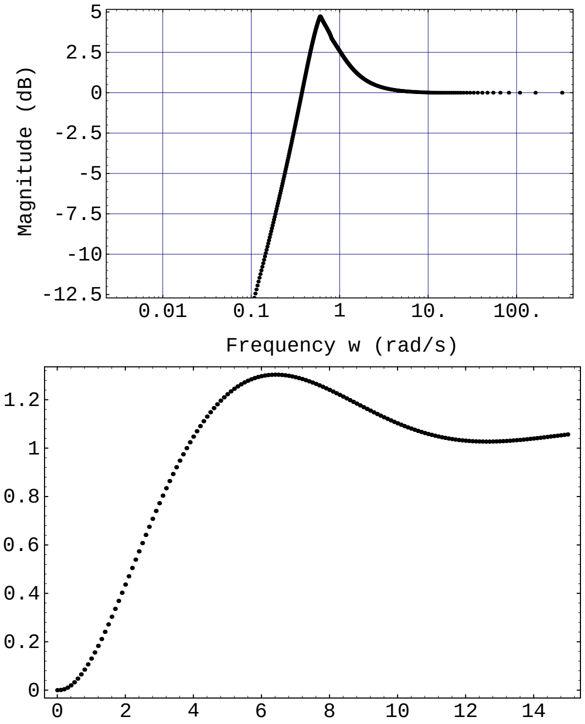

Figure 6.27: Plots of magnitude of optimal sensitivity S and of step response for design problem 2.

The set S_ω of all possible $T(j\omega)$ that satisfy both (6.39) and (6.40) is a subset of the complex plane. This set can be described as follows:

$$S_\omega = \{\ s :\ max(\ 2|s-1|\ ,\ \frac{5}{6}|s|\ ) \leq 1\ \}$$

Note that S_ω is the intersection of two disks. Another important feature of S_ω is that its boundary has " corners ".

6.5.2 Performance from competing constraints

The function Γ in (6.36) is not smooth even if the functions Γ_k are smooth. This can be seen from the *sublevel sets*

$$S_\omega(\gamma) = \{\ s :\ \Gamma(\omega, s) \leq \gamma\ \}, \tag{6.41}$$

which in many cases contain corners on their boundary (see Fig. 6.28). Such functions $\Gamma(\omega, z)$ are not differentiable in z. This is important since

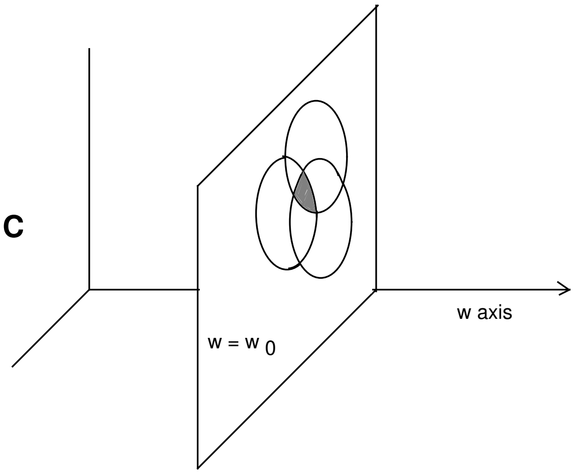

Figure 6.28: Sublevel sets S_ω from competing constraints

some algorithms for solving *OPT* require smooth functions Γ. Thus to solve *OPT* with these algorithms one must compromise (see Section 6.5.3).

Note that S_ω is convex if it is the intersection of convex sets, such as disks. Algorithms do exist that handle functions Γ with convex sublevel sets, even if these have corners. See [BB] for more details on these algorithms.

6.5.3 Solving *OPT*

To solve *OPT* for smooth Γ, gradient-descent/Newton type of iterative algorithms can be used[4] (see [HMer]). These algorithms do not apply directly when Γ is of the form (6.36). However, functions Γ from competing constraints can be handled when certain compromises are made.

To simplify, the discussion we take $m = 2$ in (6.36) to obtain the performance function

$$\Gamma(\omega, z) = \max(\Gamma_1(\omega, z), \Gamma_2(\omega, z)) \qquad (6.42)$$

We describe in Subsections 6.5.4 and 6.5.5 functions $\Gamma'(\omega, z)$ that are *smooth* approximations to Γ. *OPT* for Γ' can be solved, and then the optimal T is

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[4] The software OPTDesign is an implementation of these algorithms.

evaluated with the original requirements. In many cases this procedure is good enough to produce systems that satisfy all the constraints.

