gamma^* \in R$ such that $\Gamma(\cdot, f^*) = \gamma^*$.

b. There exists a positive valued measurable function λ^*, a function $F^* \in \mathcal{RH}^1_N$ and constants $\kappa^*_\ell \in R$ with $\kappa_\ell \ge 0$ for $n_1 + 1 \le \ell \le n_1 + n_2$, such that

$$\frac{\partial\Gamma}{\partial z}(\cdot, f^*) = \lambda^* \ (\ \chi \ F^* + \sum_\ell \kappa^*_\ell \overline{G^{(\ell)}} \ ). \tag{12}$$

c. If $A = \frac{\partial^2\Gamma}{\partial\overline{z}\partial z}(\cdot, f^*(\cdot))$, $B = \frac{\partial^2\Gamma}{\partial z^2}(\cdot, f^*(\cdot))$, and $\mathcal{N} := \{h \in \mathcal{E}_0 : \mathrm{Re}(\frac{\partial\Gamma}{\partial z}(\cdot, f^*)^t h) = 0 \ \}$, then

$$\sup_\theta \ \left(\overline{h}^t A h + Re(h^t B h) \right) \ \ge \ 0 \qquad (\forall h \in \mathcal{N} \setminus \{0\}) \ . \tag{13}$$

Conversely, If a., b., and c. with strict inequality hold, then f^* is a directional solution to Constr-OPT.

In the scalar case we again have the following statement

Theorem 4 *With the notation and assumptions of Theorem 3, if $N = 1$ then conditions a. and b. are sufficient for f^* to be a directional solution.*

The proofs of Theorems 1 through 4 will be postponed until Section 5, since they depend on preparatory technical results that are stated and proved in Sections 3 and 4.

Results characterizing local directional solutions to Constr-OPT over subsets of $\mathcal{H}^\infty_N$ rather than $\mathcal{A}_N$ are identical to Theorems 1 and 3, if the function f^* given in the hypothesis of the theorem is assumed to be continuous. This is required in the proof of necessity of condition "b." in order to assure the existence of the functions F and λ. We do not state those results.

3 Distance problems with Equality Constraints

The approach to study Constr-OPT followed in this paper is based primarily on the study of sup-norm approximation problems in terms of a distance function that depends on a given weight function. Let w be a continuous, real valued function on the unit circle ∂D, such that $w > 0$, and let $\mathcal{E} \subset \mathcal{H}^\infty_N$. For any function $q \in \mathcal{L}^\infty_N$ let

$$\mathrm{dist}_w(q, \mathcal{E}) := \inf_{f\in\mathcal{E}} \sup_\theta \; w(e^{i\theta}) \, \|q(e^{i\theta}) - f(e^{i\theta})\|_{C^N} \; , \tag{14}$$

be the distance to q from $\mathcal{E}$. A function $f^* \in \mathcal{E}$ is called a *best approximation* to q from $\mathcal{E}$ if

$$\mathrm{dist}_w(q, \mathcal{A}_N) = \sup_\theta w\|q - f^*\|_{C^N}$$

Note that when q is continuous and when $\mathcal{E} \subset \mathcal{A}_N$, finding $\mathrm{dist}_w(q,\mathcal{E})$ and best approximations f^* is a problem of the type Constr-OPT, where the performance function has the form

$$\Gamma(e^{i\theta}, z) = w(e^{i\theta})\| \; q(e^{i\theta}) - z \; \|_{C^N} \; . \tag{15}$$

In many cases it is possible to assert the existence and uniqueness of best approximations. These and other statements are conclusions in the following proposition.

Proposition 1 *Let $\{G^{(\ell)} : 1 \le \ell \le n\}$ be a given linearly independent subset of $\mathcal{H}^1_N$, and let $c_1, \ldots, c_n$ be constants in C. Set*

$$\mathcal{E} = \{f \in \mathcal{H}^\infty_N : \int f^t \overline{G^{(\ell)}} = c_\ell, \; 1 \le \ell \le n\} \tag{16}$$

Let w be a given positive valued continuous function on the circle ∂D, and let q be a function in $\mathcal{L}^\infty_N \setminus \mathcal{E}$.

Then the following statements hold:

i. The set $\{f^* \in \mathcal{E} : \mathrm{dist}_w(q,\mathcal{E}) = \sup_\omega w\|q - f^*\|\}$ of best approximations to q from $\mathcal{E}$ is nonempty and convex.

ii. Let $\mathcal{E}_\perp$ be the pre-annihilator of $\mathcal{E}$,

$$\mathcal{E}_\perp = \{H \in \mathcal{L}^1_N : H = \chi F + \sum_{\ell=1}^{n} \kappa_\ell \overline{G^{(\ell)}}, \text{ for some } F \in \mathcal{H}^1_N, \; \kappa_\ell \in C\} \tag{17}$$

and let $\mathcal{B} = \{H \in \mathcal{E}_\perp : \int \frac{1}{w} \| \; H \; \|_{C^N} \le 1\}$. Then

$$\inf_{f\in\mathcal{E}} \; \sup_\theta \; w \, \| \; q - f \; \|_{C^N} \;\; = \;\; \sup_{H\in\mathcal{B}} \; \left| \int (\; q^t \; H \;) \right| \tag{18}$$

iii. If q is continuous then the supremum in (18) is attained at some $H^* \in \mathcal{B}$.

