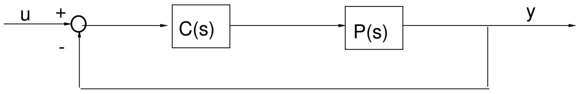

Figure 3.1: The closed loop system $\mathcal{S}$.

- The closed loop function $T = \frac{PC}{1+PC}$
- The sensitivity function $S = (1 + PC)^{-1}$
- The tracking error for input u, Su
- The closed loop compensator $Q := CS$
- The open loop transfer function $L := PC$
- The closed loop plant PS

3.1.2 Frequency domain performance requirements

Two types of requirements imposed on the system $\mathcal{S}$ are discussed in this book: internal stability and performance. The internal stability requirement was studied in Chapter 2. Performance requirements are inequalities involving the functions of the system. In this chapter we focus on performance requirements in the frequency domain, defined below.

Definition. A *frequency domain performance requirement* is an inequality that involves the designable transfer function T, and an interval in ω for which the inequality is required to hold.

The function

3.1.3 Disk inequalities

All the frequency domain performance requirements treated in this chapter can be written in the form of a *disk inequality*:

$$|K(j\omega) - T(j\omega)| \leq R(j\omega), \qquad \omega_a \leq \omega \leq \omega_b \tag{3.1}$$

Disk inequalities are easy to plot as regions in 3-D space (see Fig. 3.2).

In many cases the requirements give one disk inequality on one frequency band, and other disk inequalities on other frequency bands. When the various frequency bands are disjoint, it is possible to piece these disk inequalities together into a single one, valid for all frequencies:

$$|K(j\omega) - T(j\omega)| \leq R(j\omega), \quad (\forall\ \omega) \tag{3.2}$$

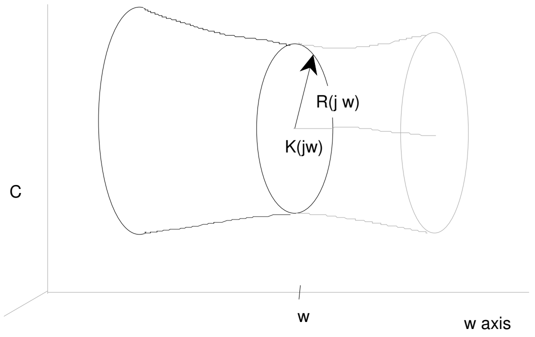

Figure 3.2: Tube like region $\{(\omega, z) : \ |k(j\omega) - z| \le r(j\omega)\}$ determined by a disk inequality $|k(j\omega) - T(j\omega)| \le r(j\omega)$.

In these cases the inequality has a nice graphical representation, as shown in Fig. 3.3. We shall see in Chapter 4 how to use disk inequalities to pose optimization problems of the simplest kind discussed in this book. Many system design problems can be solved by finding solutions to these optimization problems.

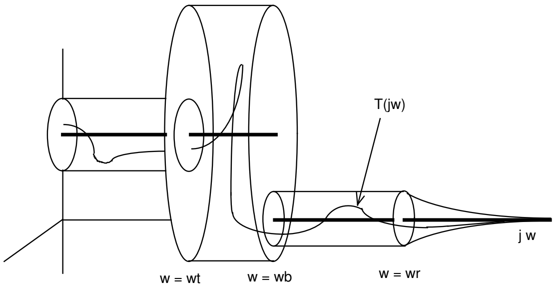

Figure 3.3: Region determined by piecing together disk inequalities.

We see in Fig. 3.2. how a disk inequality is represented graphically by a tube-like domain . If the frequency ω is fixed, then a single disk inequality is represented in the complex plane by a region S_ω consisting of a solid disk. However, if two or more disk inequalities apply at the same frequency ω_0, the region S_ω is the intersection of the disks that corresponds to individual constraints. Thus in this case, S_ω is not a disk. This situation, which is common in systems design, is treated in Chapter 5.

3.2 Measures of performance

A set of basic frequency domain performance requirements (gain-phase margin, tracking error, bandwidth and closed loop roll-off) are presented in this section. More requirements are introduced later in the chapter; however, it is possible to state physically sensible design problems with the requirements given in this section alone.

