6.2 Design Example 1

The system is a slide drive, driven by a DC motor (see Fig. 6.5). This was studied by Brett Larson in his master's thesis [La][1], written under the guidance of Prof. Fred Bailey. The following description appears in page 3 of the thesis:

> " ...an electrical voltage $u(t)$ drives a DC motor with output speed $\omega_1(t)$. The output of the motor goes through the gear train which steps the speed down to $\omega_2(t)$. The gear train is connected to a table which slides along a rail with the load on top. "

The fundamental goal in this problem is to control the speed of the table very precisely.

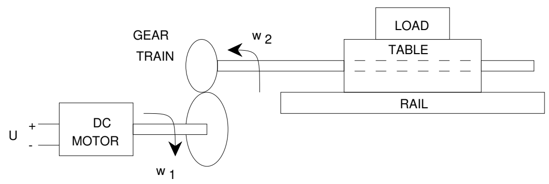

Figure 6.5: Slide drive apparatus.

6.2.1 Electro-mechanical and electrical models

Ignoring the dynamics in the gear train the problem can be represented by the equivalent circuit in Figure 6.6.

The following equations correspond to the DC motor.

$$\begin{aligned} \tau_M &= K \\ e_G &= K\omega_1 \end{aligned} \tag{6.3}$$

[1] See also [SS]

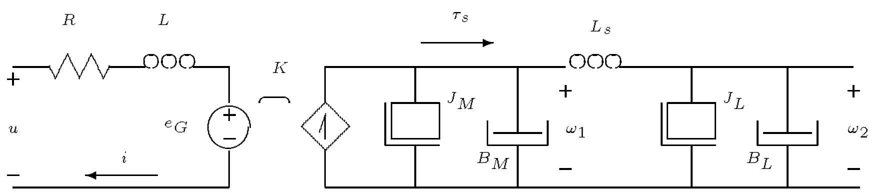

Figure 6.6: Electro-mechanical model

The parameters for this problem are

$$
\begin{aligned}
R &= 1\Omega \\
L &= 5mH \\
K &= 0.2V/rps = 0.2N-m/amp \\
J_M &= 2.010^{-3}Kg-m^3 \\
B_M &= 2.010^{-3}N-m/rps \\
K_s &= 1000N-m/rad \\
0.5J_M &\leq J_L \leq 5.0J_M \\
J_{L_{nom}} &= J_M \\
B_L &= 1N-m/rps
\end{aligned} \tag{6.4}
$$

Figure 6.7 represents this problem with the mechanical parts replaced by their electrical equivalents, where

$$
\begin{aligned}
C_m &= J_M \\
R_M &= 1/B_M \\
L_S &= 1/K_s \\
C_L &= J_L \\
R_L &= 1/B_L
\end{aligned} \tag{6.5}
$$

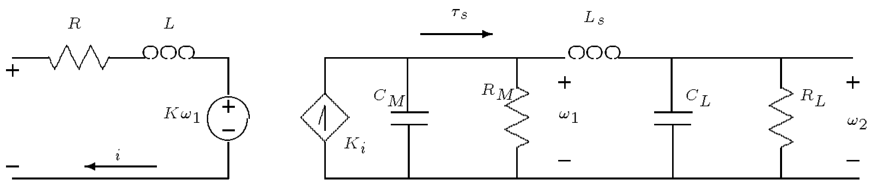

Figure 6.7: Electrical model

6.2.2 Mathematical model

Choosing the state vector as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} i \\ \omega_1 \\ \tau_x \\ \omega_2 \end{pmatrix} \tag{6.6}$$

we have that the system of state equations for the system is

$$\begin{aligned} x'(t) &= Ax(t) + bu(t) \\ y(t) &= cx(t) + du(t) \end{aligned} \tag{6.7}$$

Take ω_1 as the output and scale frequency by 100 to obtain the following coefficient matrices, where $\alpha = 1/J_L$:

$$\begin{aligned} A &= \begin{pmatrix} -2 & -0.4 & 0 & 0 \\ 1 & -0.01 & -5 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & \alpha & \alpha \end{pmatrix} \\ b &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ c &= (0\ 1\ 0\ 0) \\ d &= (0) \end{aligned} \tag{6.8}$$

The parameter α varies between 1 and 10. The maximum load corresponds to $\alpha = 1$. The plant transfer function from input u to output ω_1 is computed from the formula

$$P_\alpha(s) = c(sI - A)^{-1}b \tag{6.9}$$

We obtain,

$$P_\alpha(s) = \frac{2s^2+2\alpha s+20\alpha}{s^4+(2.01+\alpha)s^3+(50.42+12.01\alpha)s^2+(100+70.52\alpha)s+104.2\alpha} \tag{6.10}$$

