Example. Find $T \in \mathcal{RH}^\infty$ with $a = -1$ as its only pole location, such that

$$T(0) = 1, \qquad T(1) = -2 \tag{8.5}$$

Solution. Note that $n = 2$ (number of points in *INT*), so Newton's representation for an interpolant is

$$T(s) = c_0 + c_1 \frac{s}{s+1} \tag{8.6}$$

Combining (8.5) and (8.6) we see that (8.3) becomes

$$\begin{aligned} c_0 &= v_1 \\ c_0 + c_1 \frac{1}{2} &= v_2 \end{aligned} \tag{8.7}$$

and we obtain $c_0 = 1$ and $c_1 = -6$.

A special situation occurs when the interpolation values $v_1, \ldots, v_n$ are all zero. In this case, system (8.4) gives $c_0 = c_1 = \ldots = c_{n-1} = 0$, so the function T given by Newton's representation method is the zero function. A nontrivial interpolant for this case is easily obtained by setting

$$T_0(s) = \gamma \frac{(s-s_1)\cdots(s-s_n)}{(s+a)^n} \tag{8.8}$$

where γ is any nonzero real constant. Observe that the function T_0 in (8.8) has no zeros in the *RHP* other than $s_1, \ldots, s_n$.

8.1.2 Parameterization of all interpolants

Given a set *INT* of interpolation conditions, by INT_0 we denote the set of homogeneous interpolation conditions

$$INT_0 \qquad T(s_1) = 0,\ T(s_2) = 0,\ \ldots,\ T(s_m) = 0$$

obtained from *INT* by setting the right hand sides to 0. The set of conditions INT_0 plays a fundamental role in parameterizing all functions $T \in \mathcal{RH}^\infty$ that satisfy *INT*.

Theorem 6 *Let a set INT of interpolating conditions be given. Also given are functions T_1, T_0 in $\mathcal{RH}^\infty$ such that T_1 satisfies INT, T_0 satisfies INT_0, $s_0, \ldots, s_n$ are the only zeros of T_0, and T_0 has relative degree 0.*

Then every T in $\mathcal{RH}^\infty$ that satisfies INT has the form

$$T(s) = T_1(s) + T_0(s)F(s) \tag{8.9}$$

for some F in $\mathcal{RH}^\infty$. Conversely, if a function T is given by the formula (8.9), then T belongs to $\mathcal{RH}^\infty$ and satisfies INT.

Proof. Let T be a function in $\mathcal{RH}^\infty$ that satisfies INT. Then the difference $T(s) - T_1(s)$ has zeros at the locations $s = s_1, \ldots, s = s_n$. Therefore the rational function $F(s) = (T(s) - T_1(s))/T_0(s)$ has no poles in the closed RHP, and is proper since T_0 has relative degree 0. This says that $F(s)$ belongs to $\mathcal{RH}^\infty$, and we have proved the first part of the theorem. To prove the second part, suppose that T in $\mathcal{RH}^\infty$ satisfies (8.9). By direct substitution of the values $s = s_0, s_1, \ldots, s_n$ in the formula (8.9) we see that T satisfies INT. ■

It follows from Theorem 6 that it is enough to find particular functions T_0 and T_1, as in formula (8.9), to determine all $\mathcal{RH}^\infty$ functions that satisfy *INT*.

Example. We will find a formula for all functions $T \in \mathcal{RH}^\infty$ that satisfy

$$T(1) = 2, \quad T(3) = 4 \tag{8.10}$$

We can choose any negative number as the location of the pole of the rational functions T_0 and T_1 in formula (8.9). We select $a = -2$;

$$T_0(s) = \frac{(s-1)(s-3)}{(s+2)^2} \tag{8.11}$$

and

$$T_1(s) = c_0 + c_1\frac{s-1}{s+2} \tag{8.12}$$

By taking $s = 1$ and $s = 3$ in (8.12) and combining with (8.10), we obtain $c_0 = 2$ and $c_1 = 5$. We know from Theorem 1 that all functions $T \in \mathcal{RH}^\infty$ that satisfy (8.10) are of the form

$$T(s) = 2 + 5\frac{s-1}{s+1} + \frac{(s-1)(s-3)}{(s+1)^2}F(s) \tag{8.13}$$

where $F \in \mathcal{RH}^\infty$.