3.2.1 Gain-phase margin

In classical control, the gain margin and phase margin are the chief measures of how robustly stability is achieved. In order to formulate constraints in terms of the closed loop transfer function T we use a composite measure of stability, which seems at least as good as the gain margin and phase margin taken together. Define the *gain-phase margin* of the system $\mathcal{S}$ to be

$$m = \inf_{\omega} |1 + PC(j\omega)| \tag{3.3}$$

Graphically m is just the distance of the Nyquist plot of PC to the point -1 in the complex plane. Simple algebra converts (3.3) to

$$|T(j\omega) - 1| \leq \frac{1}{m} \quad \text{for all} \quad \omega \tag{3.4}$$

We can easily compare the gain-phase margin m with the gain margin g and the phase margin ϕ by looking at the Nyquist plot in Fig. 3.4. Typically m is more conservative than either ϕ or g.

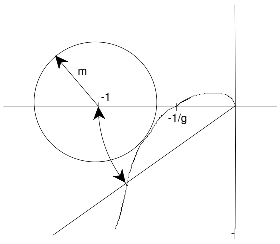

Figure 3.4: The gain margin g, phase margin ϕ, and g-p. margin m

If a given stable system has m near 0, then it is close to the unstable case, which is undesirable. If α_m denotes the largest value of $1/m$ considered to be acceptable in inequality (3.4), then we can formulate a constraint in terms of α_m as (see Fig. 3.5):

$$|1 - T(j\omega)| \leq \ \alpha_m \ \text{(for all } \omega) \ . \tag{3.5}$$

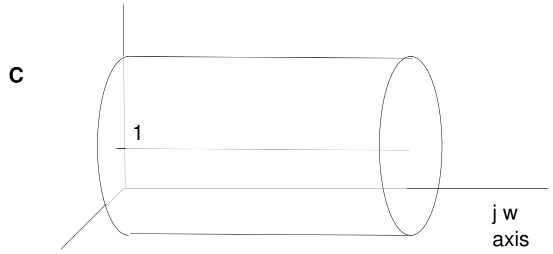

Figure 3.5: Region defined by the gain-phase margin constraint.

Fancier notions are very natural. For example, if W is a given nonnegative function of frequency, we can define

$$m_W \triangleq \inf_{\omega} |W(j\omega)(1 + PC)(j\omega)| \tag{3.6}$$

to be the *weighted gain-phase margin.* The corresponding constraint is derived from this definition in the obvious way and produces $\alpha_m(j\omega) = W(j\omega)/m_W$.

Gain-phase margin constraint

For a given α_m in the interval $(0, 1)$,

$$|1 - T(j\omega)| \leq \ \alpha_m \ \text{(for all } \omega) \ . \tag{3.7}$$

3.2.2 Tracking error

Fundamentally, tracking error is a time domain concept. The issue is to measure how close the output $y(t)$ of the closed loop system is to a given input $u(t)$. Indeed, we want the error function

$$\epsilon_u(t) = u(t) - y_u(t) \tag{3.8}$$

to be "small" for each function u in a big class of possible inputs. To use frequency domain techniques, one must translate time domain criteria back to the frequency domain. We do this below.

Consider system $\mathcal{S}$ with given time domain input $u(t)$ and output $y(t)$, and let $U(s)$ and $Y(s)$ be the Laplace transform of $u(t)$ and $y(t)$. Then $Y(s) = T(s)U(s)$, and the Laplace transform of ϵ_u is

$$E_u(s) = (1 - T(s))U(s) \tag{3.9}$$

For $T \in \mathcal{RH}^\infty$, a condition that generates small ϵ_u compared to the size of u is

$$\sup_\omega |1 - T(j\omega)| < \alpha_{tr} \tag{3.10}$$

for some small α_{tr}. Thus good tracking is generated by requiring (3.10) to hold[1].

It turns out that (3.10) is too stringent to be obtainable in a practical engineering problem since the function T rolls off at high frequency. What is realistic is to require the system to track low frequency functions well, that is, to require that $T(j\omega)$ be close to 1 over some specified frequency range $[-\omega_{tr}, \omega_{tr}]$. Thus we introduce one of the key constraints of control.

