A Fibered Polynomial Hull Without an Analytic Selection

J. William Helton and Orlando Merino

Department of Mathematics
University of California, San Diego
La Jolla, California 92093-0112

This note shows that certain polynomial hulls in $\mathbf{C}^3$ have no analytic selection thus settling a standing question about such hulls.

Recall that the *polynomial hull* $\mathcal{P}(\mathcal{S})$ of a set $\mathcal{S}$ in $\mathbf{C}^M$ is the set

$$\mathcal{P}(\mathcal{S}) = \{w \in \mathbf{C}^M : \ |p(w)| \leq \max_{v \in \mathcal{S}} |p(v)| \quad \text{for all polynomials } p \text{ on } \quad \mathbf{C}^M\} \ .$$

We shall be considering sets $\mathcal{S}$ in $\mathbf{C}^{1+N}$ that are fibered over the unit circle $\partial\mathbf{D}$ in $\mathbf{C}$ the complex plane. Thus $\mathcal{S}$ has the form

$$\{(e^{i\theta}, \ \mathcal{S}_\theta) : \ 0 \leq \theta \leq 2\pi\}$$

where each $\mathcal{S}_\theta$ is a subset of $\mathbf{C}^N$. Let $\mathbf{D}$ denote the unit disk in the complex plane and let H^∞_N denote the $\mathbf{C}^N$ valued functions bounded and analytic on $\mathbf{D}$. Clearly (from the maximum principle) if f is any H^∞_N function satisfying

$$f(e^{i\theta}) \in \mathcal{S}_\theta, \quad \text{for almost all} \quad \theta$$

then the graph $\{z, f(z)) : \ z \in \mathbf{D}\}$ of f lies in $\mathcal{P}(\mathcal{S})$. Such a function f is called an *analytic selection of* $\mathcal{P}(\mathcal{S})$. A significant question about polynomial hulls is which hulls have analytic selections.

An obvious necessary condition is that $\mathcal{P}(\mathcal{S})$ in $\mathbf{C}^{1+N}$ is a set whose projection onto the first coordinate is a set containing $\mathbf{D}$. We shall refer to such $\mathcal{P}(\mathcal{S})$ as having *nontrivial fiber over the unit disk*. Also if the $\mathcal{S}_\theta$ are *not connected* it is easy to make up examples where $\mathcal{P}(\mathcal{S})$ has no analytic selection.

(Q) Question: Are these conditions sufficient for $\mathcal{P}(\mathcal{S})$ to have an analytic selection?

Wermer [Wer] showed that in general the answer is no by giving a highly pathological example (for $N=1$). However when the $\mathcal{S}_\theta$ are nicely behaved the story is different. For $N=1$ Slodkowski [S] and independently Wegert [Weg] and Helton-Marshall [HM] showed that the answer is yes. This article gives a simple very well behaved $\mathcal{S}$ in $\mathbf{C}^3$ for which the answer to (Q) is no.

Now we write down the $\mathcal{S}$ that provides our example.

Write $\mathbf{C}^2$ as $\{z=(x_1,y_1,x_2,y_2)=(z_1,z_2)\}$. Let $\mathcal{C}$ denote the semicircle

$$\mathcal{C}=\{(z_1,z_2):\ |z_1|=1,\ \operatorname{Im} z_1\geq 0,\ z_2=0\}$$

embedded in the first complex coordinate of $\mathbf{C}^2$. Our example is based on rotating $\mathcal{C}$ inside of real 3-dimensional space in a way which varies with θ. Here $\mathbf{R}^3$ is embeded in $\mathbf{C}^2$ in the usual way $(y_2=0)$. Let r_θ denote the rotation of $x+iy$ in the complex plane by θ, that is, $r_\theta(z)=e^{i\theta}z$. Let R^1_θ denote the map which acts on $\mathbf{C}^2$ by the rotation of z in the z_1 plane by θ (z_2 is kept fixed), that is,

$$R^1_\theta(z_1,z_2)=(r_\theta(z_1),z_2)=(e^{i\theta}z_1,z_2).$$

Also, let R^2_θ denote rotation in the (y_1,x_2) plane by θ.

Theorem 1. *The polynomial hull $\mathcal{P}(\mathcal{S})$ of $\mathcal{S}$ with fibers*

$$\mathcal{S}_\theta=R^1_{\theta/2}(R^2_{\theta/2}(\mathcal{C}))$$

has nontrivial fibers over the disk, since $\mathcal{P}(\mathcal{S})$ contains the graph

$$\{(z, \pm z^{1/2}, 0) : \ z \in \mathbf{D}\}$$

of $z^{1/2}$. Moreover, each $\mathcal{S}_\theta$ is connected, and the sets $\mathcal{S}_\theta$ vary continuously with θ. However, $\mathcal{P}(\mathcal{S})$ contains no H^∞ analytic selection. Note that the sets $\mathcal{S}_\theta$ vary continuously with θ.