8.1.3 Interpolation with a relative degree condition

If a rational function T is strictly proper, then as $|s|$ gets large, $|T(s)|$ approaches 0. One can think of T as having a zero at $s = \infty$. The order we associate with this "zero" is the same number as the *relative degree of* T, $d(T)$ (the difference of degrees between denominator and numerator). One way to incorporate this special zero in interpolation problems is to specify the relative degree of the interpolant together with INT.

Given INT and an integer $r \geq 0$, we seek T in $\mathcal{RH}^\infty$ with relative degree $d(T) = r$ that satisfies INT. To produce one interpolant, modify the representation (8.3) to include the requirement on the relative degree of T:

$$\begin{aligned} T(s) = \ & \tfrac{1}{(s-a)^r}\Big(c_0 + c_1\tfrac{s-s_1}{s-a} + c_2\tfrac{(s-s_1)(s-s_2)}{(s-a)^2} + \cdots \\ & \cdots + c_{n-1}\tfrac{(s-s_1)\cdots(s-s_{n-1})}{(s-a)^{n-1}}\Big) \end{aligned} \tag{8.14}$$

To find $c_0,\ c_1,\ \ldots,\ c_{n-1}$, proceed as before by forming a triangular system of equations and solving it.

A parameterization of all interpolants that satisfy both INT and the relative degree condition $d(T) = r$ is obtained after a small modification to Theorem 6. Details of the proof of Theorem 7 below are left to the reader.

Theorem 7 *Assume the hypotheses of Theorem* 1, *with the additional assumptions that* $d(T_1) \leq r$ *and* $d(T_0) = r$. *Then a rational function* T *is a function in* $\mathcal{RH}^\infty$ *that has relative degree* r *and satisfies* INT *if and only if there exists a rational* $F \in \mathcal{RH}^\infty$ *such that*

$$T(s) = T_1(s) + T_0(s)F(s) \tag{8.15}$$

8.2 Plants with simple RHP zeros and poles

Internal stability can be expressed as the statement that certain functions associated with $\mathcal{S}$ are bounded and have no poles in the closed RHP. This fact is presented below as a lemma needed for the main theorems of this section, but it is interesting in itself. In fact, it can be used as an alternative definition of internal stability that generalizes to MIMO systems (see [BB], [FDT], and [Fr]).

Lemma 1 *The closed loop system of Figure* 2.1 *is internally stable if and only if*

$$Q,\ T,\ S,\ SP \quad \textit{belong to} \quad \mathcal{RH}^\infty \tag{8.16}$$

Proof. Key relations are $S = 1 - T$ and $T = QP = PQ$. These imply that T is in $\mathcal{RH}^\infty$ if and only if S and $PQ = QP$ are in $\mathcal{RH}^\infty$. If the system is internally stable, we only have to check that $Q = CS$ and SP belong to $\mathcal{RH}^\infty$. Note that

$$S(s_p) = \frac{1}{(1+PC)}(s_p) = 0 \tag{8.17}$$

at the poles s_p of C. Thus $Q = CS$ is in $\mathcal{RH}^\infty$. Similar reasoning implies that

$$SP = S\frac{1}{1+PC} \tag{8.18}$$

is in $\mathcal{RH}^\infty$. This proves one half of the lemma. Now suppose that the relations in (8.16) hold. If there is pole-zero cancellation in PC at a point $s = s_0$ in the *RHP*, then either $Q = C/(1+PC)$ or $SP = P/(1+PC)$ has a pole at $s = s_0$, both of which contradict the hypothesis. ∎

To illustrate the use of this lemma, we verify (8.16) for the system $\mathcal{S}_1$ with plant $P(s) = 1/(s-1)$ and compensator $C_1(s) = (s-1)/(s+1)$. In Section 2.3 we saw that $T_1(s) = 1/(s+2)$, which belongs to $\mathcal{RH}^\infty$. We also have that

$$P(s)S_1(s) = P(s)(1 - T_1(s)) = \frac{1}{s-1}\left(1 - \frac{1}{s+2}\right) = \frac{s+1}{(s-1)(s+2)} \tag{8.19}$$

We see that the closed loop plant PS_1 has a pole at $s = 1$. By Lemma 1 we can assert that the system $\mathcal{S}_1$ is not internally stable.