The parameter α accounts for the variable load in this example. Here α varies between $\alpha = 10$, which is the minimum load case, and $\alpha = 1$, which is the maximum load case. The nominal case corresponds to $\alpha = 5$. The nominal plant is

$$\begin{aligned} P_5(s) &= \frac{2s^2+10s+100}{s^4+7.01s^3+110.47s^2+452.6s+521} \\ &= \frac{(s+2.5\pm j6.61437)}{(s+2.33827\pm j0.272238)(s+1.16673\pm j9.62573)} \end{aligned} \tag{6.11}$$

The nominal plant is stable (so are all the other plants P_α for all $\alpha \in [1, 10]$).

6.2.3 Statement of the problem

The set of requirements given here is taken from [La].

1. $|S(j\omega)| < -20\ dB$, for $|\omega| < 0.12\ rad/sec$.
2. $|S(j\omega)| < 10\ dB$, for $|\omega| < 1.0\ rad/sec$.
3. $|T(j\omega)| < -20\ dB$, for $|\omega| > 5.0\ rad/sec$.
4. $|T(j\omega)| < -40\ dB$, for $|\omega| > 10.0\ rad/sec$.
5. For stability, a phase margin greater than 30 deg.

According to Larson, these requirements should be met over the full range of load perturbations. We will instead do the (much easier) problem of solving *Design* for the nominal plant only [2]. The analysis of the performance of the system for loads other than the nominal will not be discussed here, except for the comparison of the closed loop transfer functions shown in Fig. 6.17.

6.2.4 Reformulation of requirements

Now we rewrite the requirements in 6.2.3 using the notation of Chapter 3. The translation is quite direct. The only constraint that does not have an obvious translation is the phase-margin requirement; to treat it we rewrite it as a gain-phase margin constraint with $m = 0.5$:

$$|1 - T(j\omega)| < \frac{1}{0.5} = 2, \qquad \forall\omega$$

[2] To treat this problem, Larson approximated the nominal plant with the simpler plant $P(s) = 2.2188/(s+3.4)^2$.

It is clear that the gain phase margin constraint is the only one that is binding at frequencies $1 < \omega < 5$. Also note that $m = 0.5$ insures a gain margin of 6 dB (see Fig. 6.8).

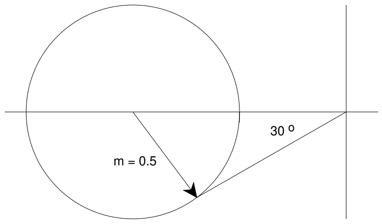

Figure 6.8: Derivation of m in Design Example 1.

An important observation is that the set of constraints in 6.2.3 lacks a closed loop roll-off constraint. We supply one by requiring

$$|T(j\omega)| < \frac{0.01}{|P(j10.)|}|P(j\omega)|, \qquad \text{for } \omega > 10.$$

This insures that constraint 4. in 6.2.3 is satisfied, since $|P(j\omega)|$ is decreasing in the band $10. < \omega$. The complete set of performance requirements is:

$$\begin{array}{rcll} |1-T(j\omega)| & < & 0.12, & 0 \leq \omega \leq 0.1 \\ |1-T(j\omega)| & < & 0.32, & 0 \leq \omega \leq 1.0 \\ |1-T(j\omega)| & < & 2, & 1 < \omega < 5.0 \\ |T(j\omega)| & < & 0.1, & 5 < \omega < 10.0 \\ |T(j\omega)| & < & 0.01, & 10 < \omega \\ |T(j\omega)| & < & 0.01|P(j\omega)|/|P(j10.)|, & \omega > 10 \end{array} \qquad (6.12)$$

Now we turn to internal stability. The plant $P(s)$ has no zeros or poles in the closed RHP. It also has relative degree $d(P) = 2$. This implies that the set INT of interpolation conditions to be satisfied by T, for internal stability with proper compensator, is

$$INT \qquad rd(T) = 2$$

6.2.5 Formulation of the performance function

A circular performance function

$$\Gamma(\omega, z) = \frac{1}{R(j\omega)^2}|K(j\omega) - z|^2 \qquad (6.13)$$

can be formed from the performance requirements by setting

$$K(j\omega) = \begin{cases} 1, & |\omega| < 5.0 \\ 0, & |\omega| \geq 5.0 \end{cases} \tag{6.14}$$

and

$$R(j\omega) = \begin{cases} 0.12, & |\omega| < 0.10 \\ 0.32, & 0.10 < |\omega| < 1.0 \\ 2.0, & 1.0 < |\omega| < 5.0 \\ 0.1, & 5.0 < |\omega| < 10.0 \\ 0.01|P(j\omega)|/|P(j10)|, & |\omega| > 10.0 \end{cases} \tag{6.15}$$