Proof:

(i) The existence of best approximations follows from a normal families argument just as in the classical unconstrained Nehari problem, see [G, pp.134-135]. The rest of the statement is a consequence of the convexity in z (for fixed $e^{i\theta}$) of the function

$$z = (z_1, \ldots, z_N)^t \to w(e^{i\theta}) \; \| \; q(e^{i\theta}) - z \; \|_{C^N}$$

(ii) For any $H \in \mathcal{B}$ and $f \in \mathcal{E}$, we have that $\int f^t H = 0$. Then,

$$
\begin{aligned}
\left| \int q^t H \right| &= \left| \int (q-f)^t H \right| \\
&\leq \int w \parallel q - f \parallel_{C^N} \tfrac{1}{w} \parallel H \parallel_{C^N} \\
&\leq \{ \sup_\theta w \parallel q - f \parallel_{C^N} \} \{ \int \tfrac{1}{w} \parallel H \parallel_{C^N} \} \\
&\leq \sup_\theta w \parallel q - f \parallel_{C^N} .
\end{aligned}
\tag{19}
$$

Thus

$$
\sup_{H \in \mathcal{B}} \left| \int q^t H \right| \leq \inf_{f \in \mathcal{E}} \sup_\theta w \parallel q - f \parallel_{C^N} . \tag{20}
$$

To prove the reverse inequality, consider the linear functional defined on $\mathcal{L}^1_N$ given by $\Lambda_q : H \rightarrow \int q^t H$. If n_0 is the positive number defined by the l.h.s. of (20), then the inequality

$$
| \Lambda_q(H) | \leq n_0 \int \frac{1}{w} \parallel H \parallel_{C^N} \tag{21}
$$

holds for every H in the space $\mathcal{E}_\perp$. By the Hahn-Banach theorem, the functional Λ_q extends to a functional Λ on $\mathcal{L}^1_N$ such that $\Lambda = \Lambda_q$ on $\mathcal{E}_\perp$, and

$$
\Lambda(H) \leq n_0 \int \frac{1}{w} \parallel H \parallel_{C^N} \quad (\forall H \in \mathcal{L}^1_N) \tag{22}
$$

Since $(\mathcal{L}^1_N)^* = \mathcal{L}^\infty_N$, there exists $p \in \mathcal{L}^\infty_N$ such that $\Lambda = \Lambda_p$. The functional $\Lambda_q - \Lambda_p$ annihilates $\mathcal{E}_\perp$, hence $f^* = q - p$ is in $\mathcal{E}$. From equation (22) we have that

$$
\begin{aligned}
n_0 &= \sup\{ | \int q^t H | \ : \ H \in \mathcal{E}_\perp, \ \int \tfrac{1}{w} \parallel H \parallel_{C^N} \leq 1 \} \\
&= \sup\{ \ | \int (q - f^*)^t H | \ : \ H \in \mathcal{L}^1_N, \ \int \tfrac{1}{w} \parallel H \parallel_{C^N} \leq 1 \}
\end{aligned}
\tag{23}
$$

We also have that

$$
\begin{aligned}
\sup_\theta w \parallel q - f^* \parallel_{C^N} &= \sup\{ \ | \int w(q - f^*)^t H_0 | \ : \ H_0 \in \mathcal{L}^1_N, \ \int \parallel H_0 \parallel_{C^N} \leq 1 \} \\
&= \sup\{ \ | \int (q - f^*)^t H | \ : \ H \in \mathcal{L}^1_N, \ \int \tfrac{1}{w} \parallel H \parallel_{C^N} \leq 1 \}
\end{aligned}
\tag{24}
$$

To finish the proof of (ii) combine (23) with (24).

(iii) If q is continuous, then $\Lambda_q : H \rightarrow \int q^t H$ is a continuous linear functional over $\mathcal{L}^1_N = (\mathcal{C}_N)^*$ equipped with the weak-* topology. Since $\mathcal{B}$ is weak-* compact, $|\Lambda_q|$ attains a maximum at some $H^* \in \mathcal{B}$. ■

Proposition 2 *Assume the hypotheses and notation of Proposition 1.*

i. *Set $\gamma^* = \mathrm{dist}_w(q, \mathcal{E})$. If the supremum in (18) is attained at some $H^* \in \mathcal{E}_\perp$, then there is a unique best approximation f^* in $\mathcal{E}$ to q, and the following relation holds:*

$$
w \parallel q - f^* \parallel_{C^N} = \gamma^* \tag{25}
$$

Moreover, there exists a unimodular constant $\alpha \in C$ such that

$$
q - f^* = \frac{\gamma^* \alpha}{w \parallel H^* \parallel_{C^N}} \overline{H^*} \tag{26}
$$

and

ii. If $f^* \in \mathcal{E}$ is such that (26) and (25) hold for some $\gamma^* > 0$ and some $H^* \in \mathcal{E}_\perp$, then f^* is a best approximation to q from $\mathcal{E}$.

Proof:

(i) Let $f^* \in \mathcal{E}$ and $H^* \in \mathcal{B}$ be optimizers for the left-hand-side and right-hand-side of (18) respectively. Then,

$$\begin{aligned} |\int (q-f^*)^t H^*| &= |\int q^t H^*| \\ &= \sup_\theta w\|q-f^*\|_{C^N} \\ &= \int (\sup_\theta w\|q-f^*\|_{C^N}) \tfrac{1}{w}\|H^*\|_{C^N} \\ &\geq \int w\|q-f^*\|_{C^N} \tfrac{1}{w}\|H^*\|_{C^N} \end{aligned} \tag{27}$$

It is clear that equality must hold throughout in (27), and this in particular implies that (25) holds. Also, from (27) we have that for some $\alpha \in C$ with $|\alpha| = 1$,

$$\overline{\alpha}(q-f^*)^t H^* = \|q-f^*\|_{C^N}\|H^*\|_{C^N} = \frac{\gamma^*}{w}\|H^*\|_{C^N} \tag{28}$$

Now (28) holds if and only if

$$q - f^* = \frac{\gamma^* \alpha}{w\|H^*\|_{C^N}} \overline{H^*} \tag{29}$$

thus proving (26). Note that these equations determine f^* uniquely in terms of H^* and γ^*.