Now consider the system $\mathcal{S}_2$ with the same plant and compensator $C_2(s) = 2$. Recall that $T_2(s) = 2/(s+1)$, so

$$\begin{array}{lll} P(s)S_2(s) & = \frac{1}{s-1}\frac{s-1}{s+1} & = \frac{1}{s+1} \\ Q_2(s) & = C_2(s)S_2(s) & = 2\frac{s-1}{s+1} \end{array} \tag{8.20}$$

Therefore (8.16) is satisfied, and the system $\mathcal{S}_2$ is internally stable.

The description of internal stability in terms of interpolation conditions is given in the following theorem.

Theorem 8 *Consider the closed loop system $\mathcal{S}$, where the plant P is given. Let ρ_ℓ ($\ell = 1, \ldots, n$) denote the poles and z_ℓ ($\ell = 1, \ldots, m$) denote the zeros of the plant P in the closed* RHP. *Suppose that all unstable zeros and poles of the plant are simple. If $\mathcal{S}$ is internally stable, then there exists an integer $d_c \geq 0$ such that the closed loop transfer function $T = PC(1+PC)^{-1}$ must satisfy interpolation conditions*

$$\left\{ \begin{array}{ll} T(\rho_\ell) = 1 & (\ell = 1, \ldots, n) \\ T(z_\ell) = 0 & (\ell = 1, \ldots, m) \\ d(T) = d(P) + d_c & \end{array} \right. \tag{8.21}$$

Conversely, if T is any function $\mathcal{RH}^\infty$ satisfying (8.21) for some $d_c \geq 0$, then the closed loop system $\mathcal{S}$ associated with T and P is internally stable.

Proof. Suppose the system $\mathcal{S}$ is internally stable. Then $P^{-1}T = C(1+PC)^{-1} = Q \in \mathcal{RH}^\infty$. Clearly, if $P(z_\ell) = 0$ then z_ℓ is a pole of P^{-1}. Thus $T(z_\ell) = 0$. Also $SP = (1+PC)^{-1}P \in \mathcal{RH}^\infty$, so poles of P are zeros of $(1+PC)^{-1} = 1 - T$. Then $T(\rho_\ell) = 1$. We now set $d_c = d(C)$ in equation (8.21) to finish the first part of the proof.

Now suppose that $T \in \mathcal{RH}^\infty$ satisfies (8.21). The relation $d(T) - d(P) = d_c \geq 0$ combined with (8.21) insures that the compensator is proper. This fact, and both P and T having the same RHP zeros imply that $Q = TP^{-1}$ is proper and has no RHP poles, i.e., $P^{-1}T = Q \in \mathcal{RH}^\infty$. Also $S = 1 - T \in \mathcal{RH}^\infty$. Again, (8.21) says that poles of P are zeros of $1 - T$, thus $(1-T)P = SP \in \mathcal{RH}^\infty$. Therefore $\mathcal{S}$ is internally stable. ■

Theorem 8 can be combined with Theorem 7 to obtain a formula describing all possible internally stable systems $\mathcal{S}$ associated with a fixed plant. The proof of Theorem 9 is left to the reader.

Theorem 9 *Let P be a proper real rational function, with simple RHP zeros $z_1, \ldots, z_n$ and RHP poles $\rho_1, \ldots, \rho_m$, and let d_c be a nonnegative integer. Let T_0, T_1 be $\mathcal{RH}^\infty$ functions such that*

$$\begin{cases} T_1(\rho_\ell) = 1, & (\ell = 1, \ldots, n) \\ T_1(z_\ell) = 0, & (\ell = 1, \ldots, m) \\ d(T_1) \geq d(P) + d_c \end{cases} \tag{8.22}$$

and

$$\begin{cases} T_0(\rho_\ell) = 0, & (\ell = 1, \ldots, n) \\ T_0(z_\ell) = 0, & (\ell = 1, \ldots, m) \\ d(T_0) = d(P) + d_c \end{cases} \tag{8.23}$$

and suppose that all the RHP zeros of T_0 are listed in (8.23).