6.5.4 Smoothing the performance function

One method to obtain a smooth performance function Γ from (6.42), if Γ_1 and Γ_2 are smooth, is based on optimizing the following performance function:

$$\Gamma'(\omega, T(j\omega)) = \Gamma_1(\omega, T(j\omega))^2 + \Gamma_2(\omega, T(j\omega))^2 \tag{6.43}$$

Suppose that T is such that

$$\Gamma'(\omega, T(j\omega)) \leq 1 \tag{6.44}$$

Since for any positive numbers x and y,

$$\max(x, y) \leq \sqrt{x^2 + y^2}$$

it follows that both

$$\Gamma_1(\omega, T(j\omega)) \leq 1 \tag{6.45}$$

and

$$\Gamma_2(\omega, T(j\omega)) \leq 1 \tag{6.46}$$

hold. Therefore, if solving OPT with the function Γ' as performance yields an optimal value $\gamma^* \leq 1$, then the optimal T satisfies both (6.45) and (6.46). One can try variations on the formula (6.43), such as[5] using an exponent higher than 2 in 6.43.

6.5.5 Weight functions

Choose weight functions of frequency W_1 and W_2, and form

$$\Gamma'(\omega, z) = W_1(j\omega)\Gamma_1(\omega, z) + W_2(j\omega)\Gamma_2(\omega, z) \tag{6.47}$$

These weight functions can be chosen to emphasize either Γ_1 or Γ_2 at each frequency ω. A performance function of the type (6.47) is used in Design Example 3 below.

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[5] This is more demanding computationally; on the other hand, $\sqrt[n]{x^n + y^n} \approx max(x, y)$ if n is large.

6.6 Design Example 3

6.7 References

[La] Brett Larson. Siso Robust Controller Design via the H^∞ Method. Master Thesis, under the direction of Prof. F. Bailey. University of Minnessota, April 1989.

[HMer] J. W. Helton and O. Merino.

[BB] Stephen P. Boyd and Craig P. Barrat, *Linear Controller Design*, Prentice Hall, 1991.

[FPE] G. F. Franklin, J. D. Powell and A. Emami-Naeini. *Feedback Control of Dynamic Systems.* Addison-Wesley, 1986.

Chapter 7

Time domain requirements

This chapter discusses time domain requirements in more detail than was discussed in Chapter 5 (Section 7.1), and presents mathematical results on translating these requirements to the frequency domain (Section 7.2). These results are used in a solved example (Section 7.3).

7.1 Requirements on the step response function

In the time domain, the performance of systems is usually described in terms of output functions, produced by specific input functions. Typical input functions used for this are the unit impulse, unit step, ramp, and sinusoidal functions. We concentrate our study on the step response function $c(t)$, that is, the output produced when the input is the unit step function.

7.1.1 Step response indices

A step response function $c(t)$ is usually described in terms of a set of numbers, or *indices*, that depend on the function. These include the overshoot, undershoot, settling time, and rise time [1]. The definitions below are taken

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[1] Another common index is the time delay. It is usually defined as the time it takes the step response curve $c(t)$ to reach 0.5. This definition is ambiguous unless $c(t)$ is a priori known to be extremely well behaved.

from [BB, Chapter 8]. Here $c(t)$ represents the step response function of a *given* system $\mathcal{S}$.

- **Overshoot**: $M_{os} = \sup_{t \geq 0} c(t) - 1$.
- **Undershoot**:$M_{us} = \inf_{t \geq 0} c(t)$.
- **Rise time**: $t_{rise} = \inf\{t_0 : c(t) \geq 0.8 \; for \; t \geq t_0\}$.
- **Settling time**: $t_{settle} = \inf\{t_0 : |c(t) - 1| \leq 0.05 \; for \; t \geq t_0\}$.

The indices above are depicted in Fig 7.1.

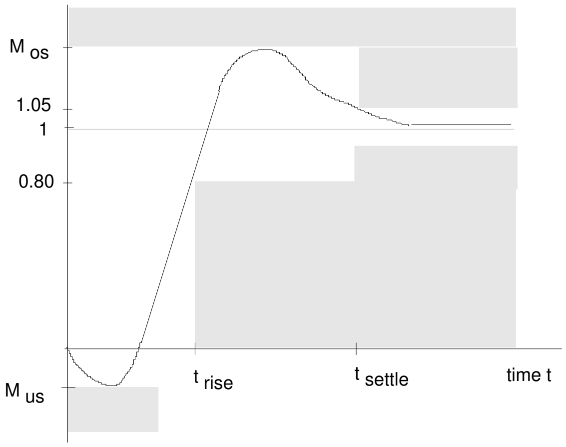

Figure 7.1: The step response function indices

7.1.2 Time domain performance requirements

The time domain indices listed above give information about a particular curve $c(t)$. To express goals of the design process in the time domain we need the definition below.