(ii). Suppose that (26) and (25) hold, and that f^* is not a best approximation to q from $\mathcal{E}$. Let $h \in \mathcal{E}$ be such that

$$\sup_\theta \; w \parallel q-h \parallel_{C^N} \; < \; \sup_\theta \; w \parallel q - f^* \parallel_{C^N} . \tag{30}$$

Combining the relation

$$\begin{aligned} \sup_\theta \quad & w^2 \parallel q-h \parallel^2_{C^N} \\ = \quad & \sup_\theta \; w^2 \parallel q - f^* \; + \; f^* - h \parallel^2_{C^N} \\ = \quad & \sup_\theta \left\{ \; w^2 \parallel q-f^* \parallel^2_{C^N} \; + \; 2w^2 \; Re(\overline{q-f^*})^t(f^*-h) \; + \; w^2 \parallel f^*-h \parallel^2_{C^N} \right\} \end{aligned} \tag{31}$$

with (30) and with the hypothesis (25) we obtain that

$$\sup_\theta \; \mathrm{Re}\,(\overline{q-f^*})^t(f^*-h) < 0 \, . \tag{32}$$

Relations (32) and (26) give

$$\sup_\theta \; \mathrm{Re}\,(\; \overline{\alpha}(H^*)^t(f^*-h)\;) \; < 0 \tag{33}$$

But $f^* - h$ is in the annihilator of $\mathcal{E}_\perp$, hence the function $\mathrm{Re}((f^*-h)^t H^*)$ has zero average. Since by (33) this function is negative almost everywhere, we obtain a contradiction. ∎

4 Distance problems with Inequality Constraints

Let $\{G^{(\ell)} : 1 \leq \ell \leq n_1 + n_2\}$ be a given linearly independent subset of $\mathcal{RH}^1_N$, and let

$$\mathcal{E} = \left\{ f \in \mathcal{RH}^\infty_N : \begin{array}{ll} \int f^t \overline{G^{(\ell)}} \; = c_\ell, & 1 \leq \ell \leq n_1 \\ \int f^t \overline{G^{(\ell)}} \; \geq c_\ell, & n_1 + 1 \leq \ell \leq n_1 + n_2 \end{array} \right\} . \tag{34}$$

Let w be a given positive valued continuous function in $\mathcal{RL}^\infty$, and consider the weighted distance

$$\mathrm{dist}_w(q,\mathcal{E}) = \inf_{f\in\mathcal{E}} \sup_\theta \|q-f\|_{C^N}$$

from a function q in $\mathcal{RL}_N^\infty \setminus \mathcal{E}$ to the set $\mathcal{E}$. The set $\mathcal{E}$ is said to be minimal with respect to the weighted distance to q if for any set of indices $\mathcal{I}$ which is a proper subset of $\{1,\ldots,n\}$, the set $\mathcal{E}_\mathcal{I} = \{f \in \mathcal{H}_N^\infty : \int f^t\overline{G^{(\ell)}} = c_\ell,\ \ell\in\mathcal{I}\}$ satisfies

$$\mathrm{dist}_w(q,\mathcal{E}_\mathcal{I}) < \mathrm{dist}_w(q,\mathcal{E})$$

In other words, $\mathcal{E}$ is minimal if no constraint can be eliminated from the list of constraints without changing the weighted distance to q.

Proposition 3 *Let $\{G^{(\ell)} : 1\le \ell\le n_1+n_2\}$ be a given linearly independent subset of $\mathcal{RH}_N^1$. Set*

$$\mathcal{E} = \left\{ f\in\mathcal{RH}_N^\infty : \begin{array}{ll} \int f^t\overline{G^{(\ell)}} & = c_\ell, \quad 1\le\ell\le n_1 \\ \int f^t\overline{G^{(\ell)}} & \ge c_\ell, \quad n_1+1\le\ell\le n_1+n_2 \end{array} \right\}. \tag{35}$$

Let w be a given positive valued continuous function in $\mathcal{RL}^\infty$, and let q be a function in $\mathcal{RL}_N^\infty\setminus\mathcal{E}$. Suppose that the set $\mathcal{E}$ is minimal with respect to the weighted distance to q. Then the following statements hold:

i. The set $\{f^*\in\mathcal{E} : \mathrm{dist}_w(q,\mathcal{E}) = \sup_\theta w\|q-f^*\|\}$ of best approximations to q from $\mathcal{E}$ is nonempty and convex. All the constraints are active at any best approximation $f^*\in\mathcal{E}$.

ii. Let $\mathcal{E}_+$ be the set given by

$$\begin{array}{l} \mathcal{E}_+ = \{H\in\mathcal{RL}_N^1 : H = \chi F + \sum_{\ell=1}^{n_1+n_2}\kappa_\ell\overline{G^{(\ell)}}, \quad \text{for some } F\in\mathcal{RH}_N^1, \\ \qquad\qquad\qquad \kappa_1,\ldots,\kappa_{n_1}\in R, \kappa_{n_1+1},\ldots,\kappa_{n_1+n_2}\in R^+\} \end{array} \tag{36}$$

and let $\mathcal{B} = \{H\in\mathcal{E}_+ : \int \frac{1}{w}\parallel H\parallel_{C^N}\le 1\}$. Then

$$\inf_{f\in\mathcal{E}}\ \sup_\theta\ w\parallel q-f\parallel_{C^N} \ = \ \sup_{H\in\mathcal{B}} \left|\int q^t\ H\right| \tag{37}$$

iii. If q is continuous then the supremum in (37) is attained.