If $T \in \mathcal{RH}^\infty$ is the closed loop transfer function of an internally stable system $\mathcal{S}$ with plant P and $d(C) = d_c$, then there exist $H \in \mathcal{RH}^\infty$ such that

$$T(s) = T_1(s) + T_0(s)H(s) \tag{8.24}$$

Conversely, if (8.24) holds for some $H \in \mathcal{RH}^\infty$, then the system $\mathcal{S}$ associated with P and T is internally stable, and the degree of its compensator is $d(C) = d_c$.

Example. Consider the plant $P(s) = (s-1)/(s-3)$, and set $d_c = 0$. We now find a formula for all T that yields an internally stable system $\mathcal{S}$ with degree 0 compensator and plant $P(s) = (s-1)/(s-3)$.

The plant P has RHP zero $s = 1$ and RHP pole $s = 3$. Set $T_1(s) = 2(s-1)/(s+1)$ and $T_0(s) = (s-1)(s-3)/(s+1)^2$. Then T_1 and T_0 satisfy (8.22) and (8.23) respectively, and by Theorem 9 we have

$$T(s) = \frac{2(s-1)}{s+1} + \frac{(s-1)(s-3)}{(s+1)^2}H(s), \ (H \in \mathcal{H}^\infty).$$

8.3 Parameterization: the general case

In this section, the results already obtained on parameterization of internally stable systems is generalized to the case where the plant may have RHP zeros or poles with multiplicity greater than 1. Results in this section are stated without proof.

8.3.1 Higher order interpolation

The interpolation condition we now consider specifies values of the derivatives of the function at selected points.

$$INT_h \left\{ \begin{array}{llll} T(s_1) = v_{1,1}, & T^{(1)}(s_1) = v_{1,2} & \ldots, & T^{(m_1-1)}(s_1) = v_{1,m_1} \\ T(s_2) = v_{2,1}, & T^{(1)}(s_2) = v_{2,2} & \ldots, & T^{(m_2-1)}(s_2) = v_{2,m_2} \\ T(s_n) = v_{n,1}, & T^{(1)}(s_n) = v_{n,2} & \ldots, & T^{(m_n-1)}(s_n) = v_{n,m_n} \\ d(T) = r & & & \end{array} \right. \tag{8.25}$$

where[1] $r \geq 0$, $\{s_1, s_2, \ldots, \ s_n\} \subset RHP$ and

$$\{v_{11}, \ldots, v_{1m_1}, v_{21}, \ldots, v_{2m_2}, \ldots, v_{n1}, \ldots, \ v_{nm_n}\} \subset C$$

are given. As in previous sections, we consider the problems of producing one interpolant T_1 for INT_h, and of obtaining a parameterization of all interpolants for INT_h.

We use two $\mathcal{RH}^\infty$ functions, T_0 and T_1. The assumptions on these are that T_1 satisfies INT_h, and that T_0 satisfies the corresponding set of homogeneous conditions INT_h^0. Furthermore, suppose that all the RHP zeros of T_0 are listed in INT_0^h, and that $T_0^{(m_\ell)}(s_\ell) \neq 0$, $(\ell = 1, \ldots, n)$.

A variation of the method we have been using works for higher order interpolation problems as well. The difference here is that the formula for T now contains summands with numerators of the form 1, $(s-s_1)$, $(s-s_1)^2$, …, $(s-s_1)^{m_1-1}$, $(s-s_2)(s-s_1)^{m_2-1}$, ….

---

[1] The inclusion $\{s_1, \ldots, s_n\} \subset RHP$ is not required for interpolation. However, it is essential for internal stability.

Theorem 10 *Assume the hypotheses of Theorem 2. Then a function T is in $\mathcal{RH}^\infty$, has relative degree r and satisfies INT^h if and only if there exists $H \in \mathcal{RH}^\infty$ such that*

$$T(s) = T_1(s) + T_0(s)H(s). \tag{8.26}$$

Example. Find $T \in \mathcal{RH}^\infty$ with relative degree $d(T) = 0$ such that

$$T(0) = 1 \ , \ T^{(1)}(0) = 2 \ , \ T(3) = 4 \ , \ T(4) = 0 \tag{8.27}$$

Solution. The expression for T is

$$T(s) = c_0 + c_1 \frac{s}{s+1} + c_2 \frac{s^2}{(s+1)^2} + c_3 \frac{s^2(s-3)}{(s+1)^3}. \tag{8.28}$$