Definition 2 *A time domain performance requirement on the step response function $c(t)$ is an inequality involving $c(t)$ and an interval on which the inequality is defined.*

Time domain performance requirements for other functions, such as the impulse response, ramp response, etc., are defined in similar fashion. Observe that time domain performance requirement and index are two different concepts; while the performance requirement is an inequality stated *in advance* of choosing a system, the performance index applies to a *given* system.

7.1.3 Classical time domain requirements

We now give a set of time domain performance requirements, which corresponds to the indices typically used.

Overshoot requirement: Given $M > 0$,

$$c(t) - 1 < M, \ \forall t > 0 : \tag{7.1}$$

Undershoot requirement: Given $M > 0$,

$$c(t) > M, \ \forall t > 0 : \tag{7.2}$$

Rise time requirement: Given $t_r > 0$, $\Delta_r > 0$,

$$c(t) > \Delta_r, \ \forall t > t_r : \tag{7.3}$$

Settling time requirement: Given $t_s > 0$ and $\Delta_s > 0$,

$$|c(t) - 1| < \Delta_s, \qquad \forall t > t_s \tag{7.4}$$

7.1.4 Envelope-type requirements

A mathematically natural way to establish requirements on $c(t)$ is to give a set of curves that act as bounds for $c(t)$.

General step response envelope requirement: Given functions of time $g_1(t)$ and $g_2(t)$,

$$g_1(t) \leq c(t) \leq g_2(t) \ (\forall t \geq 0) \tag{7.5}$$

We refer to g_1 and g_2 as bounds for $c(t)$. See Fig. 7.2.

Another constraint can be defined on the slope of the curve $c(t)$. In fact the rise time can be used to give an indirect measure of this slope. To restrict the behavior of $c'(t)$, functions g_3 and g_4 can be chosen to require that

$$g_3(t) \leq c'(t) \leq g_4(t) \tag{7.6}$$

The corresponding picture is given in Fig. 7.3.

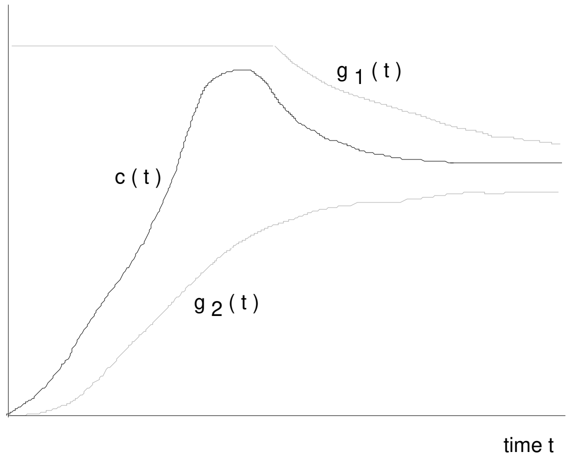

Figure 7.2: Bounds for $c(t)$.

7.2 Translation of time-domain requirements

In this section we look at the question of how the inequality $f(t) \geq 0$ translates to the frequency domain.

7.2.1 Completely monotonic functions

Let F be the Laplace transform of a function of time $f(t)$:

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt \tag{7.7}$$

Below we find criteria that indicate when the function $F(s)$ arises from a nonnegative function $f(t)$.

Suppose that the function f in (7.7) satisfies $f(t) \geq 0$. To gain some insight, we formally differentiate both sides of (7.7). By differentiating under the integral sign we obtain

$$F^{(n)}(s) = \int_0^\infty (-t)^n e^{-st} f(t) dt \tag{7.8}$$

Thus for s any positive real number,

$$(-1)^n F^{(n)}(s) = \int_0^\infty t^n e^{-st} f(t) dt \geq 0 \tag{7.9}$$

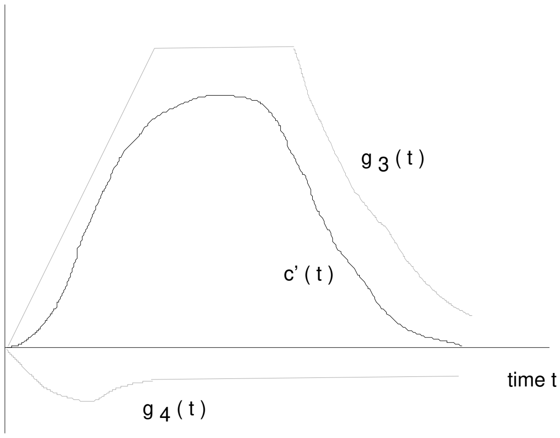

Figure 7.3: Bounds for $c'(t)$.

Theorem 2 formalizes this and gives a converse statement.