---

[1] A standard theorem (Plancherel theorem, [Y]) applied to (3.9) yields immediately that for $T \in RH^\infty$ the error satisfies a useful bound:

$$\begin{aligned} \int_0^\infty |\epsilon_u(t)|^2 dt &= \int_{-\infty}^\infty |E_u(j\omega)|^2 d\omega = \int_{-\infty}^\infty |1 - T(j\omega)|^2 |U(j\omega)|^2 d\omega \\ &\leq \left\{ \sup_\omega |1 - T(j\omega)|^2 \right\} \int_{-\infty}^\infty |U(j\omega)|^2 d\omega \\ &= \left\{ \sup_\omega |1 - T(j\omega)|^2 \right\} \int_0^\infty |u(t)|^2 dt \end{aligned} \tag{3.11}$$

for all functions u on $[0, \infty)$ having finite *energy* $\int_0^\infty |u(t)|^2 dt$. The inequality is sharp in the sense that there are input functions that make this inequality as close as you desire to equality. That is,

$$\sup_u \frac{\int_0^\infty |\epsilon_u(t)|^2 dt}{\int_0^\infty |u(t)|^2 dt} = \sup_\omega |1 - T(j\omega)|^2 \tag{3.12}$$

Consequently, if

$$\sup_\omega |1 - T(j\omega)|^2 \leq c \tag{3.13}$$

then the normalized error in the left-hand-side of (3.12) is no larger than c. Thus specs of the form (3.13) guarantee that

$$\text{energy}(\epsilon_u) \leq c \ \text{energy}(u)$$

for all u with finite energy. This is a very strong form of tracking.

Tracking Error Constraint

For given $\alpha_{tr} > 0$, $\omega_{tr} > 0$,

$$|1 - T(j\omega)| < \alpha_{tr} \quad (|\omega| < \omega_{tr}) \,. \tag{3.14}$$

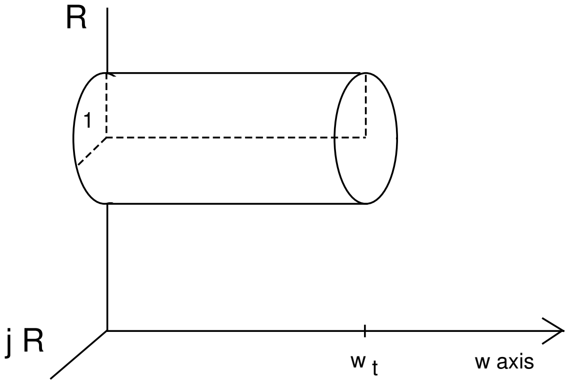

Figure 3.6: Region defined by the tracking error constraint

3.2.3 Bandwidth

Bandwidth of a system is commonly defined as the frequency ω_b at which $|T(j\omega)|$ falls below a given constant times the low frequency value of the input. This constant usually is taken to be $\alpha_b = \frac{1}{\sqrt{2}}$. A particular bandwidth is required to insure that the system at high frequency is not upset by noise, plant uncertainty, actuator sluggishness, etc.

Bandwidth Constraint

Given $\alpha_b > 0$, $\omega_b > 0$,

$$|T(j\omega)| < \alpha_b, \ \text{for} \ |\omega| > \omega_b \,. \tag{3.15}$$

In practice a more refined constraint is required for very high frequency, since it is desirable that T rolls off to 0. This is discussed in Section 3.2.4.

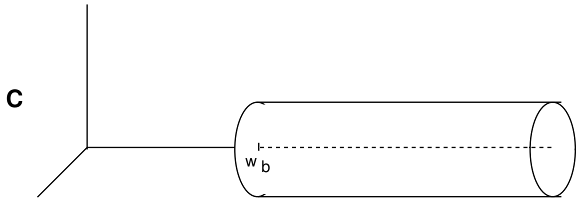

Figure 3.7: Region defined by the bandwidth constraint.

3.2.4 Closed loop roll-off

The open loop transfer function $L = PC$ of a given system rolls off at high frequency, since the plant P and the compensator C are strictly proper. That is, $|P(j\omega)C(j\omega)| \rightarrow 0$ as $\omega \rightarrow \infty$.

From the relation

$$\begin{aligned} |T(j\omega)| &\leq \frac{|P(j\omega)C(j\omega)|}{1 - |P(j\omega)C(j\omega)|} && (3.16) \\ &\approx |P(j\omega)C(j\omega)| \text{ (large } |\omega|) && (3.17) \end{aligned}$$

we see that both T and PC roll off at the same rate. To obtain an inequality useful for the design process, we must eliminate the compensator C from the right-hand-side of relation (3.17). To do this, use the fact that compensators roll off at high frequency with the asymptotic form[2]

$$C(j\omega) \leq \frac{\alpha_r}{|\omega|^n} \qquad (3.18)$$

for some α_r and some n. At the outset of design, an engineer should specify n and α_r for the class of compensators that is to be built [3]. Once α_r and n are specified, combine (3.17) and (3.18) to obtain the closed loop roll-off constraint.