Fig. 6.9 shows the requirements envelope. We now have center and radius

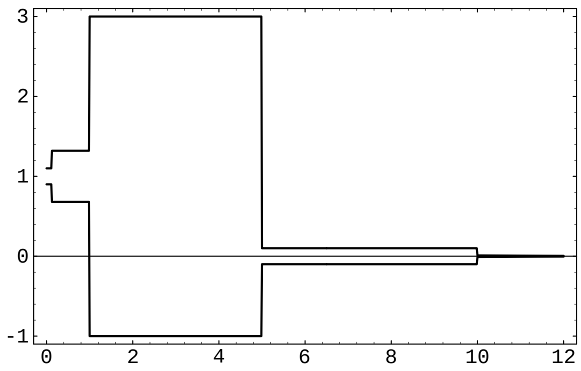

Figure 6.9: Requirements envelope.

functions with large jumps, and for good numerical behavior of algorithms, we attempt smoothing by averaging near the jumps. However, this leads to an envelope which is larger than the envelope from the set of constraints (see Fig. 6.10). One way to fix this problem is to enlarge the bands on which the constraints are tighter, and then do the averaging at jumps. The envelope obtained in this fashion can be contained in the original envelope. However, there are large changes in the smooth envelope near the frequencies that previously corresponded to jumps. If possible this situation must be avoided, since *large changes in the envelope over small frequency bands correspond to large changes in the optimal designable function T over the same bands.* This occurs for rational T when T has poles near the $j\omega$ axis, that is, T is nearly unstable.

We now try a simple modification of the envelope. The boundary of the envelope, which is mostly piecewise constant, is replaced by a continuous

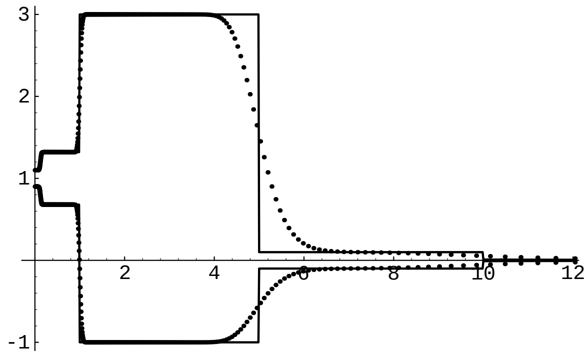

Figure 6.10: Requirement envelopes: before and after smoothing.

function which is piecewise linear at low and mid-frequency ranges. Keep in mind that the resulting envelope must be contained in the original one.

To do this modification, note that on the frequency band $1 < \omega < 5$ the radius is relatively large, and that the center drops from 1 to 0. Set the center in this band to drop linearly from 1 at $\omega = 1.0$, down to 0 at $\omega = 0.1$:

$$K(j\omega) = \begin{cases} 1 & \text{if } |\omega| < 1.0 \\ -0.25(\omega - 1) + 1 & \text{if } 1.0 < |\omega| < 5.0 \\ 0 & \text{if } |\omega| \geq 5.0 \end{cases} \tag{6.16}$$

Now pick a radius in a similar way, so that it is continuous and the resulting envelope does not become larger than the original one. Set

$$\begin{aligned} \{\omega_0, \omega_1, \ldots, \omega_8\} &= \{0., 0.12, 0.5, 1., 2., 3., 5., 10.\}, \\ \{y_0, y_1, \ldots, y_8\} &= \{0.1, 0.1, 0.32, 0.32, 1.75, 1.5, 0.1, 0.01\}, \\ m_\ell &= (y_{\ell+1} - y_\ell)/(\omega_{\ell+1} - \omega_\ell),\ \ell = 0, \ldots, 7 \end{aligned} \tag{6.17}$$

and

$$R(j\omega) = \begin{cases} m_\ell(|\omega| - \omega_\ell) + y_\ell & \text{if } \omega_\ell \leq |\omega| < \omega_{\ell+1},\ \ell = 0, \ldots, 7 \\ 0.01|P(j\omega)|/|P(j10.)| & \text{if } |\omega| \geq 10.0 \end{cases} \tag{6.18}$$

The resulting requirements envelope is depicted in Fig. 6.11.