We also need the derivative of T,

$$T^{(1)}(s) = \frac{c_1}{(s+1)^2} + \frac{2c_2 s}{(s+1)^3} + \frac{c_3(-6s+6s^2)}{(s+1)^4}. \tag{8.29}$$

Setting $s = 0$ in (8.28) and in (8.29) gives $c_0 = 1$ and $c_1 = 2$. Now $s = 3$ in (8.28) gives

$$4 = c_0 + c_1 \frac{3}{4} + c_2 \frac{9}{16} = 1 + 2 \frac{3}{4} + c_2 \frac{9}{16}. \tag{8.30}$$

Solving for c_2 in (8.30) yields $c_2 = \frac{8}{3}$. Finally $s = 4$ in (8.28) leads to

$$0 = c_0 + c_1 \frac{4}{5} + c_2 \frac{16}{25} + c_3 \frac{32}{125} \ , \tag{8.31}$$

and this implies that $c_3 = \frac{-1615}{48}$.

8.3.2 Plants with multiple RHP zeros and poles

A RHP pole or zero of the plant with multiplicity higher than 1 in an internally stable system can be translated as an interpolation condition on derivatives of T at that location. More precisely:

Theorem 11 *Consider the closed loop system $\mathcal{S}$. Let ρ_ℓ ($\ell = 1, \ldots, n$) denote the poles and z_ℓ ($\ell = 1, \ldots, m$) denote the zeros of the plant P in the closed* RHP, *so that these poles and zeros have multiplicity n_ℓ, $\ell = 1, \ldots, n$ and m_ℓ, $\ell = 1, \ldots, m$, respectively. If $\mathcal{S}$ is internally stable, then for some* *integer* $d_c \geq 0$ *the closed loop transfer function* $T = PC(1+PC)^{-1}$ *satisfies the interpolation conditions*

$$INT^h \left\{ \begin{array}{llll} T(\rho_\ell) = 1, & T^{(1)}(\rho_\ell) = 0 & \ldots, & T^{(n_\ell - 1)}(\rho_\ell) = 0 \quad (\ell = 1, \ldots, n) \\ T(z_\ell) = 0, & T^{(1)}(z_\ell) = 0 & \ldots, & T^{(m_\ell - 1)}(z_\ell) = 0 \quad (\ell = 1, \ldots, m) \\ d(T) = & d(P) + d_c & & \end{array} \right. \tag{8.32}$$

Conversely, if T *is any function* $\mathcal{RH}^\infty$ *that satisfies* INT^h *for some* $d_c \geq 0$, *then the closed loop system* $\mathcal{S}$ *associated with* T *is internally stable and its compensator has relative degree* d_c.

We now state the result with the parameterization of internally stable systems.

Theorem 12 *Assume the hypotheses of Theorem 11, and let* T_1, T_0 *in* $\mathcal{RH}^\infty$ *be such that* T_1 *satisfies* INT^h, *and* T_0 *satisfies* INT_0^h. *Furthermore, suppose that all the RHP zeros of* T_0 *are listed in* INT_0^h, *and that* $T_0^{n_\ell}(\rho_\ell) \neq 0$, $(\ell = 1, \ldots, n)$, $T_0^{m_\ell}(z_\ell) \neq 0$, $(\ell = 1, \ldots, m)$.

If $T \in \mathcal{RH}^\infty$ *is the closed loop transfer function of an internally stable system* $\mathcal{S}$ *with plant* P *and* $d(C) = d_c$, *then there exist* $H \in \mathcal{RH}^\infty$ *such that*

$$T(s) = T_1(s) + T_0(s)H(s) \tag{8.33}$$

Conversely, if (8.33) holds for some $H \in \mathcal{RH}^\infty$, *then the system* $\mathcal{S}$ *associated with* P *and* T *is internally stable, and the degree of its compensator is* $d(C) = d_c$.