---

[2] Since behavior near $j\infty$ is what is important in (3.18), α_r can be taken to be a weight function of frequency that is not too close to 0 at any given frequency.

[3] We thank L. Desoer for emphasizing this constraint to us.

Closed loop roll-off constraint

For given $\omega_r > 0$, $\alpha_r > 0$ and $n > 0$,

$$|T(j\omega)| \leq \frac{\alpha_r |P(j\omega)|}{|\omega|^n} \quad (|\omega| > \omega_r) \ . \tag{3.19}$$

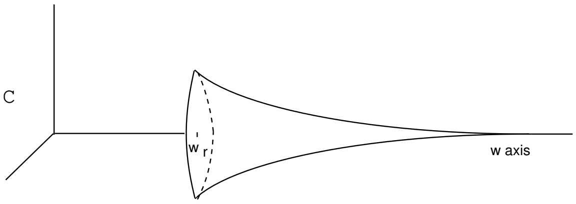

Figure 3.8: Region defined by the closed loop roll-off constraint.

3.2.5 Fundamental tradeoffs

The most fundamental tradeoff in control is between bandwidth constraints and performance such as tracking or gain-phase margin which dominate considerations at low frequency. Bandwidth and roll-off are consistent constraints while tracking error competes strongly with them. Thus a basic challenge in control is to get tracking over a broad enough frequency band, subject to the roll-off constraints dictated by actuator sluggishness and uncertainty in plants, sensors, and the environment. In a particular engineering problem, to obtain any precise feel for this tradeoff requires the use of a computer program.

3.2.6 Choosing sets of performance requirements

The designer must choose a set of performance requirements in order to state and then solve a design problem. This set should be selected with care, since a bad choice leads to problems that do not make sense from either the numerical or the physical point of view. We illustrate this with examples a, and b below.

Example a

Consider the problem *Design* for the plant $P(s) = 1/(1-s)$ and the performance requirement

$$|T(j\omega)| < 0.707, \qquad for\ \omega > 1.0. \tag{3.20}$$

An obvious mistake here is the absence of constraints valid on the band $0 \leq \omega \leq 1.0$. It is easy to find functions T that satisfy (3.20) and that produce internally stable systems $\mathcal{S}$, but that have undesirable behavior at low frequency.

Example b

Now consider *Design* for $P(s)$ with performance requirements

$$\begin{array}{rcll} |T(j\omega)| & < & 0.707, & \text{for } \omega > 1.0 \\ |1 - T(j\omega)| & < & 2.0, & \text{for } 0 \leq \omega \leq 1.0 \end{array} \tag{3.21}$$

For any physical system, T must roll off to 0 at very high frequency. This is not enforced by the constraints (3.21). One way to remedy this is to require roll-off on T, for example with the constraint

$$|T(j\omega)| < \alpha_c |P(j\omega)|, \ for\ \omega > \omega_c$$

The examples above illustrate two basic principles for choosing a set of frequency domain performance requirements:

- At each frequency ω, there must be a constraint in the set of requirement that is active at this frequency.

- Behavior of the system at very high frequency must be specified with a roll-off constraint on T.

There are cases where performance measures not described in the present section are important. In particular, if the plant has a zero or a pole right on the $j\omega$ axis or near it, the *compensator bound* and *plant bound* constraints must be used to set up the problem correctly. See Section 3.4 for details on this.

3.3 Disk inequalities

The performance measures introduced in Section 3.2 can be expressed as *disk inequalities*. In this section, we present an example in which several disk inequalities are put together and expressed as one disk inequality that is valid for all frequencies. Being able to combine disk inequalities as a single one is a necessary skill for students of control who want to solve system design problems with the methods proposed in this book. We note that it is not always possible to combine several disk inequalities as a single one; however, the type of problem that results if this is done leads to the simplest optimization problem[4] when solving the problem *Design*. Such problems are treated on Chapter 4 and on first reading one may want to go from this section to Chapter 4.

Consider a case where the requirements are given by constraints only on the tracking error, gain-phase margin, and bandwidth [5]. We will formulate a single disk constraint from the given constraints.

Suppose the following constraints are given:

- $|1 - T(j\omega)| < 0.25$ for $\omega < 0.5$ (tracking for unit step input).
- $|1 - T(j\omega)| \leq 2.$ for all ω (gain-phase margin).
- $|T(j\omega)| \leq 0.707 \quad \text{for } |\omega| > 3$ (bandwidth).