Proof of Proposition 3.

(i) The existence of solutions and convexity of the set of solutions is the same proof as in Proposition 1 part (i). Let f^* be a best approximation to q from $\mathcal{E}$. Note that if one or more than one constraints are not active at f^*, then they can be dropped without changing the fact that f^* is a directional minimizer. But then this contradicts the minimality of the set of constraints. Hence all constraints must be active at any best approximation f^*.

(ii) Let f^* be a best approximation to q whose existence is guaranteed by (i). Let

$$\begin{array}{rll} \mathcal{E}_0 = & \{f\in\mathcal{H}_N^\infty : & \int f^t\overline{G}_\ell = c_\ell,\ \ell=1,\ldots,n_1+n_2\} \\ \mathcal{B}_0 = & \{H\in\mathcal{L}_N^1 : & H=\chi F+\sum_{\ell=1}^{n_1+n_2}\kappa_\ell\overline{G^{(\ell)}},\ \text{for some } F\in\mathcal{H}_N^1,\ \kappa_\ell\in R, \\ & & \text{and } \int \frac{1}{w}\parallel H\parallel_{C^N}\le 1\} \end{array} \tag{38}$$

By (i) all the constraints are active at f^*. Thus f^* is a solution when the optimization is carried over $\mathcal{E}_0$, that is, when all the inequality constraints are changed to equality constraints. We now can take advantage of Proposition 1 and use equality (18) to obtain

$$\begin{aligned}\inf_{f\in\mathcal{E}}\ \sup_\theta\ \ & w \parallel q - f \parallel_{C^N} \\ & = \inf_{f\in\mathcal{E}_0}\ \sup_\theta\ w \parallel q - f \parallel_{C^N} \\ & = \sup_{H\in\mathcal{B}_0}\ \left| \int (\ q^t\ H\ ) \right| \\ & \geq \sup_{H\in\mathcal{B}}\ \left| \int (\ q^t\ H\ ) \right|\end{aligned} \tag{39}$$

To finish the proof of (37) we have to prove that the following equality holds:

$$\sup_{H\in\mathcal{B}_0}\ \left| \int (\ q^t\ H\ ) \right| = \sup_{H\in\mathcal{B}}\ \left| \int (\ q^t\ H\ ) \right| \tag{40}$$

In other words, we need to show that the supremum of $|\int q^t H|$ over functions $H = \chi F + \kappa_1 \overline{G^{(1)}} + \ldots + \kappa_{n_1+n_2}\overline{G^{(n_1+n_2)}}$ in $\mathcal{B}$ does not change if $\kappa_{n_1+1},\ldots,\kappa_{n_1+n_2}$ are allowed to be any real number (not necessarily nonnegative). Some of the difficulty in showing this is due to the fact that the supremum is not necessarily attained at some H. To accomplish our goal we have to study the properties of functions $H \in \mathcal{B}$ that nearly optimize $|\int q^t H|$. This is done in lemmas 1–3.

We now state the three lemmas that we need in the proof of the proposition. The proofs of the lemmas can be found right after the completion of the proof of the proposition. The first lemma states that elements H in $\mathcal{B}_0$ that nearly optimize $|\ \int q^t H\ |$ cannot have coefficients κ_ℓ that are small in size.

Lemma 1 *Assume the hypotheses and notation of Proposition 3, and suppose that $n_2 > 0$. There exist $\eta > 0$, $\epsilon > 0$ such that if $H = \chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}} \in \mathcal{O}$, where*

$$\mathcal{O} = \{H \in \mathcal{B}_0 : \left| \operatorname{dist}_w(q,\mathcal{E}) - \left| \int (\ q^t\ H\ )\right| \right| < \epsilon\}$$

then $|\kappa_\ell| > \eta$ for all ℓ, $1 \le \ell \le n_1+n_2$.

If $n_2 > 0$, the signum $sgn(H)$ of a function $H = \chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}$ in $\mathcal{B}_0$ is by definition the signum of the tuple $(\kappa_{n_1+1},\ldots,\kappa_{n_1+n_2})$. That is,

$$sgn(\chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}) := (sgn(\kappa_{n_1+1},\ldots,\kappa_{n_1+n_2})$$

Our next lemma states that the signum of a function H in $\mathcal{B}_0$ that nearly optimizes $|\ \int q^t H\ |$ can have only one of two possible values.

Lemma 2 *The number ϵ in Lemma 1 can be chosen so that if H_1 and H_2 are in $\mathcal{O}$, then $(\int q^t H_1)(\int q^t H_2) > 0$ if and only if $sgn(H_1) = sgn(H_2)$. Hence there exist $\sigma^* = (\sigma_1,\ldots,\sigma_{n_2})$ with $\sigma_1 = +1$ and $\sigma_\ell \in \{-1,1\}$, such that any $H \in \mathcal{O}$ satisfies either $sgn(G) = \sigma^*$ or $sgn(G) = -\sigma^*$.*

The next lemma states that if not all constraints are active at $f^* + h \in \mathcal{E}$ and if $||h||$ is small enough, there exists H that nearly optimizes $|\ \int q^t H\ |$ and such that the sign of $\int q^t H$ is the opposite of the sign of $\int h^t H$.