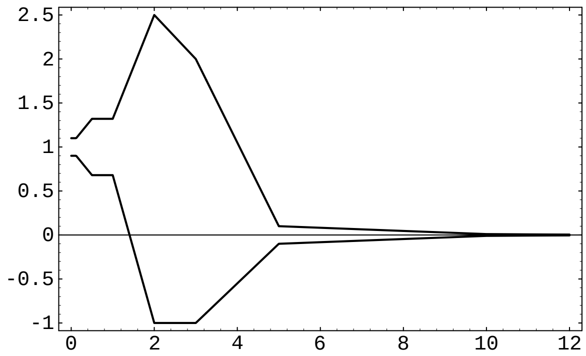

Figure 6.11: Modified envelope.

6.2.6 Optimization

We now have all the elements necessary to state and solve the optimization problem:

$$\begin{aligned} \gamma^* &= \inf_{T \text{ sat. } INT} \sup_\omega \tfrac{1}{R(\omega)^2} |K(j\omega) - T(j\omega)|^2 \\ &= \inf_{T_1} \sup_\omega \tfrac{1}{R_1(\omega)^2} |K_1(j\omega) - T_1(j\omega)|^2 \end{aligned} \tag{6.19}$$

where $K_1 = (K - A)/B$, $R_1 = R/B$ and $T = A + BT_1$.

Calculating the answer by computer [3] yields $\gamma^* = 0.98$. By Proposition 3 in Chapter 4 this means that there exist solutions to *Design*. Plots of the optimal T and optimal L are shown in Figs. 6.12 – 6.15.

6.3 Time domain performance requirements

Time domain requirements on the closed loop system $\mathcal{S}$ must be translated to the frequency domain in order to solve design problems with the methods used in this book. In this section, two time domain requirements are introduced, and a technique for treating them is given. The technique is applied later in a second design problem (Section 6.4). We warn the reader that the technique introduced in this section may not be powerful enough to solve some problems; the methods presented later may give better results.

[3] See Part II for the input of a computer session with the package OPTDesign

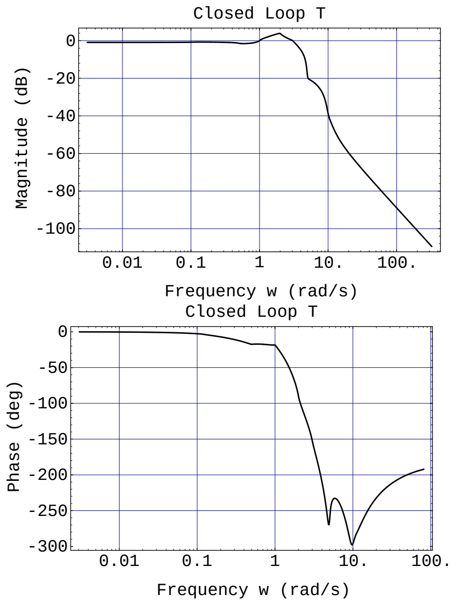

Figure 6.12: Bode plots of optimal closed loop T in design example 1.

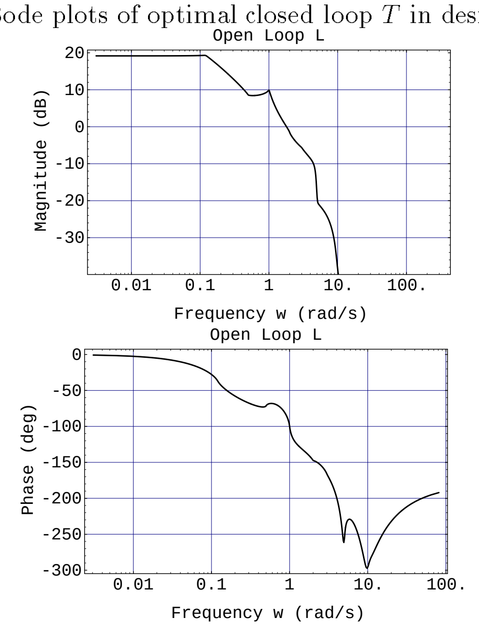

Figure 6.13: Bode plots of optimal open loop L in design example 1.

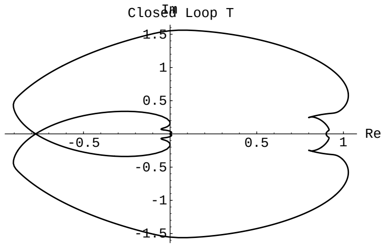

Figure 6.14: Polar plot of optimal T in design example 1.

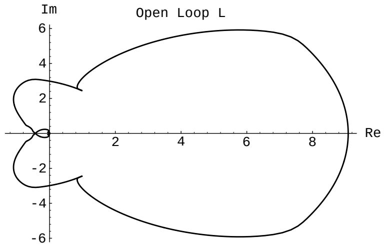

Figure 6.15: Polar plot of optimal L in design example 1.