Note that for ω fixed, the large disks from the gain-phase margin constraint contain the smaller disks from the bandwidth and the tracking error constraints. Thus the smaller disks define the region for either low or high frequencies. For midrange frequencies, the gain-phase margin constraint is the only constraint that is applicable. We collect this information in Table 3.1. The notation there is the same as the notation in inequality (3.1). The region defined by these constraints is drawn in Fig. 3.9.

Constraint	Band	$K(j\omega)$	$R(j\omega)$
Tracking	$0. \leq \|\omega\| < 0.5$	1	0.25
g-p margin	$0.5 < \|\omega\| < 3$	1	2.
Bandwidth	$3 < \|\omega\|$	0	0.707

---

[4] That is, an optimization problem with circular performance function.

[5] This is not a physical example, since there is no decay requirement on the magnitude of T.

Table 3.1

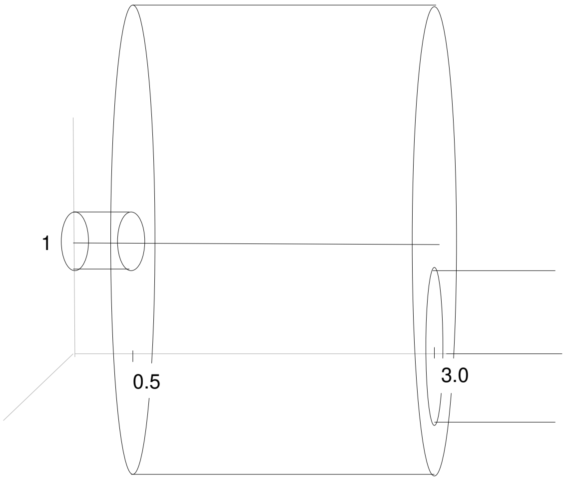

Figure 3.9: The region defined by the constraints in Table 3.1.

3.4 More performance measures

In Section 3.2 we showed how to convert the most basic specifications in a control problem to precise performance measures (suitable for computer optimization). A beginner who masters these alone will have a very good feel for the basics of control[6]. However, there are common control problems that cannot be formulated correctly using only the constraints in Section 3.2; for example, if the plant has a pole or a zero on the $j\omega$ axis. In this section we give additional performance measures, in particular the ones that deal with zeros or poles of the plant on the $j\omega$ axis.

3.4.1 Peak magnitude

One common constraint is that the magnitude of the closed loop transfer function T not become too large, because if it does then T is close to unstable. Also, it has been seen in practical design (see [Hor], page 190) that the peak magnitude of T is related to a large overshoot in the step response function. In many cases a peak value of 1.1–1.5 is acceptable, while

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[6] Also, the examples in Chapter 4 and Section 5.2 will be clear to him.

in others this is too high. We now define the peak magnitude constraint; the corresponding graphical representation is Fig. 3.10.

Peak Magnitude Constraint

For given $\alpha_M > 0$,

$$|T(j\omega)| \leq \alpha_M \quad (\text{all } \omega)\ . \tag{3.22}$$

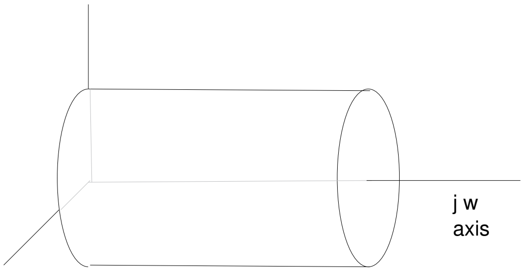

Figure 3.10: Region defined by the peak magnitude constraint.

3.4.2 Compensator bound

Desoer and Gustafson proposed another very sensible criterion for good compensation in [DG1]. It is that

$$|C(j\omega)(1 - T(j\omega))| \leq \alpha_c \ \ (\text{all } \omega) \tag{3.23}$$

for some prescribed number α_c. In terms of the closed loop system $\mathcal{S}$, this says that the "closed loop compensator" has magnitude bounded by α_c. The *closed loop compensator* is what the compensator puts out in response to an input to the system. The reason that (3.23) is imposed on the system is that if it is violated, then the closed loop system can saturate, a small current into the system can cause arcing at the output of C, or other problems may appear. Doyle and Stein discuss in [DS] drawbacks of high loop gain in somewhat different terms. Their main concern is that (3.23) holds at high frequencies (low frequencies are less important).