Theorem 2 *Let* $F \in \mathcal{RH}^\infty$ *be a strictly proper function, and let* $f : (0, \infty) \to R$ *be such that* $\mathcal{L}[f](s) = F(s)$.

(i) If $f(t) \geq 0$ *for all* $t \geq 0$, *then*

$$(-1)^n F^{(n)}(s) \geq 0 \ (\forall n \in N) \ (\forall s > 0) \tag{7.10}$$

(ii) If there exists a real number $s_0 > 0$ *such that*

$$(-1)^n F^{(n)}(s_0) \geq 0, \tag{7.11}$$

then $f(t) \geq 0$ *for all* $t \geq 0$.

Definition 3 *A function* $F(s)$ *is completely monotonic if it is continuous on* $0 \leq s < \infty$, *has derivatives of all orders on* $0 < s < \infty$, *and satisfies (7.10).*

Thus Theorem 2 says that strictly proper $\mathcal{RH}^\infty$ functions F that are completely monotonic are precisely those that are the Laplace transform of nonnegative functions. Furthermore, to check that F is completely monotonic it suffices to verify (7.10) at only one point $s = s_0$. Theorem 2, in more general form, is due to D. V. Widder and to S. Bernstein (circa 1930). See [W] for details.

Example 1. The function $F(s) = 1/(s+1)^3$ is a strictly proper $\mathcal{RH}^\infty$ function, thus it has an inverse transform $f(t) = \mathcal{L}[F](t)$. Since

$$F^{(n)}(1) = (-1)^n \frac{(n+2)!}{2^{n+4}}$$

it follows from Theorem 2 that the function F is completely monotonic, and that $f(t)$ is nonnegative.

7.2.2 Known boundary values

Let F be a strictly proper $\mathcal{RH}^\infty$ function, and suppose that its values $F(j\omega)$ on the $j\omega$ axis are known. The question posed now is, when is F the Laplace transform of a nonnegative function $f(t)$? Theorem 3 below gives a partial answer to this question.

We need the following notation. For $F \in \mathcal{RH}^\infty$ and $\omega \in R$ set

$$\Delta_n(F;\omega) = \mathbf{Det}\begin{pmatrix} F(0) & F(j\omega) & \cdots & F(j(n-1)\omega) \\ \overline{F(j\omega)} & F(0) & \cdots & F(j(n-2)\omega) \\ \overline{F(j2w)} & \overline{F(j\omega)} & \cdots & F(j(n-3)\omega) \\ \vdots & \vdots & & \vdots \\ \overline{F(j(n-1)\omega)} & \overline{F(j(n-2)\omega)} & \cdots & F(0) \end{pmatrix} \tag{7.12}$$

Let us compute the determinant (7.12) for $n = 2$.

$$\Delta_2(F;\omega) = \mathbf{Det}\begin{pmatrix} F(0) & F(j\omega) \\ \overline{F(j\omega)} & F(0) \end{pmatrix} = F(0)^2 - |F(j\omega)|^2. \tag{7.13}$$

Some calculations give the following formula for $n = 3$.

$$\Delta_3(F;\omega) = (F(0)^2 - |F(jw)|^2)^2 - |F(jw)^2 - F(0)F(j2w)|^2. \tag{7.14}$$

Theorem 3 *Let F be a strictly proper $\mathcal{RH}^\infty$ function, and let $f = \mathcal{L}^{-1}[F]$. If the function f in nonnegative then $\Delta_n(F;\omega) \geq 0, \forall \omega \in R, \forall n \in N$.*

For emphasis we state Theorem 3 above in a different form. If there exist n and ω such that $\Delta_n(F,\omega) < 0$, then $\mathcal{L}^{-1}[F](t)$ changes sign at least once on the interval $0 < t < \infty$.

7.2.3 Application to system design

Consider the closed loop system $\mathcal{S}$ given in Fig. 1.1. The step response curve $c(t)$ has as its Laplace transform the function

$$\mathcal{L}[c](s) = T(s)/s \tag{7.15}$$

From the assumption that the functions c and c' are both of exponential type so that their transforms are well defined, and from the properties of Laplace transform we have

$$\mathcal{L}[\frac{dc}{dt}](s) = T(s) \tag{7.16}$$

Now it is possible to characterize in terms of the closed loop transfer function T all systems $\mathcal{S}$ with non-decreasing step response function.