Lemma 3 *Assume the hypotheses and notation of Lemma 2. There exists $\delta > 0$ such that if $f^* + h$ is in $\mathcal{E} \setminus \mathcal{E}_0$, $||h||_\infty < \delta$ then there exists $H \in \mathcal{O}$ so that $sgn(G) = \sigma^*$ and*

$$\left(\int q^t H \right) \left(\int h^t H \right) < 0 \tag{41}$$

If there is only one inequality constraint ($n_2 = 1$) and $H = \chi F + \kappa \overline{G^{(1)}}$ is in $\mathcal{B}_0$, then either $H \in \mathcal{B}$ or $-H \in \mathcal{B}$. Since $| \int (q^t H)| = | \int (q^t(-H))|$, we have that (40) holds in this case. From now on we assume that $n_2 > 1$. To finish the proof of (ii) we suppose for the sake of contradiction that (40) does not hold, i.e. there exists an integer ℓ_0, $n_1 + 1 < \ell_0 \leq n_2$ such that the set

$$\mathcal{B}_{\ell_0} = \left\{ H \in \mathcal{B} \ : \ H = \chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}, \ \kappa_\ell \geq 0 \text{ for } \ \ell < \ell_0 \right\} \tag{42}$$

satisfies

$$\sup_{H \in \mathcal{B}} \left| \int (q^t \ H) \right| = \sup_{H \in \mathcal{B}_{\ell_0+1}} \left| \int (q^t \ H) \right| < \sup_{H \in \mathcal{B}_{\ell_0}} \left| \int (q^t \ H) \right| \tag{43}$$

That is, optimizing $\int q^t H$ over H whose constants κ_ℓ are nonnegative for $1 \leq \ell < \ell_0$ gives a different answer than if κ_{ℓ_0} is only allowed to be nonnegative.

By Lemma 1 and by Lemma 2, we know that there exists σ^* so that any $H \in \mathcal{B}_0$ that nearly optimizes $| \int q^t H|$ must have either $sgn(H) = \sigma^*$ or $sgn(H) = -\sigma^*$. Hence all functions H that nearly optimize $| \int q^t H|$ and have $sgn(H) = \sigma^*$ must have either $\int q^t H > 0$ or $\int q^t H < 0$.

Now we consider two cases. First, suppose that that functions $H \in \mathcal{O}$ that have $sgn(H) = \sigma^*$ satisfy $\int q^t H < 0$. Choose h such that $f^* + h \in \mathcal{E} \setminus \mathcal{E}_0$ and $||h||_\infty < \epsilon$, $\int h^t \overline{G^{(\ell)}} = 0$ for $\ell \neq \ell_0$, and such that $\int h^t \overline{G^{(\ell_0)}} > 0$. Let $H \in \mathcal{B}$ be the function given by Lemma 3. Since $\kappa_{\ell_0} < 0$ we have that $\int h^t H < 0$, and this contradicts Lemma 3.

The other case to consider is when functions $H \in \mathcal{O}$ that have $sgn(H) = \sigma^*$ satisfy $\int q^t H > 0$. Choose h such that $f^* + h \in \mathcal{E} \setminus \mathcal{E}_0$ with $||h||_\infty < \epsilon$, $\int h^t \overline{G^{(\ell)}} = 0$ for $\ell \neq \ell_0 - 1$, and such that $\int h^t \overline{G^{(\ell_0 - 1)}} > 0$. Let $H \in \mathcal{B}$ be the function given by Claim 3, so in particular we have that $\kappa_{\ell_0 - 1} > 0$. Then $\int h^t H > 0$ which contradicts Lemma 3.

Thus we have that there is no such number ℓ_0, and therefore the equality (40) holds. This finishes the proof of (ii). The proof of statement (iii) is similar to the corresponding statement in Proposition 1. ∎

Proof of Lemma 1. If the statement does not hold, then there exist an optimizing sequence $\{H_{(j)}\}$, where $H_{(j)} = \chi F_{(j)} + \sum_\ell \kappa_{(j),\ell} \overline{G^\ell}_{(j)}$, and an index ℓ_1 such that $\kappa_{(j),\ell_1}$ satisfies $\kappa_{(j),\ell_1} \to 0$ as $j \to \infty$. But by Proposition 1 this implies that the constraint can be dropped without changing the distance to q, which contradicts the minimality hypothesis on $\mathcal{E}$. ∎

Proof of Lemma 2. If the first statement of Lemma 2 does not hold then there exist H_1 and H_2 in $\mathcal{O}$ such that $(\int q^t H_1)(\int q^t H_2) > 0$ but $sgn(H_1) \neq sgn(H_2)$. By taking a suitable convex combination of H_1 and H_2 we obtain H_3 close to optimal but with one of the κ_ℓ's equal to 0. But this contradicts Lemma 1. Hence the first statement of Lemma 2 holds. The second statement of Lemma 2 is clearly a consequence of the first statement. ∎

Proof of Lemma 3. For any function h as in the hypothesis we have that

$$\mathrm{dist}_w(q, \mathcal{E}) < \mathrm{dist}_w(q-h, \mathcal{E}) \tag{44}$$

Moreover, if $\|h\|_\infty$ is small enough then $\mathcal{E}$ is minimal with respect to $q-h$. Now use (18) to conclude that there exists $H \in \mathcal{B}_0$ such that

$$|\int (q-h)^t H| > \gamma^* \tag{45}$$

The conclusion follows from (45) and from the fact that $|\int q^t H| \leq \gamma^*$. ■