Let us analyze (3.23) in terms of T. Since $PC = T/(1-T)$, we obtain

$$|T(j\omega)| \leq \alpha_c |P(j\omega)| \ \ (\text{all } \omega)\,. \tag{3.24}$$

Typically this constraint adds no serious restriction unless $|P|$ is small (e.g., as $\omega \to \infty$ or near zeros of P on the $j\omega$-axis). Inequality (3.24) adds a restriction that is binding near the zeros of P ($\omega = \infty$ included). Thus the designer should require (3.24) to hold whenever $|\omega - \omega_z| < \eta$, where $j\omega_z$ is any zero of P on the axis and η is a small positive number. Note that for large ω, inequality (3.24) is contained in the closed loop roll-off constraint (set $n = 0$ in (3.19)).

Compensator Bound Constraint

For given $\alpha_c > 0$, η_z and for any zero $j\omega_z$ of P on the $j\omega$ axis,

$$|T(j\omega)| < \alpha_c |P(j\omega)|, \quad \text{whenever} \quad |\omega - \omega_z| < \eta_z \; . \tag{3.25}$$

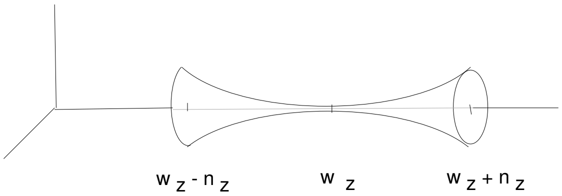

Figure 3.11: Region defined by the compensator bound constraint, if there is a zero of P at $s = j\omega_z$.

3.4.3 Plant bound

Another constraint is that the "closed loop plant" be bounded by a specified value α_p. That is,

$$|(1 - T(j\omega))P(j\omega)| \le \alpha_p \qquad (\forall \omega) \tag{3.26}$$

The closed loop plant is the output of the closed loop system of Fig. 3.1 to an input to the plant.

Inequality (3.26) is analyzed in similar fashion to inequality (3.24). Recall that internal stability of the system implies that $T(j\omega) = 1$ when

$P(j\omega) = \infty$. Inequality 3.26 is binding near the poles of P; it must hold whenever $|\omega - \omega_p| < \eta_p$, where ω_p is any pole of P on the $j\omega$ axis and η_p is a small positive number. From this we get the plant bound constraint, illustrated in Fig. 3.12.

Plant Bound Constraint

For given $\alpha_p > 0$ and $\eta_p > 0$ and all poles $j\omega_p$ of $P(j\omega)$,

$$|1 - T(j\omega)| < \frac{\alpha_p}{|P(j\omega)|} \quad \text{whenever} \quad |\omega - \omega_p| < \eta_p \ . \tag{3.27}$$

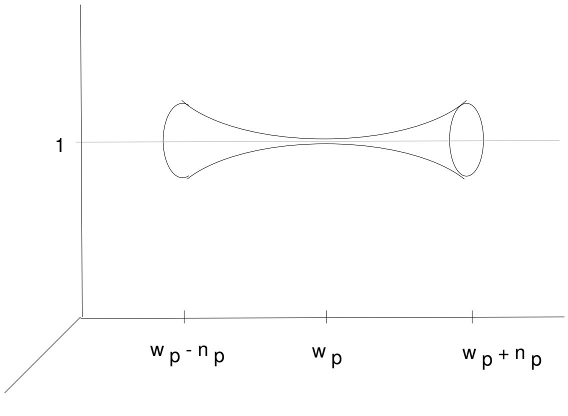

Figure 3.12: Region defined by the plant bound constraint, if P has a pole at $s = j\omega_p$.

3.4.4 Disturbance rejection

Systems are affected by disturbances, and one of the purposes of feeback control is to minimize the effect of these disturbances. As an example, consider the system given by an airplane, with input (established by the pilot) a certain yaw rate $u(t)$, and output the actual yaw rate of the airplane $y(t)$. A gust of wind produces yaw and so it affects the output of the system.