6.3.1 Two common time domain requirements

Two common time domain requirements used for system design are the overshoot requirement and the settling time requirement. Both are defined in terms of the step response curve $c(t)$, which is the response of the system in the time domain to an input that is a unit step (see Fig. 6.18). The step response function has $T(s)/s$ as Laplace transform i.e.,

$$c(t) = \mathcal{L}^{-1}[T(s)/s](t) \tag{6.20}$$

The requirements we present in this sections are

- **Overshoot requirement**:
  Given $M > 0$,
  $$c(t) - 1 < M, \ \forall t > 0 \tag{6.21}$$
- **Settling time requirement**:
  Given $t_s > 0$ and $\Delta > 0$,
  $$|c(t) - 1| < \Delta, \qquad \forall t > t_s \tag{6.22}$$

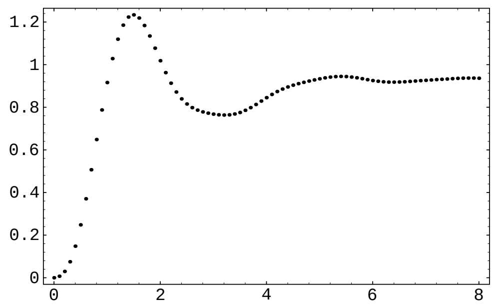

Figure 6.16: Step Response curve of optimal system in design example 1.

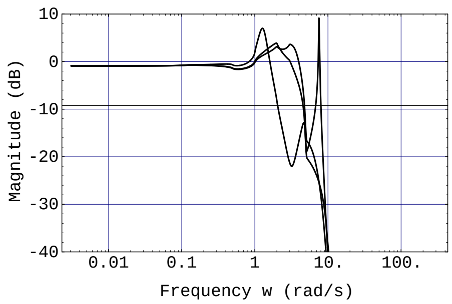

Figure 6.17: Magnitude of the closed loop transfer function for different plants in the parameter range: $\alpha = 1, 5, 10$. The compensator is optimal for $\alpha = 5$.

6.3.2 A naive method

A very common method for handling time domain requirements in the frequency domain is to use a low order rational function T_{ref}. The function T_{ref} is the transfer function of a well understood system that meets the time domain requirements. Frequency domain requirements that correspond to given time domain requirements are established in terms of the function T_{ref}.

A simple example is the closed loop system with transfer function

$$T_{ref}(s) = \frac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2} \tag{6.23}$$

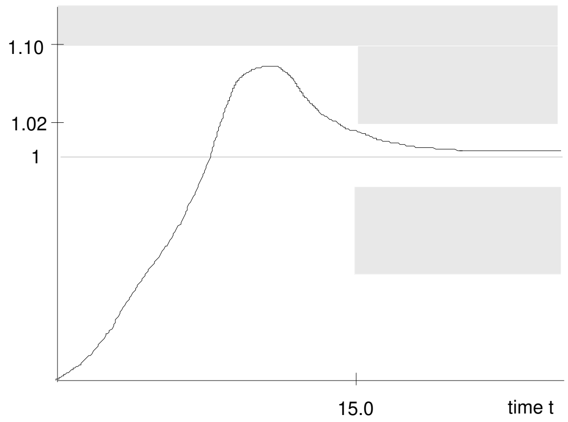

Figure 6.18: A step response function that satisfies the overshoot and settling time requirements when $M = 0.1$, $t_s = 15.0$ and $\Delta = 0.02$. In this figure, the step response curve cannot run into the shaded area without violating the constraints.

This system is known to satisfy the relations (see [FPE])

$$t_s \approx \frac{4.6}{\zeta\omega_n} \tag{6.24}$$

and

$$M_p = \mathrm{Exp}\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \tag{6.25}$$

Hence $T_{ref}(s)$ is completely determined by specifying t_s and M_p. Also determined is the open loop $L_{ref} = T_{ref}/(1 - T_{ref})$. Bode plots of T_{ref} and L_{ref} are then used to determine parameters such as bandwidth, peak magnitude, etc.

6.3.3 Limitations of the naive method

Relationships between time domain and frequency domain constants, like the ones described in Section 6.3.2, are commonly used in practice as rules of thumb for system design. However, they do not hold for general closed loop systems, and thus may give misleading information. Another important weakness of the naive method is that the function T_{ref} has nothing in it that accounts for the characteristics of the given plant P.

6.4 Design Example 2

The following design problem is taken from [FPE, p. 497]. *Design* is treated here with the naive method for converting time domain requirements just described in Section 6.3.2.