Proposition 4 *Assume the hypotheses and notation of Proposition 3.*

i. Let $\gamma^* = \mathrm{dist}(q, \mathcal{E})$. *If the supremum in (37) is attained at some* $H^* \in \mathcal{E}_+$, *then there exists a unique best approximation* f^* *in* $\mathcal{E}$ *to* q, *and the following relations hold:*

$$w \parallel q - f^* \parallel_{C^N} = \gamma^* , \tag{46}$$

and

$$q - f^* \;=\; -\frac{\gamma^*}{\parallel H^* \parallel_{C^N}}\;\overline{H^*}\,. \tag{47}$$

ii. If $f^* \in \mathcal{E}$ *is such that (47) and (46) hold for some* $\gamma^* > 0$ *and some* $H^* \in \mathcal{E}_\perp$, *then* f^* *is a best approximation to* q *from* $\mathcal{E}$.

Proof. i. By following steps identical to those in the proof of Proposition 2, we obtain that there exists $\alpha \in \{-1, 1\}$ such that

$$w \parallel q - f^* \parallel_{C^N} = \gamma^* , \tag{48}$$

and

$$q - f^* \;=\; \frac{\gamma^* \alpha}{\parallel H^* \parallel_{C^N}}\;\overline{H^*}\,. \tag{49}$$

Since any best approximation f^* must satisfy (49), uniqueness of f^* follows from this relation. To complete the proof of i., we must show that $\alpha = -1$. To do this we see that by relations (27) and (28) it is enough to prove that

$$\int q^t\; H^* \;<\; 0 \tag{50}$$

A consequence of Lemma 2 is that any $H \in \mathcal{B}$ that nearly optimizes $|\int q^t H|$ satisfies $(\int q^t H)(\int q^t H^*) > 0$, i.e., both integrals have the same sign. Hence to prove (50) it is enough to find a sequence $\{H^n\} \subset \mathcal{B}$ such that $\int q^t H^n \to \int q^t H^*$ as $n \to \infty$ and such that $\int q^t H^n < 0$.

Let $\{h^n\} \subset \mathcal{E}$ be such that $\|h^n\|_\infty \to 0$ as $n \to \infty$. Since $\mathrm{dist}_w(q - h^n, \mathcal{E}) > \mathrm{dist}_w(q, \mathcal{E})$, we have that there exists $H^n \in \mathcal{B}$ such that

$$|\int q^t H^n - \int h^n H^n| = \int (q - h^n)^t H^n > \mathrm{dist}_w(q, \mathcal{E}) \tag{51}$$

Note that since $\|h^n\|_\infty \to 0$, we have $|\int q^t H^n| \to \gamma^*$. Hence, for n large enough, $\int q^t H^n < 0$ necessarily, as otherwise $0 < \int q^t H^n - \int h^n H^n < \mathrm{dist}_w(q, \mathcal{E})$.

The proof of ii. is identical to the proof of ii. of Proposition 2. ■

5 Proof of the main results

Lemma 4 *Assume the notation and hypotheses of Theorem 1. Suppose that f^* is a directional solution of Constr-OPT over $\mathcal{E}$. Set $g = \Gamma(\cdot, f^*(\cdot))$, $a = \frac{\partial \Gamma}{\partial z}(\cdot, f^*(\cdot))$, and let*

$$w(e^{i\theta}) = \|a(e^{i\theta})\|^2_{C^N} / g(e^{i\theta}) \ .$$

and define a performance function

$$\Gamma_1(e^{i\theta}, z) = w(e^{i\theta}) \parallel \frac{1}{w(e^{i\theta})} \overline{a(e^{i\theta})} + z \parallel^2_{C^N} \tag{52}$$

Then f^* is a solution for the Constr-OPT over the set $\mathcal{E}$ for the performance Γ_1.

Proof. A simple calcultation yields

$$\Gamma_1(e^{i\theta}, z) = g(e^{i\theta}) + 2Re(a(e^{i\theta})^t z) + w(e^{i\theta}) \|z\|^2_{C^N} \tag{53}$$

so the Taylor expansion of $\Gamma_1(e^{i\theta}, z)$ about $z = 0$ coincides up to order one with the Taylor expansion of $\Gamma(e^{i\theta}, f^*(e^{i\theta}) + z)$ about $z = 0$:

$$\Gamma(\cdot, f^* + z) = g + 2Re(a(\cdot)^t z) + \overline{z}^t A z + Re(z^t B z) + \dots \tag{54}$$

We shall prove the contrapositive. Let $h \in \mathcal{E}$ be such that

$$\sup_\theta \Gamma_1(\cdot, h) < \gamma^* = \sup_\theta \Gamma_1(\cdot, 0) \tag{55}$$

Note that at peak points $e^{i\theta}$ of g the function $Re(a^t h)$ is negative. Since the set of peak points of g is closed by continuity of g, there is an open subset of the circle J and a constant $\delta_1 > 0$ such that

$$Re(a(e^{i\theta})^t h(e^{i\theta})) < -\delta_1 < 0, \qquad \forall e^{i\theta} \in J \tag{56}$$

Let $\Phi = \max_\theta w(e^{i\theta}) \|h(e^{i\theta})\|^2$. For any $e^{i\theta}$ in J and $t \geq 0$ we have that

$$\begin{aligned} \Gamma(e^{i\theta}, f(e^{i\theta}) + t h(e^{i\theta})) &= g(e^{i\theta}) + 2t Re(a(e^{i\theta})^t h(e^{i\theta})) + t^2 w(e^{i\theta}) \|h(e^{i\theta})\|^2 \\ &< g(e^{i\theta}) - \delta_1 t + t^2 \Phi \\ &< \gamma^* - \delta_1 t + t^2 \Phi \end{aligned} \tag{57}$$