A simple way of modelling this situation is shown in Figure 3.13. There, the disturbance is represented as an additive input to the system at the output to the plant. If there is no other input to the system in Fig. 3.13, the transfer function from disturbance d to plant output y is easily found to be

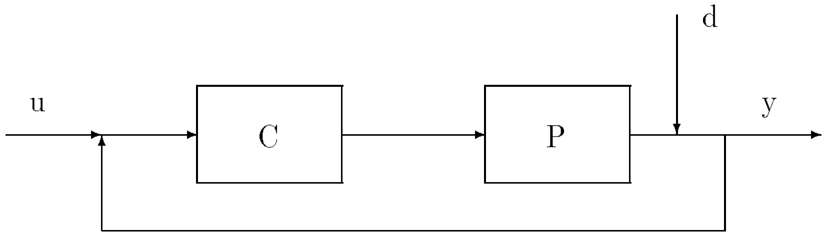

Figure 3.13: Disturbance in a feedback system.

$$T_{d\rightarrow y} = \frac{1}{1+P(s)C(s)} \tag{3.28}$$

For good behavior of the feedback system it is necessary that $T_{d\rightarrow y}$ be RHP-stable. This is always the case with internally stable systems. More can be required, for example, that there is a restriction on the size of $T_{d\rightarrow y}$. The following inequality gives a precise statement, where c is a given constant:

$$|\frac{1}{1+P(j\omega)C(j\omega)}| \leq c \quad \forall\omega \tag{3.29}$$

We have already encountered inequalities similar to (3.29) before under the name tracking error constraint, so at this point we refer the reader to the discussions of the latter.

3.4.5 More on tracking error

Consider a system $\mathcal{S}$ with input u and output y. The steady-state error for input $u(t)$ is

$$e_{ss} = \lim_{t\rightarrow\infty} u(t) - y(t) \tag{3.30}$$

Relation (3.30) has a counterpart in the frequency domain, which can be obtained with the Final Value Theorem. Applying this theorem to 3.30 we obtain

$$e_{ss} = \lim_{s\rightarrow 0} s(U(s) - Y(s)) \tag{3.31}$$

Since $Y(s) = T(s)U(s)$, it follows from (3.31) that

$$e_{ss} = \lim_{s \to 0}(1 - T(s))sU(s) \tag{3.32}$$

If α_{tr} denotes the largest acceptable value of e_{ss} in (3.32), then

$$|e_{ss}| = |\lim_{s \to 0}(1 - T(s))sU(s)| \leq \alpha_{tr} \tag{3.33}$$

If α_{tr} is not 0, one can write an inequality in the variable ω that guarantees (3.33). We call this inequality the tracking error constraint for input $U(s)$.

Tracking Error Constraint for input U

For given $\alpha_{tr} > 0$, $\omega_{tr} > 0$,

$$|(1 - T(j\omega))(j\omega)U(j\omega)| < \alpha_{tr} \quad (|\omega| < \omega_{tr}) \ . \tag{3.34}$$

We now discuss the case where the plant P has n pure integrations (a pole of order n at $s = 0$). In this case, the system $\mathcal{S}$ is called a "type n" system.

We assume that internal stability is a requirement, so RHP poles of the plant P are not canceled by zeros of the compensator C. Since the system is type n, the product PC also has a pole of order n at $s = 0$. In this situation, the closed loop transfer function T satisfies the interpolation conditions

$$T(0) = 1, \ T'(0) = 0, \ \ldots, \ T^{(n-1)}(0) = 0 \tag{3.35}$$

Let us now take $m \leq n + 1$ and $U(s) = 1/s^m$. Because of (3.35), the function

$$(1 - T(s))sU(s) = (1 - T(s))s^{1-m} \tag{3.36}$$

has no poles on the closed RHP, except possibly a pole of order 1 at $s = 0$. Therefore the Final Value Theorem applies and one has

$$e_{ss} = \lim_{s \to 0}(1 - T(s))s^{1-m} \tag{3.37}$$

By combining (3.35) with (3.37), and applying L'hopital's rule for limits we obtain

Proposition 2 *An internally stable type n system $\mathcal{S}$ produces a steady state error e_{ss} for input $U(s) = 1/s^m$ given by*

$$e_{ss} = \begin{cases} 1 - T(0) & \text{if } m = 1 \\ \\ -\frac{T^{(m-1)}(0)}{(m-1)!} & \text{if } 1 < m \leq n+1 \end{cases} \tag{3.38}$$

Hence $e_{ss} = 0$ if $1 \leq m \leq n$.

Proposition 2 implies that for internally stable type n systems, the most appropriate input $U(s) = 1/s^m$ in the constraint (3.34) is $U(s) = 1/s^{n+1}$. Other inputs U that may be suitable are functions that are stable on the closed RHP, except for a pole of order $n+1$ at $s = 0$.