Theorem 4 *If the system $\mathcal{S}$ has a strictly proper closed loop transfer function $T \in \mathcal{RH}^\infty$, then the step response curve $c(t)$ is nondecreasing if and only if the function T is completely monotonic.*

Proof. The function $c(t)$ is nondecreasing if and only if its derivative $c'(t)$ is nonnegative for all t, which in turn holds if and only if T is completely monotonic. ∎

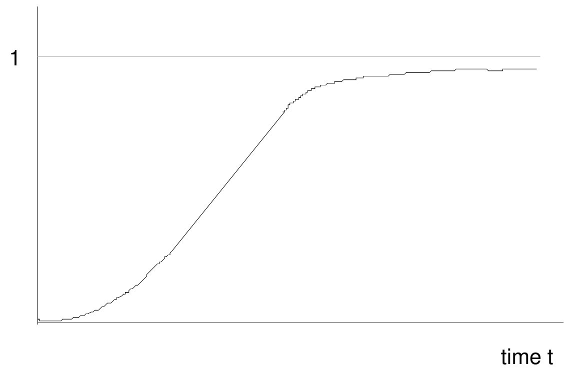

Figure 7.4: A nondecreasing step response function.

Time domain performance requirements on the step response, such as

$$c(t) \leq g(t), \qquad t \geq 0,$$

can be translated to the frequency domain as the statement,

$$G(s) - \frac{T(s)}{s} \text{ is a completely monotonic function.} \tag{7.17}$$

Note that there is a singularity at the point $s = 0$ in the expression (7.17). However, this is a removable singularity under the appropriate hypotheses.

Theorem 5 *Let $\mathcal{S}$ be an internally stable system with plant $P(s)$ having a pole at $s = 0$. Let $g(t)$ be a function whose Laplace transform is of the form $G(s) = 1/s + G_0(s)$, with $G_0(s) \in \mathcal{RH}^\infty$. Then the function $G(s) - \frac{T(s)}{s}$ is in $\mathcal{RH}^\infty$, and the inequality $c(t) \leq g(t)$ holds for all $t \geq 0$ if and only if $G(s) - \frac{T(s)}{s}$ is completely monotonic.*

Proof. Since $G(s) - \frac{T(s)}{s}$ is the Laplace transform of $g(t) - c(t)$, the only thing to prove is the statement $G(s) - \frac{T(s)}{s}$ is in $\mathcal{RH}^\infty$. Note that $s = 0$ is the only possible source of trouble. By internal stability and since there is a pole of P at $s = 0$, the closed loop transfer function T satisfies the interpolation condition $T(0) = 1$. This says that $\frac{(1-T(s))}{s}$ is in $\mathcal{RH}^\infty$. The same is true then for the function

$$G(s) - \frac{T(s)}{s} = G_0(s) - \frac{T(s) - 1}{s}.$$

■

An application of Theorem 5 and Theorem 3 is presented next in Section 7.3.

7.3 Design Example 4

We now use the results given in this chapter to improve upon the answer obtained in Section 4 of Chapter 5.

7.3.1 Statement of the problem

Solve *Design* for the plant

$$P(s) = \frac{0.036(s + 25)}{s^2(s + 0.02 + 2j)(s + 0.02 - 2j)}$$

The step response function has requirements step response overshoot

$$c(t) < 1.15 \qquad (\forall t \geq 0)$$

and settling time

$$|c(t) - 1| < 0.01, \qquad (t \geq 14.0)$$

We need to find a closed loop system $\mathcal{S}$ that is internally stable and meets the performance requirements.

7.3.2 Strategy for defining the performance function

The optimal system obtained in Subsection 6.4 does not satisfy the settling time requirement. Now the performance function is redefined to obtain an improved solution. To do this, we set the performance to be of the form

$$\Gamma'(\omega, z) = W_1(j\omega)\Gamma_1(\omega, z) + W_2(j\omega)\Gamma_2(\omega, z) \tag{7.18}$$

where Γ_1 is the performance function in Subsection 6.4. The function Γ_2 and the weight functions W_1 and W_2 are selected in the following way:

a. Choose a function g for the step response envelope requirement

$$c(t) \leq g(t), \qquad \forall t \geq 0 \tag{7.19}$$

The idea is to force the settling time and overshoot requirements upon our design by the choice of a suitable function g.

b. The envelope constraint (7.19) is replaced with a (weaker) statement in the frequency domain of the form

$$|G(j\omega) - T(j\omega)/(j\omega)|^2 \leq \gamma_0^2 \quad \forall \omega \tag{7.20}$$

where $\mathcal{L}[g] = G$ (based on Theorems 3 and 5). Call $\Gamma_2(\omega, T(j\omega))$ the left-hand-side of (7.20).

c. Substitute the optimal T_1^* obtained in Subsection 6.4 into (7.20). Determine frequency bands where (7.20) is not satisfied. Then choose W_1 and W_2 to enhance Γ_2 at these frequency bands.