Example. If $P(s) = \frac{(s+3)(s-2)}{s^2}$ and $d_c = 1$, then the interpolation conditions on any $T \in \mathcal{RH}^\infty$ for internal stability of the system $\mathcal{S}$ are given by

INT $\quad T(0) = 1,\ T'(0) = 0,\ T(2) = 0,\ d(T) = 1.$

If $a = -1$ is selected as the location of the poles of T, then the technique described in Subsection 8.3.1 yields the interpolant

$$\frac{-15s^2}{4(1+s)^3} + \frac{s}{(1+s)^2} + \frac{1}{1+s} \tag{8.34}$$

Thus we have the parameterization

$$T(s) = \frac{-15s^2}{4(1+s)^3} + \frac{s}{(1+s)^2} + \frac{1}{1+s} + \frac{s^2(s-2)}{(1+s)^4}\, T_1(s), \qquad \forall T_1 \in \mathcal{RH}^\infty. \tag{8.35}$$

8.4 Exercises

1. 1. Find an interpolant for each set of conditions below:

   a. $T(1) = 0$, pole location $s = -1$.
   b. $T(2+j) = 3$, $T(2-j) - 3$, pole location $s = -10$.
   c. $T(0) = 0$, $T(1) = 0$, $T(2) = 0$, pole location $s = -5$.
   d. $T(1) = 2$, $T(3+j) = 0$, $T(3-j) = 0$, pole location $s = -1$.

   2. Prove that the system of equations (8.4) does have a solution in $c_0, \ldots, c_{n-1}$; and that this solution is unique.

   3. Prove that if the function $T \in \mathcal{RH}^\infty$ satisfies $T(s_1) = 0, \ldots, T(s_n) = 0$, then either T is identically zero or degree $\{\text{denominator}(T)\} \geq n$.

   4. Prove that given *INT* with n data points, *there exists a unique* $T \in \mathcal{RH}^\infty$ that satisfies *INT*, with the following property: if p and q are polynomials such that $T = p/q$, then *degree*(q) $< n$.

2. 1. Find *all* interpolants satisfying conditions stated in Exercises 1.a–d.

   2. Prove Theorem 6.

3. 1. Find an interpolant for each set of conditions below:

   a. $T(1) = 0$, pole location $s = -1$, relative degree $d(T) = 2$.
   b. $T(2+j) = 3$, $T(2-j) = 3$, pole location $s = -10$, $d(T) = 1$.
   c. $T(0) = 2$, $T(j) = 1 + 2j$, $T(-j) = 1 - 2j$, pole location $s = -1$, $d(T) = 1$.

   2. Prove the statements below:

   a. If both $T_1, T_2 \in \mathcal{RH}^\infty$ have relative degree r, then $T_1 + T_2$ has relative degree $\leq r$. Give an example where strict inequality occurs.
   b. If $T_1, T_2 \in \mathcal{RH}^\infty$ have relative degrees r_1, r_2 respectively, then the product T_1T_2 has relative degree $r_1 + r_2$.

   3. Find *all* functions $T \in \mathcal{RH}^\infty$ that satisfy the conditions stated in Exercise 1a–c.

   4. Prove Theorem 2.

4. Consider the following system:

Define the "closed loop compensator" Q to be $Q = C(1 + CP)^{-1}$.

a. Check that Q is the $e_1 \to e_2$ transfer function.
b. What interpolation conditions must Q satisfy for the system to be internally stable?
c. Do all Q meeting these interpolation conditions correspond to an internally stable system?

5. Find a formula for all closed loop transfer functions T that correspond to an internally stable system $\mathcal{S}$, if the relative degree of the compensator is 0 and the plant $P(s)$ is given by

   a. $1/(s-3)$
   b. $(s^2-4)/(s^2-1)$
   c. $(s-5)/(s^2+1)$
   d. $(s+2)/(s+4)$

6. Do 6a – d, but now with $d(C) = 1$.

7. Use the results in Section 8.3 to find the closed loop T that arise from internally stable systems with degree of compensator $d_c = 0$, and plant

   a. $(s^2-2s+1)/(s^2-4)$
   b. $1/(s-2)^2$
   c. $(s+2)/(s-2)^2$
   d. Do 8a – c with $d(C) = 1$.

8. Supply a proof for the results stated in Section 8.3.

8.5 References and further reading

References to amplify our presentation of internal stability, interpolation and control (but with a different point of view) are

[BB] Stephen P. Boyd and Craig P. Barrat, *Linear Controller Design*, Prentice Hall, 1991.

[DC] Carl DeBoor and Richard Conte, *Elementary Numerical Analysis: an algorithmic approach.* McGraw Hill, New York, 1980.

[F] Bruce Francis. *A Course in $\mathcal{RH}^\infty$-Control.* Springer Verlag, 1985.