Let $\gamma_1 = \max\{g(e^{i\theta}) : e^{i\theta} \in \partial D \setminus J\}$. Then $\gamma_1 < \gamma^*$. Now pick $\delta_3 > 0$ so small that for all $t > 0$ small enough,

$$\sup_{e^{i\theta} \in \partial D \setminus J} \Gamma(\cdot, f + th) < \gamma^* - \delta_1 \tag{58}$$

By combining (57) with (58) we can see that for all $t > 0$ small enough,

$$\sup_{e^{i\theta} \in \partial D} \Gamma(\cdot, f + th) < \gamma^* - \delta_1 t + t^2 \Phi \tag{59}$$

Now the left-hand-side of (59) is smaller than γ^* for all small $t > 0$. ∎

Lemma 5 *Assume the hypotheses and notation of Lemma 4, and suppose that statement (b) in Theorem 2 holds. If $h \in \mathcal{E}$ is such that $Re(a^t h)$ is not the zero function then $\sup_\theta Re(a^t h) > 0$.*

Proof. By (b) we have that

$$Re(a^t h) = Re(\lambda(\chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}))^t h = \lambda Re h^t(\chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}) \tag{60}$$

If $\sup_\theta Re(a^t h) \leq 0$ we can eliminate the positive factor λ to get the inequality

$$Re h^t(\chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}) \leq 0 \tag{61}$$

On the other hand,

$$\int h^t(\chi F + \sum_\ell \kappa_\ell \overline{G^{(\ell)}}) = 0 \tag{62}$$

But a nonpositive function integrates to 0 only when it is the zero function, hence $Re(a^t h) = 0$. This proves the claim. ∎

Proof of Theorem 1. By Lemma 4 and by Proposition 1 applied to the function Γ_1 we have that statements (a) and (b) in the statement of the Theorem hold. Now let $h \in \mathcal{E}$. Since f^* is a directional solution we have that

$$\sup_\theta \Gamma(\cdot, f^* + th) = \sup\{g + 2tRe(a^t h) + t^2(\overline{h}^t Ah + Re(h^t Bh)) + t^2\mathcal{O}(t)\} \geq \gamma^* \tag{63}$$

By statement (a), $g = \gamma^*$ so we can rewrite (63) as

$$\sup_\theta\{2Re(a^t h) + t(\overline{h}^t Ah + Re(h^t Bh)) + t\mathcal{O}(t)\} \geq 0 \tag{64}$$

If $Re(a^t h)$ is not the function zero, then $sup_\theta Re(a^t h) > 0$ by Lemma 5, and by taking $t > 0$ small enough we conclude that (64) holds with strict inequality. Thus this case does not impose any restrictions on the second order terms of the Taylor expansion of Γ.

If $Re(a^t h)$ *is* the function zero, then the expansion of Γ gives

$$\sup_\theta \Gamma(\cdot, f + th) = \sup_\theta\{\gamma^* + t^2(\overline{h}^t Ah + Re(h^t Bh)) + t^2\mathcal{O}(t)\} \geq \gamma^* \tag{65}$$

Hence

$$\sup_\theta(\overline{h}^t Ah + Re(h^t Bh) + t\mathcal{O}(1)) \geq 0 \tag{66}$$

Letting t approach 0 in (66) we obtain statement (c).

Assume now that (a), (b) and (c') (i.e., (c) with strict inequality) hold, and pick $h \in \mathcal{E}$. We already saw that if $Re(a^t h)$ is not identically 0 then h is not a descent direction. If on the other hand $Re(a^t h)$ is the zero function, then (c') and a Taylor expansion of $\Gamma(\cdot, f + th)$ yields the inequality (for some small $t > 0$),

$$\sup_\theta(\overline{h}^t Ah + Re(h^t Bh) + \mathcal{O}(t)) > 0 \tag{67}$$

hence h is not a descent direction. ∎

Proof of Theorem 2 Let $a = \frac{\partial \Gamma}{\partial z}(\cdot, f^*(\cdot))$, and suppose that $h \in \mathcal{E}_0$ is a descent direction. We claim first that $Re(ah) \leq 0$. Indeed, if $Re(ah)$ is positive for some θ_0, from the relation

$$\Gamma(e^{i\theta_0}, f^*(e^{i\theta_0}) + th(e^{i\theta_0})) = \Gamma(e^{i\theta_0}, f^*(e^{i\theta_0})) + tRe(ah) + \mathcal{O}(t^2)$$

it follows that there exists an ϵ such that for $0 < t < \epsilon$,

$$\Gamma(e^{i\theta_0}, f^*(e^{i\theta_0})) < \Gamma(e^{i\theta_0}, f^*(e^{i\theta_0}) + th(e^{i\theta_0})).$$

As $\Gamma(\cdot, f^*)$ is a constant, then

$$\sup_\theta \Gamma(e^{i\theta}, f^*(e^{i\theta})) = \Gamma(e^{i\theta_0}, f^*(e^{i\theta_0})) < \Gamma(e^{i\theta_0}, f^*(e^{i\theta_0}) + th(e^{i\theta_0})) \leq \sup_\theta \Gamma(e^{i\theta}, f^*(e^{i\theta}) + th(e^{i\theta})),$$

and this proves our claim.