7.3.3 Generating an envelope requirement

The function g must have a maximum value of 1.15 approximately, and it must satisfy $g(14.0) \approx 0.01$. One way to do this is to choose nonnegative parameters α_1, α_2, α_3 and κ in

$$g(t) = 1 + \alpha_1\, e^{-\kappa t} + \alpha_2\, t\, e^{-\kappa t} + \alpha_3\, t^2\, e^{-\kappa t} + \alpha_4\, t^3\, e^{-\kappa t}$$

After some experimentation we obtain the following bound for the step response:

$$g(t) = 1 + 0.05\, t\, e^{-0.65\, t} + 0.005\, t^2\, e^{-0.65\, t} + 0.02\, t^3\, e^{-0.65\, t} \tag{7.21}$$

Fig. 7.5 shows a plot of the function g(t). Thus we expect to satisfy the requirements on the step response and settling time by requiring (7.19) to hold for the step response function and the function g in (7.21).

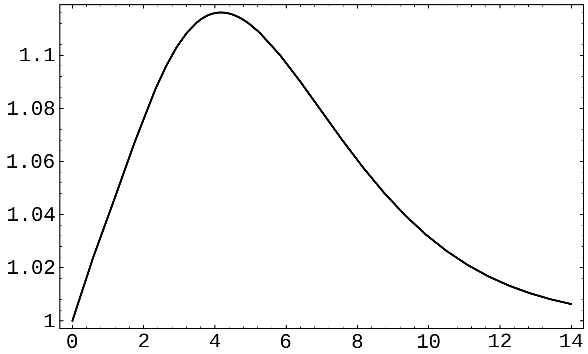

Figure 7.5: Bound g for envelope requirement in Design Example 4.

7.3.4 Weak form of requirements envelope

The internal stability requirement on the system $\mathcal{S}$ and the order 2 pole of the plant at $s = 0$ imply that designable closed loop transfer functions satisfy

$$\lim_{s \to 0} \frac{1 - T(s)}{s} = 0 \tag{7.22}$$

Then the function

$$\mathcal{L}[g - c](s) = \frac{1}{s} + \frac{0.05}{(s + 0.65)^2} + 2\frac{0.005}{(s + 0.65)^3} + 6\frac{0.02}{(s + 0.65)^4} - \frac{T(s)}{s} \tag{7.23}$$

is in $\mathcal{RH}^\infty$, and the value of this function at $s = 0$ is

$$\mathcal{L}[g - c](0) = \frac{0.05}{(0.65)^2} + 2\frac{0.005}{(0.65)^3} + 6\frac{0.02}{(0.65)^4} =: \gamma_0 \approx 0.827 \tag{7.24}$$

The function $g - c$ is nonnegative if and only if $\mathcal{L}[g - c](s)$ is completely monotonic, by Theorem 5. A necessary condition for this, by Theorem 3,

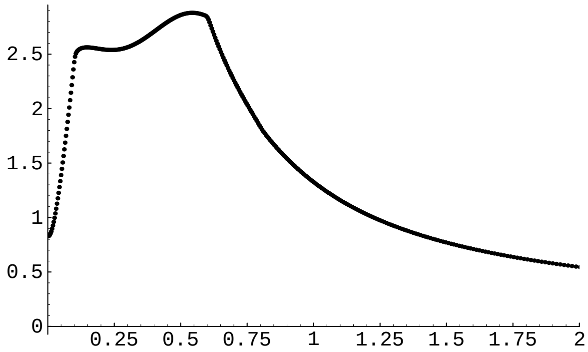

Figure 7.6: Plot of $\Gamma_2(\omega, T_1^*(j\omega))$.

is that at all frequencies ω,

$$\left|\frac{1}{j\omega} + \frac{0.05}{(j\omega+0.65)^2} + 2\frac{0.005}{(j\omega+0.65)^3} + 6\frac{0.02}{(j\omega+0.65)^4} - \frac{T(j\omega)}{j\omega}\right|^2 \leq \gamma_0^2 \tag{7.25}$$

Now set $\Gamma_2(\omega, T(j\omega))$ to be the left-hand-side of (7.25).

7.3.5 Choosing weight functions

To choose W_2 we look at the plot of $\Gamma_2(\omega, T_1^*(j\omega))$ depicted in Fig. 7.6. It is clear from this figure that inequality 7.25 is false over the band $0 < \omega < 1.25$. This implies that Γ_2 should be emphasized on this frequency band. Note that for any choice of strictly proper T the function values $\Gamma_2(\omega, T(j\omega))$ become very small at high frequency. We select constant weights

$$W_1(j\omega) = 1 \qquad (\forall\omega) \tag{7.26}$$

and

$$W_2(j\omega) = 9 \qquad (\forall\omega) \tag{7.27}$$

7.3.6 Results of the optimization

A computer run[2] produces an optimal closed loop T with Bode plots shown in Fig. 7.7. The following is a rational approximation to T^*:

$$\begin{aligned} T_1(s) \quad = \quad & \frac{5.52947+855.26\,s+4867.54\,s^2+7480.54\,s^3+\cdots}{5.52947+855.26\,s+5303.79\,s^2+13156.5\,s^3+17071.8\,s^4+\cdots}\cdots \\ & \frac{+4972.77\,s^4+1303.81\,s^5+197.695\,s^6+5.26629\,s^7}{+12858.6\,s^5+5832.42\,s^6+1582.24\,s^7+246.287\,s^8+21.0034\,s^9+1.\,s^{10}} \end{aligned} \tag{7.28}$$