6.4.1 Statement of the problem

The problem is to design a satellite's attitude control. The plant is the rational function

$$P(s) = \frac{0.036(s+25)}{s^2(s+0.02+2j)(s+0.02-2j)}$$

The requirements on the step response function $c(t)$ are overshoot requirement,

$$c(t) < 1.15 \qquad (\forall t \geq 0),$$

and settling time,

$$|c(t) - 1| < 0.01, \ \forall t, \ t \geq 14.0$$

We need to find a closed loop system $\mathcal{S}$ that is internally stable and meets the performance requirements.

6.4.2 Translation of time domain requirements

Set $t_s = 14.$ and $M_p = 0.15$. Substituting these into equations (6.24) and (6.25) gives $\zeta \approx 0.52$ and $\omega_n \approx 0.64$. Thus (6.23) becomes

$$T_{ref}(s) = \frac{0.404}{0.404 + 0.657s + s^2} \tag{6.26}$$

The corresponding step response function $c_{ref}(t) = \mathcal{L}^{-1}(\frac{T_{ref}(s)}{s})$ is given by

$$c_{ref}(t) = 1 - (0.6039\, sin(0.544t) + cos(0.544t))\, e^{-0.328571t} \tag{6.27}$$

Figs. 6.19 – 6.22 show plots associated with T_{ref}. Frequency domain requirements are set now from these plots.

Midrange frequencies: tracking and bandwidth

We use a combination of the plots of $|T_{ref}|$ and $|1 - T_{ref}|$ (Fig 6.22) to set frequency dependent tracking and bandwidth constraints:

$$\begin{aligned} |1 - T(j\omega)| &< |1 - T_{ref}(j\omega)|, \quad |\omega| < 0.8 \\ |T(j\omega)| &< |T_{ref}(j\omega)|, \qquad\quad |\omega| > 0.8 \end{aligned} \tag{6.28}$$

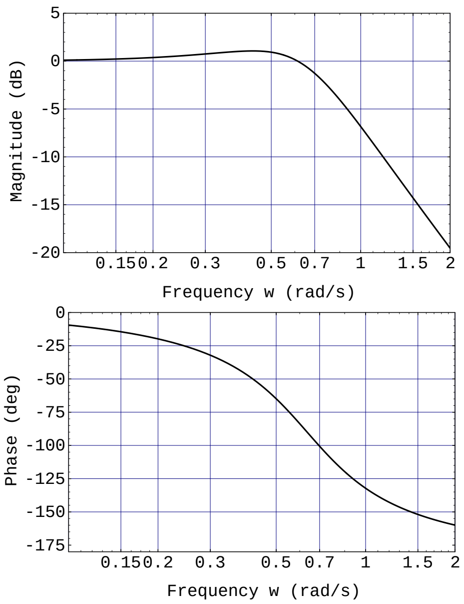

Figure 6.19: Bode plots of T_{ref}.

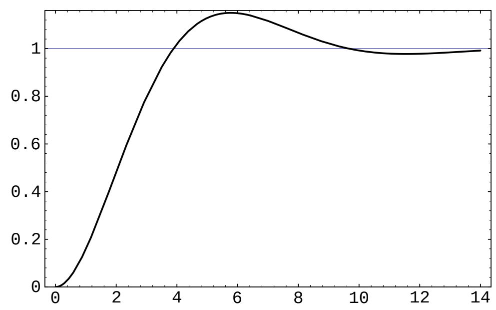

Figure 6.20: Step response for T_{ref}.

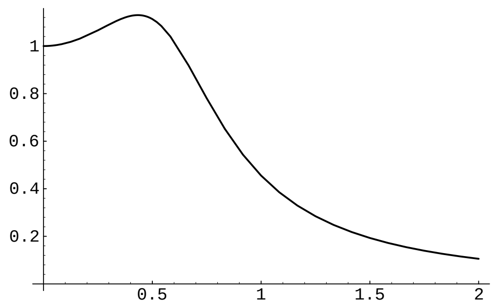

Figure 6.21: Plot of $|T_{ref}|$.

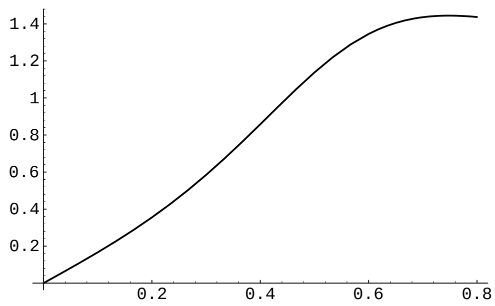

Figure 6.22: Plot of $|1 - T_{ref}|$.