Common time domain specifications on type n systems are on the position, velocity and acceleration error constants [7] K_0, K_1 and K_2, where

$$K_m = \lim_{s \to 0} s^m P(s) C(s) \tag{3.39}$$

One can show that an internally stable type n system $\mathcal{S}$ satisfies $K_0 = T(0)/(1 - T(0))$ and $K_m = \frac{-m!}{T^{(m)}(0)}$ if $1 \leq m \leq n$.

3.5 A fully constrained problem

In Section 3.3 we obtained a single disk inequality from performance requirements bandwidth, tracking and gain-phase margin. We consider here a similar problem, but now with requirements on bandwidth, closed loop roll-off, plant bound, tracking and gain-phase margin. Again, we will derive a single disk inequality from the requirements.

The given plant is

$$P(s) = \frac{1 - s/5}{s(1 - s)}$$

with requirements

a. $|1 - T(jw)| < 0.3$, for $0 \leq \omega < 1.0$.

b. $|1 - T(jw)| < 1/|P(j\omega)|$, for $0 \leq \omega < 1.0$.

c. $|1 - T(jw)| < 2.$ for all ω.

d. $|T(j\omega)| < 0.707$, for $2 < \omega$.

e. $|T(j\omega)| < 4|P(j\omega)|$, for $2 < \omega$.

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[7] denoted by K_p, K_v and K_a in most books

The constraints a, b and c are active at all frequencies $0 \leq \omega < 1$. Clearly constraint c is contained in constraint a, so we restrict attention to a and b. Solve the equation

$$\frac{1}{|P(j\omega)|} = 0.3$$

to obtain the solution $\omega_0 \approx 0.29$. Thus b is more stringent than a on the frequency band $0 \leq \omega \leq 0.29$, and b is less stringent than a on the band $0.29 \leq \omega < 1$. We see that c is the only constraint that applies on the band $1 \leq \omega \leq 2$. For $\omega \geq 2$ it is clear that every T that satisfies d also satisfies c, so it is enough to study constraints d and e on this band. We solve the equation

$$0.707 = 4|P(j\omega)|$$

and obtain as solution $\omega_1 \approx 2.41$. Therefore constraint d is more stringent on the band $2 \leq \omega < 2.41$, whereas e is more stringent on $2.41 < \omega$. We summarize what we have in Table 3.2, that defines the center function K and the radius function K at each frequency. See Figs. 3.14, 3.15 and 3.16 for a graphical representation of the functions K and R.

Constraint	Band	$K(j\omega)$	$R(j\omega)$
Plant	$\lvert\omega\rvert < 0.29$	1	$1/\lvert P(j\omega)\rvert$
Tracking	$0.29 < \lvert\omega\rvert < 1$	1	0.3
g-p margin	$1 < \lvert\omega\rvert < 2$	1	2.
Bandwidth	$2 < \lvert\omega\rvert < 2.41$	0	0.707
Roll off	$\lvert\omega\rvert > 2.41$	0	$4.\lvert P(j\omega)\rvert$

Table 3.2

The format of Table 3.2 is convenient for representing constraints in a way that is easy to read, thus useful in similar problems. Note that in this example we could write the constraints as a single disk inequality. This was possible here because overlapping constraints can be written as a disk constraint again. We will see in Chapters 4 and 5 that this is not always the case. That is, sometimes one cannot define a radius and center functions that incorporate all the information contained in the requirements.

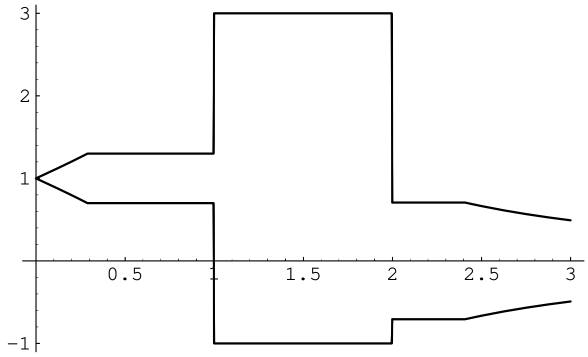

Figure 3.14: Section of the envelope defined by the center and radius functions. The horizontal axis is frequency ω, and the two functions are $K(j\omega) + R(j\omega)$ and $K(j\omega) - R(j\omega)$.

3.6 References and further reading

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