We can see in Fig. 7.8 that the rational function T_1 satisfies $\max_\omega |T^*(j\omega) - T_1(j\omega)| < 0.02$. The compensator associated with T_1 is

$$\begin{aligned} C(s) \quad = \quad & \frac{146.286(s+0.02\pm j\,1.)(s+30.3597)(s+2.50175\pm j\,3.28011)\cdots}{(s+25.)(s+4.8979\pm j\,7.8766)(s+4.62512)\cdots} \\ & \frac{\cdots(s+0.956961\pm j\,0.515942)(s+0.255874)(s+0.00671959)}{\cdots(s+2.25488\pm j\,2.03546)(s+0.989326\pm j\,0.532551)(s+0.0941209)} \end{aligned}$$

The plot of the step response curve associated with T_1 (or with T^*) shown in Fig. 7.9 shows that the original design goals are satisfied.

7.4 References

[BB] Stephen P. Boyd and Craig P. Barrat, *Linear Controller Design*, Prentice Hall, 1991.

[W] D. V. Widder. The Laplace Transform. Princeton University Press. 1941.

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[2] with the software package OPTDesign.

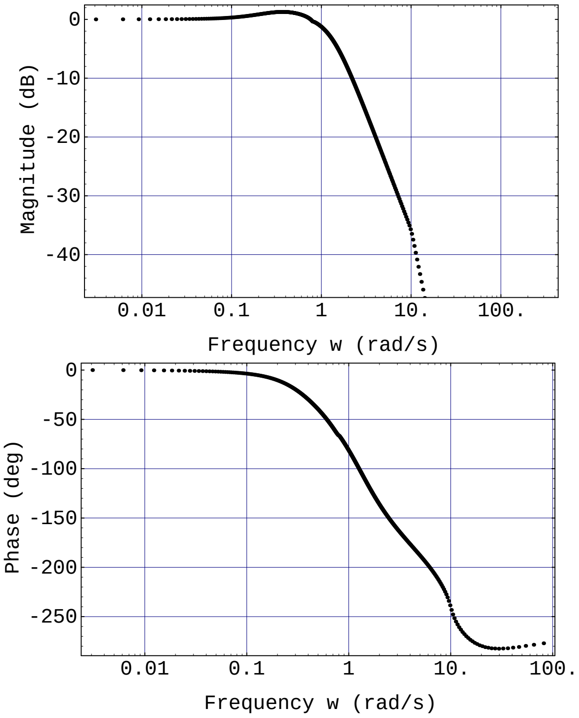

Figure 7.7: Bode plots of optimal closed loop T^* in Design Example 4.

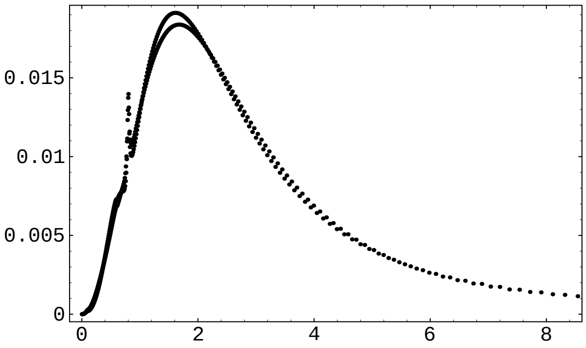

Figure 7.8: Plot of $|T^*(j\omega) - T_1(j\omega)|$.