High frequency: closed loop roll off

We specify high frequency behavior of the designable T with the roll-off constraint: $|T(j\omega)| < \alpha_r |P(j\omega)|$, valid for $\omega > \omega_r$. The value of α_r is chosen to have continuous envelope at the point $\omega = \omega_r$.

$$|T(j\omega)| \leq |\frac{T_{ref}(j10.)}{P(j10.)}||P(j\omega)|, \qquad |\omega| \geq 10.0 \tag{6.29}$$

Low frequency: plant bound constraint

Since the plant P has a pole at the origin, it is necessary to consider the plant bound constraint $|(1 - T(j\omega))P(j\omega)| < \alpha_p$, valid for $|\omega| < \delta_p$. Somewhat arbitrarily choose $\delta_p = 0.1$, and α_p is selected to make $|1 - T(j\delta)| \leq |1 - T_{ref}(j\delta)|$:

$$|T(j\omega) - 1| \leq |P(j0.1)(1 - T_{ref}(j0.1))|\frac{1}{|P(j\omega)|}, \quad |\omega| < 0.1 \tag{6.30}$$

6.4.3 Forming the performance function Γ

A circular performance function

$$\Gamma(\omega, z) = \frac{1}{R(j\omega)^2}|K(j\omega) - z|^2 \tag{6.31}$$

can be formed from inequalities (6.28) through (6.30). Inequalities (6.28) lead to center and radius functions with a jump discontinuity at frequency $\omega = 0.8$. To avoid numerical difficulties we compromise and set a constraint defined on the band $0.6 < \omega < 0.8$ that smooths the transition between the constraints in (6.28). Simply set the radius function K to drop linearly, from $K(j0.6) = 1.$ down to $K(j0.8) = 0.$. The radius function R is also set to be linear on the band $0.6 < \omega < 0.8$, with endpoints given by the constraints (6.28).

$$K(j\omega) = \begin{cases} 1, & |\omega| < 0.5 \\ \frac{-1}{0.2}(|w| - 0.8), & 0.6 \leq |\omega| < 0.8 \\ 0, & |w| \geq 0.8 \end{cases} \tag{6.32}$$

and

$$R(j\omega) = \begin{cases} \frac{|P(j0.1)(1-T_{ref}(j0.1))|}{|P(j\omega)|}, & |\omega| < 0.1 \\ |1 - T_{ref}(j\omega)|, & 0.1 < |\omega| < 0.6 \\ \frac{|T_{ref}(j0.8)| - |1-T_{ref}(j0.6)|}{0.2}(|\omega| - 0.6) & \\ +|1 - T_{ref}(j0.6)|, & 0.6 < |\omega| < 0.8 \\ |T_{ref}(j\omega)|, & 0.8 < |\omega| < 10.0 \\ |\frac{T_{ref}(j10.)}{P(j10.)}||P(j\omega)|, & |\omega| > 10.0 \end{cases} \tag{6.33}$$

The corresponding requirements envelope is shown in Fig. 6.23.

6.4.4 Optimization

Proceed as in Section 6.2.6 to fomulate OPT. Solving by computer produces an optimum $\gamma^* = 1.269$, indicating that there are no solutions to *Design* for the given set of constraints.

Since the frequency domain requirements were obtained from T_{ref}, there is a certain amount of flexibility in resetting them, which we now do. Empirical observations play a role here: for example, it is known that in practice large bandwidth is associated with short rise time. We relax the low frequency constraints below to see if the new constraints lead to a satisfactory answer.

Consider the following modified constraints obtained by increasing the

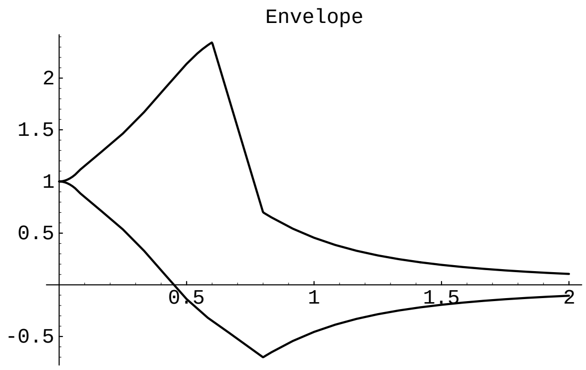

Figure 6.23: Requirements envelope.