[V] M. Vidyasagar. *Control Systems Synthesis.* The MIT Press, 1984.

[BGR] J. Ball, I. Gohberg, L. Rodman. *Interpolation of Rational Matrix Functions.* Operator Theory Advances and Applications vol. 45, Birkhauser, New York, 1990.

Part III

H^∞ Theory

In this part of the book we discuss the general optimization problem OPT, which is the main mathematical tool on which our control design method is based. The discussion is restricted to scalar valued analytic functions. In control applications this corresponds to the SISO case.

In Chapter 9 the mathematical problem OPT is formulated, and its relevance to Control is treated. We introduce performance functions and sublevel sets of a performance function.

In Chapter 10 we give a test for determining if a given function is a solution to OPT. This one simple characterization of optima is the key to understanding OPT both conceptually and numerically. This test has a beautiful graphical interpretation that is also easy to implement on a computer.

Chapter 11 contains background facts on analytic functions and lists three fundamental properties of analytic functions, which every control engineer should know.

Chapter 12 gives the proof of the characterization given in Chapter 10 of solutions to OPT. The proof follows quickly from the three basic properties listed in Chapter 3.

Chapter 13 concerns computations of solutions to OPT. We begin with computer diagnostics, follow with a sample computer run, and then we discuss an algorithm for finding solutions.

Chapter 9

H^∞-Optimization and Control

9.1 The problem OPT

Let X be a subset of the complex plane C, and let $f : X \to C$ be a continuous function. The function f is said to be

- *bounded* (uniformly) if there exists $m > 0$ such that $|f(s)| < m$ for all $s \in X$. The symbol $\sup_{s \in X} |f(s)|$ denotes the smallest of such m.
- *real* (on the real axis) if it satisfies[1] $f(s) = \overline{f(\overline{s})}$.
- *analytic* if it can be expanded as a power series about any point s_0 in the interior of X and convergent in some (possibly small) disk centered at s_0.

We denote with A_{RHP} the set of all functions $f(s)$ that are real, bounded and continuous in the closed Right–Half–Plane (*R.H.P.*), and that are *analytic* on the open *R.H.P.* Therefore, $f(j\omega)$ is a bounded, continuous and real function of ω, for any $f(s) \in A_{RHP}$. The set A_{RHP} is a (real) vector space with the usual properties of addition

$$(f+g)(s) = f(s) + g(s)$$

and scalar multiplication

$$(\alpha f)(s) = \alpha f(s), \quad (\forall \alpha \in R)$$

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[1] This definition makes sense only for sets X that are symmetric about the x-axis.

It is well known from the theory of analytic functions that an f continuous on a closed set and analytic on the interior of the set achieves the maximum modulus on the boundary of the set. As a consequence of this we have that any f in A_{RHP} satisfies

$$\sup_{\omega \in R} |f(j\omega)| = \sup_{s \in RHP} |f(s)|$$

A positive function of $\omega \in \mathbf{R}$ and $z \in C$ is called a *performance function*. We will use the notation $\Gamma(\omega, z)$ to denote a performance function. A performance function Γ that satisfies

$$\Gamma(\omega, z) = \Gamma(-\omega, \bar{z}) \tag{9.1}$$

is said to be *real symmetric*. Performance functions that arise from engineering applications are real symmetric, consequently all the performance functions considered in this book are real symmetric. So from now on, by the word "performance" what we really mean is "real symmetric performance".

The main optimization problem we consider[2] is,

OPT Given a performance function $\Gamma(\omega, z)$, find $\gamma^* \geq 0$ and f^* in A_{RHP} such that

$$\gamma^* = \inf_{f \in A_{RHP}} \sup_{\omega} \Gamma(\omega, f(j\omega)) = \sup_{\omega} \Gamma(\omega, f^*(j\omega)).$$

We will refer to the function f^* in the statement of OPT as a *solution* or *minimizer*, and refer to γ^* as the *optimal value*. Solutions f^* to OPT do not always exist in A_{RHP}, however for many performance functions they do (see Section 10.4).