We now proceed to show that there exists a θ_0 such that $h(e^{i\theta_0}) = 0$. Suppose that $h \in \mathcal{E}_0$ and $Re(ah) \leq 0$. From condition (b) and the fact that $h \in \mathcal{E}_0$, $\int \frac{1}{\lambda}ah = 0$. In particular, $\int Re(\frac{1}{\lambda}ah) = 0$ and $\int Im(\frac{1}{\lambda}ah) = 0$. As $\frac{1}{\lambda}$ is real valued, then $0 = \int Re(\frac{1}{\lambda}ah) = \int \frac{1}{\lambda}Re(ah)$. Since $Re(ah) \leq 0$, then $Re(ah) = 0$ as otherwise the integral would be negative. Hence, $ah = Im(ah)$. As $\int \frac{1}{\lambda}Im(ah) = 0$, then either $Im(ah) = 0$ or $Im(ah)$ is both positive and negative. As ah is continuous, then ah must be zero at some θ_0. As by hypothesis a is never zero, then there exists θ_0 such that $h(e^{i\theta_0}) = 0$.

Finally, since $\Gamma(e^{i\theta_0}, f^*(e^{i\theta_0}) + th(e^{i\theta_0})) = \Gamma(e^{i\theta_0}, f^*(e^{i\theta_0}))$ for all $t > 0$, we have that

$$\sup_\theta \Gamma(\cdot, f^* + th) \geq \sup_\theta \Gamma(\cdot, f^*).$$

Since h was assumed to be a descent direction, we obtained a contradiction. ∎

Proof of Theorem 3. The proof is quite similar to the proof of Theorem 1, so we only make some remarks. Suppose first that f^* is an optimizer over $\mathcal{E}$, and that f^* is not a directional optimizer over any set defined by a proper subset of the inequalities that define $\mathcal{E}$. Then proceed as in the proof of Theorem 1 to conclude that f^* satisfies (a), (b), and (c). If now f^* is a directional optimizer over $\mathcal{E}$ and if is also an optimizer over a set $\mathcal{E}'$ such that $\mathcal{E} \subset \mathcal{E}'$ defined by fewer inequalities, find such a set so that its set of defining inequalities cannot be reduced further. Apply the first part of this proof to conclude that (a), (b) and (c) hold with respect to $\mathcal{E}'$, thus they hold with respect to the smaller set $\mathcal{E}$ (with some obvious changes, such as adding constants $\kappa_\ell = 0$ in (b)).

The sufficiency of (a), (b) and (c) with strict inequality is identical to the corresponding proof of Theorem 1.

Proof of Theorem 4 Let h be a descent direction, and set $a = \frac{\partial \Gamma}{\partial z}(\cdot, f^*)$. Using the same argument as in the proof of Theorem 2, we have that $Re(ah) \leq 0$. Also, by condition (b) and admissibility of h, $\int \frac{1}{\lambda}ah \geq 0$. Hence, $\int \frac{1}{\lambda}Re(ah) \geq 0$. But, as $\lambda > 0$ and $Re(ah) \leq 0$, we again have that $Re(ah) = 0$. Now, as all Fourier coefficients are real, $\int \frac{1}{\lambda}Im(ah) = 0$ and so again $Im(ah)$ is either zero or both positive and negative, so there exists a θ_0 such that $h(e^{i\theta_0}) = 0$. To complete the proof we can proceed as in the proof of Theorem 2 to obtain a contradiction. ∎

References

[B] S. Boyd and B. Barat. Linear Controller Design: Limits of performance. Prentice-Hall, Englewood Cliffs, N.J., 1991. Academic Press.

[EDD] N. Elia, M. A. Daleh, and I. J. Diaz-Bobillo. Controller Design via Infinite Dimensional Linear Programming. Proceedings of the ACC, 1993.

[FF] C. Foias and A. E. Frazho. The Commutant Lifting Approach to Interpolation Problems. Operator Therory: Advances and Applications. Vol 44. Birkhauser 1990.

[G] J. Garnett. Bounded Analytic Functions. Academic Press, NY 1981.

[H] J. W. Helton. Worst Case Design in the Frequency Domain. IEEE TAC AC-30, pp.1154-1170.

[HMar] J. W. Helton and D. Marshall. Frequendy domain design and analytic selections. Indiana U. Math. J. **39** (1990) 157-184.

[HH] J. W. Helton and R. Howe. A Bang-Bang Principle for the Frequency Domain, J. Operator Theory, **15** (1986), 359-375.

[HMer1] J. W. Helton and O. Merino. Conditions for Optimality over $\mathcal{H}^\infty$. SIAM J. Cont. Opt., Vol. 31, No. 6, November 1993.

[HMer2] J. W. Helton and O. Merino. A Novel Approach to Accelerating Newton's Method for sup-norm optimization arising in H^∞-control. J. Optim. Theory Apps. 1993.

[HMer3] J. W. Helton and O. Merino. Classical Control with $\mathcal{H}^\infty$ methods. Book in preparation.

[HMW] J. W. Helton, O. Merino and T. E. Walker. Algorithms for Optimizing over Analytic Functions. Indiana U. Math. J., Vol 42, No.3 (1993)

[HW] J. W. Helton and H. Woerdeman. SIAM J. Lin Alg. Jan 1994.

[RS] H. Rotstein and A. Sideris. H^∞ Optimization with Time Domain Constraints. IEEE TAC Vol39, No4, April 1994.

[S] R. L. Streit. An algorithm for the solution of systems of complex linear equations in the ℓ_∞ norm with constraints on the unknowns, ACM Trans. Math. Software, **11**, No.3 (1985).

[Y] N. Young. An Introduction to Hilbert Space. Cambridge University Press, Cambridge, 1988.

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