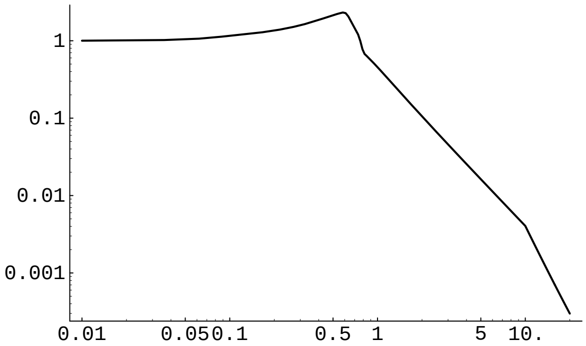

Figure 6.24: Log-log plot of the function $K + R$.

radius function on the low frequency range:

$$R(j\omega) = \begin{cases} 1.5\frac{|P(j0.1)(1-T_{ref}(j0.1))|}{|P(j\omega)|}, & |\omega| < 0.1 \\ 1.5|1 - T_{ref}(j\omega)|, & 0.1 < |\omega| < 0.6 \\ \frac{|T_{ref}(j0.6)| - 1.5|1-T_{ref}(j0.8)|}{0.2}(|\omega| - 0.6) & \\ +|1 - T_{ref}(j0.6)|, & 0.6 < |\omega| < 0.8 \\ |T_{ref}(j\omega)|, & 0.8 < |\omega| < 10.0 \\ |\frac{T_{ref}(j10.)}{P(j10.)}||P(j\omega)|, & |\omega| > 10.0 \end{cases} \tag{6.34}$$

The envelope corresponding to the center function K and the radius R in (6.34) is shown in Fig. 6.25. Solving by computer with the modified

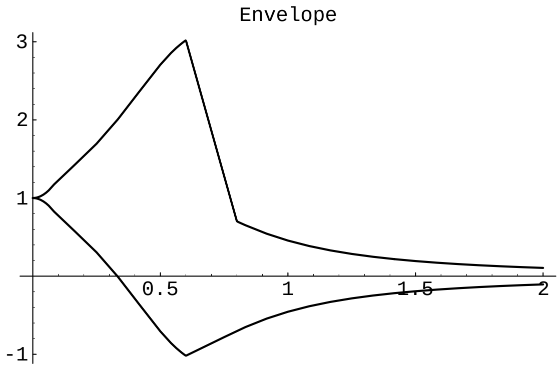

Figure 6.25: Modified envelope.

constraints gives $\gamma^* = 0.73$, which indicates that there are solutions to the modified optimization problem OPT. Plots 6.26 – 6.27 correspond to the optimal system $\mathcal{S}$. Observe that the time domain constraints are not met; overshoot is 0.25, and $c(14.0) \approx 0.05$.

6.5 Design problems with competing constraints

6.5.1 Competing constraints

Design problems can easily lead to situations in which several inequalities, binding at the same frequency, are expected by the designer to be satisfied simultaneously by the functions of the system. In this case, the inequalities are called competing constraints.

In general, a set of competing constraints at frequency ω can be stated in the form

$$\begin{array}{rcl} \Gamma_1(\omega, T(j\omega)) & \leq & 1 \\ \Gamma_2(\omega, T(j\omega)) & \leq & 1 \\ & \vdots & \\ \Gamma_m(\omega, T(j\omega)) & \leq & 1 \end{array} \tag{6.35}$$

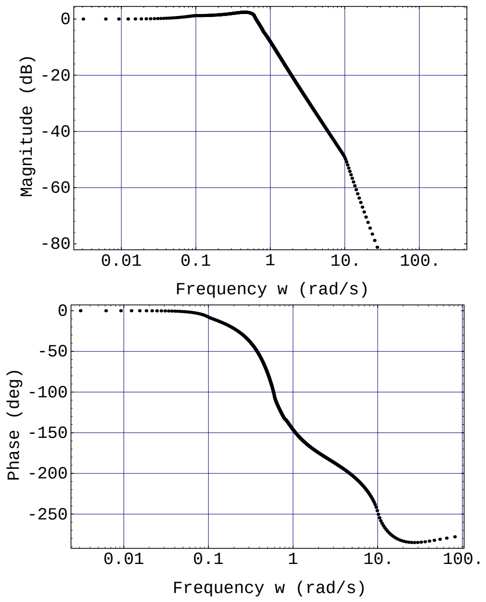

Figure 6.26: BodePlots of optimal T for Design Problem 2.

The inequalities in (6.35) can be written as a single inequality as follows:

$$\Gamma(\omega, T(j\omega)) := \max_{1 \leq k \leq m} \Gamma_k(\omega, T(j\omega)) \leq 1 \tag{6.36}$$

Example

Suppose a system $\mathcal{S}$ is to be designed, such that at frequency ω the designable transfer function T satisfies

$$|T(j\omega) - 1| \leq 1/2 \tag{6.37}$$

and

$$|T(j\omega)| \leq 6/5 \tag{6.38}$$

These inequalities can be written as

$$2|T(j\omega) - 1| \leq 1 \tag{6.39}$$

$$\frac{5}{6}|T(j\omega)| \leq 1 \tag{6.40}$$