First we need a lemma. For $k = 1, 2$, let Proj_k be the map on $\mathbf{C}^3$ given by $\mathrm{Proj}(\zeta, z_1, z_2) = z_k$.

Lemma. $\mathcal{P}(\mathcal{S}) \subset \mathbf{D} \times \mathcal{P}(\mathrm{Proj}_1(\mathcal{S})) \times \mathcal{P}(\mathrm{Proj}_2(\mathcal{S}))$.

Proof: $\mathcal{S} \subset \partial\mathbf{D} \times \mathrm{Proj}_1(\mathcal{S}) \times \mathrm{Proj}_2(\mathcal{S})$. Thus

$$\mathcal{P}(\mathcal{S}) \subset \mathcal{P}(\partial\mathbf{D} \times \mathrm{Proj}_1(\mathcal{S}) \times \mathrm{Proj}_2(\mathcal{S})) \subset \mathbf{D} \times \mathcal{P}(\mathrm{Proj}_1(\mathcal{S})) \times \mathcal{P}(\mathrm{Proj}_2(\mathcal{S})).$$

Proof of Theorem 1. Suppose that $f = (f^1, f^2)$ in H_2^∞ has its graph contained in $\mathcal{P}(\mathcal{S})$. By the Lemma the function f^2 is in $\mathcal{P}(\mathrm{Proj}_2(\mathcal{S})) \subset [-1, 1]$ in $\mathbf{R}$. In particular Im f^2 is 0, therefore f^2 is constant, but $f^2(1) = 0$, since $f(1) \in \mathcal{C}$. We conclude that f^2 is 0.

The value of $f = (f^1, 0)$ at θ is in $\mathcal{S}_\theta \cap \{(z_1, 0)\} = \{(\pm e^{i\theta/2}, 0)\}$ except possibly for $\theta = 0$ or 2π. That is, $f^1 = \pm e^{i\theta/2}$ almost everywhere. No such function exists in H_1^∞, so we have a contradiction.

Now we check that the $\mathcal{S}_\theta$ are smoothly varying. If $0 < \theta < 2\pi$, then the maps R_θ^1 and R_θ^2 on $\mathbf{C}^2$ are jointly C^∞ in θ and (z_1, z_2). The only potential difficulty is at

$\theta = 0$, $\theta = 2\pi$. One can easily visualize $\mathcal{S}_\theta$ since it lies in $R^3 \subset C^2$. The key is the set of points $\pm z_\theta = (\pm e^{i\theta/2}, 0)$ which are the image of $\pm i$ under the map $R^1_{\theta/2} R^2_{\theta/2}$ (and which consequently lie in $\mathcal{S}_\theta$). Observe that $\pm i = \pm z_0 = \mp z_{2\pi}$ which implies that as θ moves from 0 to 2π the points $(+1,0)$ and $(-1,0)$ rotate into each other. Thus $\mathcal{S}_{2\pi}$ is a rotation of $\mathcal{C}$ in the $(z_1, 0)$ plane which has the same end points as $\mathcal{C}$; we have $\mathcal{S}_{2\pi} = \mathcal{C}$ or $\mathcal{S}_{2\pi} = -\mathcal{C}$. However $R^1_{2\pi/2}(R^2_{2\pi/2}(1,0)) = R^1_\pi((-1,0)) = (1,0)$. We have established that the $\mathcal{S}_\theta$ vary continuously in θ.

Lest one complains that our example hinges on degeneracy of $\mathcal{S}_\theta$. We now give an example where $\mathcal{S}$ is the closure of an open set in $\mathbf{C}^3$, in addition to maintaining other nice properties.

Theorem 2. *For $n \in \mathbf{N}$, let $\mathcal{S}^n$ be the set in $\mathbf{C}^3$ with fibers $\mathcal{S}_\theta + \frac{1}{n}B$, where B is the unit ball of $\mathbf{C}^2$. Then there exists $n_0 \in \mathbf{N}$ such that the polynomial hull of $\mathcal{S}^{n_0}$ has no analytic selection.*

Proof. Suppose not. Then for each n there exists f_n in H^∞ whose graph lies in $\mathcal{S}^n$. Let f_0 be a normal families limit of $\{f_n\}$. By polynomial convexity of each $\mathcal{S}_\theta$ and by Corollary 2 in [HM], the function f_0 has graph lying in $\mathcal{P}(\mathcal{S})$. Thus $\mathcal{P}(\mathcal{S})$ has a selection contrary to Theorem 1.

Bibliography

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