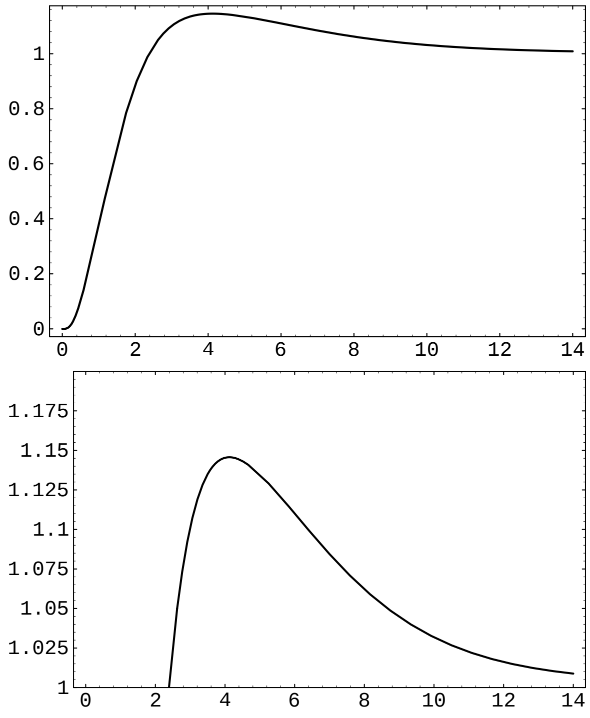

Figure 7.9: Step response function corresponding to T_1.

Chapter 8

Internal Stability II

Internal Stability was introduced in Chapter 2. This chapter continues the discussion to obtain theorems that precisely characterize internally stable systems, either in terms of zeros and poles of the plant or in terms of a formula for the closed loop transfer function. Section 8.1 develops the mathematical tools for interpolation with rational functions. In Section 8.2 a characterization of internally stable systems is given in terms of interpolation conditions on the closed loop transfer function, when the plant has simple RHP zeros and poles. The case of higher multiplicity is treated in Section 8.3.

8.1 Calculating interpolants

Systems $\mathcal{S}$ whose plant has RHP poles or RHP zeros have closed loop transfer functions that can be described by a formula. We shall see that this formula depends on those RHP poles and RHP zeros of the plant and nothing else. To derive the formula and related results, we concentrate first on the case where all of these zeros and poles have multiplicity one (called *simple* poles and zeros).

Suppose that the following sets of complex numbers are given:

$$s_1, \ldots, s_n \in RHP \quad \text{(data points)} \tag{8.1}$$

and

$$v_1, \ldots, v_n \in C \quad \text{(data values)}\ . \tag{8.2}$$

For $T(s)$ a rational function, consider the set of equalities

$$(INT) \qquad T(s_1) = v_1,\ T(s_2) = v_2,\ \ldots,\ T(s_n) = v_n$$

called an *interpolation condition on* T. If $T \in \mathcal{RH}^\infty$ satisfies INT, then T is called an *interpolant* (for the data given in (8.1) and (8.2)).

The main objective of this section is to present ways to generate interpolants. More precisely, given a set of equalities INT we show how to

- Produce one interpolant.
- Produce a parameterization (formula) of all interpolants.

First, we present an elementary way to produce one interpolant (Subsection 8.1.1) and to parameterize all interpolants (8.1.2). This is then generalized to problems with an interpolation condition at infinity (Subsection 8.1.3).

Interpolation with polynomials, rather than rational functions, is a standard subject in numerical analysis (see [DC]). The treatment here for rational functions is conceptually the same, and proofs are similar.

8.1.1 Calculating one interpolant

Let INT and a real number $a < 0$ be given. We will show how to produce an interpolant T in $\mathcal{RH}^\infty$ such that all its poles are located at $s = a$.

Let T be a proper rational function of s with a single pole location at $s = a$. One can think of T as being a sum of rational functions whose denominators are of the form $(s-a)^k$, with $k = 0, \ldots, n-1$. Since T is proper, for each factor $(s - s_0)$ that appears in a numerator there is a corresponding factor $(s - a)$ in the denominator. Thus T must have the form

$$T(s) = c_0 + c_1 \frac{s - s_1}{s - a} + c_2 \frac{(s - s_1)(s - s_2)}{(s - a)^2} + \cdots + c_{n-1} \frac{(s - s_1)\cdots(s - s_{n-1})}{(s - a)^{n-1}} \tag{8.3}$$

for some constants $c_0,\ c_1,\ \cdots,\ c_{n-1}$. The right-hand-side of (8.3) is called *Newton's representation for the function* $T(s)$.

Now we seek $T \in \mathcal{RH}^\infty$ with the form (8.3) that satisfies INT. Set $s = s_1,\ s = s_2,\ \ldots,\ s = s_n$ in (8.3) and combine with INT to obtain the system of equations

$$\begin{aligned} c_0 &= v_1 \\ c_0 + c_1 \tfrac{s_2 - s_1}{s_2 - a} \cdots &= v_2 \\ &\vdots \\ c_0 + \cdots + c_{n-1} \tfrac{(s_n - s_1)\cdots(s_n - s_{n-1})}{(s_n - a)^{n-1}} &= v_n \end{aligned} \tag{8.4}$$

This system of equations in the unknown c_k's has a unique solution. It can be solved easily by back substitution.