9.2 The fundamental H^∞ problem of control

The OPT problem is central to the design of a system where specifications are given in the frequency domain and where stability is a key issue. Suppose that our objective is to design a linear, time invariant system in the frequency domain description, with stability a major consideration. Part of the system is given and we are forced to use (called the plant), and another part f we get to choose (designable part) (see Fig. 1). The stipulation that the designable part of the circuit be stable can be restated as f having no poles in the right half plane.

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[2] As stated here, OPT corresponds to SISO system design. The case MIMO leads to OPT with admissible functions $\mathbf{f} = (f_1, \ldots, f_N)$, with each $f_\ell \in A_{RHP}$. See Chapter 15.

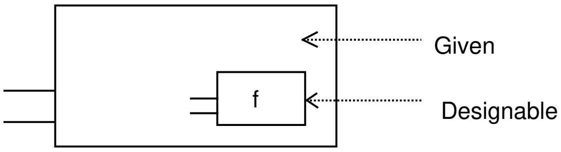

Figure 9.1: The given and designable parts of the system

Figure 9.2: The function f makes performance at least as good as c at $\omega = \omega_0$.

Suppose also that to carry out the design (i.e., choose f) we use a given performance function Γ. In many cases the performance takes the form of a positive function of frequency ω and of the designable part f at that frequency.

Thus we write $\Gamma(\omega, f(j\omega))$ for the performance at frequency ω and for designable function f. The performance is a cost function: big is bad. The "worst case" is the frequency ω at which $\sup_{\omega \in R} \Gamma(\omega, f(j\omega))$ occurs. We minimize this over all $f \in A_{RHP}$. Thus we obtained exactly the OPT problem.

The theory of OPT is based in part on the analysis of the sublevel sets

$$\mathcal{S}_\omega(c) = \{z \in \mathbf{C} \colon \Gamma(\omega, z) < c\}$$

of Γ. In fact there is a simple picture to think of in connection with a design. For each desired level of performance c, one has target sets $\mathcal{S}_\omega(c)$ associated with each frequency ω. The objective is to find a function f with no poles in the R.H.P. such that each $f(j\omega)$ belongs to $\mathcal{S}_\omega(c)$. Any such f makes the performance of the overall system at least as good as c for all ω. See Fig. 9.2.

9.3 Change of coordinates from RHP to the unit disk

Because it is more convenient[3] we now change coordinates in the domains of functions, and use ζ in the unit disk D in the complex plane, and $e^{j\theta}$ in the unit circle instead of s in RHP, and $j\omega$ in the imaginary axis.

The change of coordinates can be accomplished with a linear fractional transformation:

$$\begin{aligned} s &= \tfrac{1+\zeta}{1-\zeta} \\ j\omega &= \tfrac{1+e^{j\theta}}{1-e^{j\theta}} \end{aligned} \tag{9.2}$$

so that functions $f(s)$ of s become functions of ζ:

$$\begin{aligned} f(s) &= f(\tfrac{1+\zeta}{1-\zeta}) &= F(\zeta) \\ f(j\omega) &= f(\tfrac{1+e^{j\theta}}{1-e^{j\theta}}) &= F(e^{j\theta}) \end{aligned} \tag{9.3}$$

The transformation (9.2) maps the open RHP one-to-one and onto the open unit disk D, and maps the $j\omega$ axis with $\omega = \infty$ one-to-one and onto the unit circle.

A fundamental property of this coordinate change is that functions remains analytic, bounded and real: $f(s)$ is analytic, bounded and real for $s \in RHP$ if and only if $F(\zeta)$ is analytic, bounded and real for $\zeta \in D$. The following table gives some pairs (s, ζ) that arise from this transformation.

s	∞	j	0	1	-1
ζ	1	j	-1	0	∞

Example. We perform the change of coordinates on the function $f(s) = (s-3)/(s+2)$. We have,

$$f(s) = \frac{s-3}{s+2} = \frac{\frac{1+\zeta}{1-\zeta} - 3}{\frac{1+\zeta}{1-\zeta} + 2} = \frac{\frac{-2+4\zeta}{1-\zeta}}{\frac{3-\zeta}{1-\zeta}} = \frac{-2+4\zeta}{3-\zeta} = F(\zeta) \tag{9.4}$$

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[3] Mathematics literature treats functions on the disk rather the RHP. There are other reasons, e.g. our graphical test in Section 10 is easier on the disk, Fourier transforms and their discrete versions are available and easy to implement on the computer, etc.