Chapter 17

A sufficient condition for optimality

A sufficient condition for optimality in the problem OPT is presented in this chapter. The main result is given in Section 17.1, but the proof is only given at the end of the chapter in Section 17.4. An example is given in Section 17.2, and some qualitative results on OPT from the literature are given in Section 17.3.

17.1 Complete characterization of solutions

We will use the following notation for the second order partial derivatives:

$$\frac{\partial^2 \Gamma}{\partial \overline{z} \partial z}(e^{j\theta}, z) \ \left(\text{resp. } \frac{\partial^2 \Gamma}{\partial z \partial z}(e^{j\theta}, z)\right) \tag{17.1}$$

to represent the $N \times N$ matrix-valued function with (ℓ, k) entries given by

$$\frac{\partial^2 \Gamma}{\partial \overline{z_\ell} \partial z_k}(e^{j\theta}, z) \ \left(\text{resp. } \frac{\partial^2 \Gamma}{\partial z_\ell \partial z_k}(e^{j\theta}, z)\right)$$

The next result states a sufficient condition for a function to be a solution to OPT.

Theorem 22 *Let Γ be of class C^3 and $f^* = (f_1, \ldots, f_N)$ in A_N be such that*

$$a(e^{j\theta}) := \left(\frac{\partial \Gamma}{\partial z_1}(e^{j\theta}, f^*(e^{j\theta})), \ldots, \frac{\partial \Gamma}{\partial z_N}(e^{j\theta}, f^*(e^{j\theta}))\right) \neq (0, \ldots, 0), \quad \forall \theta$$

Set $A(e^{j\theta}) = \frac{\partial^2\Gamma}{\partial z \partial \overline{z}}(e^{j\theta}, f^*(e^{j\theta}))$, *and* $B(e^{j\theta}) = \frac{\partial^2\Gamma}{\partial^2 z}(e^{j\theta}, f^*(e^{j\theta}))$.

If the function f^* *is a directional solution to* OPT *then, in addition to I and II of Theorem 20, the following condition holds:*

III *For every nonzero* $h \in \mathcal{N} = \{h \in A_N : a(e^{j\theta})^t h(e^{j\theta}) = 0, \forall e^{j\theta}\}$,

$$\sup_\theta \overline{h(e^{j\theta})}^t \mathcal{A}(e^{j\theta}) h(e^{j\theta}) + Re[h(e^{j\theta})^t \mathcal{B}(e^{j\theta}) h(e^{j\theta})] \geq 0$$

Conversely, if conditions I, II, and III with strict inequality hold, then f^* *is a directional minimizer.*

The interested reader may check the reference **[HMer2]** for details.

The following theorem replaces III with a (weaker) condition that is easier to check than III. When taken together with I and II, they are sufficient for optimality.

Theorem 23 *Let* Γ, f^*, a, $\mathcal{A}$ *and* $\mathcal{B}$ *be as in Theorem 20. If, in addition to I and II of Theorem 20, there exist* $\delta > 0$ *and* $e^{j\theta}$ *such that for every* $z \in C^N$,

$$a(e^{j\theta})^t z \; = \; 0 \; \Rightarrow \; \overline{z}^t \mathcal{A}(e^{j\theta}) z + Re[z^t \mathcal{B}(e^{j\theta}) z] \; \geq \; \delta \, || \, z \, ||^2_{C^N} \qquad (17.2)$$

Then f^* *is a directional solution of OPT. Moreover, if the statement (17.2) holds for some* $\delta > 0$ *and all* $e^{j\theta} \in \partial D$ *and* $z \in C^N$, *then* f^* *is a strict local optimizer.*

We present in the Section 17.4 the proof of Theorem 23 since it has not been published before. The reader may skip it without harm. We have the following corollaries of Theorem 23:

Corollary 1 *If* f^* *satisfies I and II, and if the sublevel sets of* Γ *are strictly pseudo-convex, then* f^* *is a strict local optimizer.*

Corollary 2 *If* f^* *satisfies I and II, and if* $\Gamma(e^{j\theta}, z)$ *is strictly convex in* z, *then* f^* *is a strict local optimizer.*

17.2 An Example, continued

We continue our study of OPT for the performance function in (15.3). We will show that f_1 and f_2 are indeed local solutions, and that f_0 is not.

We begin by verifying that f_1 and f_2 are local solutions. To use Theorem 23 we need the second order partial derivatives:

$$\frac{\partial^2\Gamma}{\partial z\partial \overline{z}} = \begin{pmatrix} \begin{matrix} 1+|e^{i\theta}+0.1+0.1z_2|^2 \\ +|0.1z_2+0.1|^2 \end{matrix} & \begin{matrix} (0.1\bar{z}_1+0.1)(e^{i\theta}+0.1+0.1z_2) \\ +(e^{-i\theta}+0.1\bar{z}_1+0.1)(0.1+0.1z_2) \end{matrix} \\ \begin{matrix} (0.1\bar{z}_2+0.1)(e^{i\theta}+0.1+0.1z_1) \\ +(e^{-i\theta}+0.1\bar{z}_2+0.1)(0.1+0.1z_1) \end{matrix} & \begin{matrix} 1+|e^{i\theta}+0.1+0.1z_1|^2 \\ +|0.1z_1+0.1|^2 \end{matrix} \end{pmatrix}$$

and

$$\frac{\partial^2\Gamma}{\partial^2 z} = \begin{pmatrix} 0 & \begin{matrix} 0.1(200+e^{-j\theta}\bar{z}_1+e^{-j\theta}\bar{z}_2 \\ +0.2(\bar{z}_1+\bar{z}_2+\bar{z}_1\bar{z}_2)) \end{matrix} \\ \begin{matrix} 0.1(200+e^{-j\theta}\bar{z}_1+e^{-j\theta}\bar{z}_2 \\ +0.2(\bar{z}_1+\bar{z}_2+\bar{z}_1\bar{z}_2)) \end{matrix} & 0 \end{pmatrix}$$

With the functions $\mathcal{A}$ and $\mathcal{B}$ defined as in Theorem 22, we have that

$$\begin{aligned} \mathcal{A}(e^{j\theta}) &= \begin{pmatrix} 18.82-5.8cos(\theta) & 3.1e^{j\theta}-2.9\overline{e^{j\theta}}-17.98 \\ 3.1\overline{e^{j\theta}}-2.9e^{j\theta}-17.98 & 21.22+6.2cos(\theta) \end{pmatrix} \\ \mathcal{B}(e^{j\theta}) &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

If θ is fixed, then clearly $a(e^{j\theta})^t z = 0$ when z has the form $z^t = (z_1, -z_1) = z_1(1,-1)$ for some $z_1 \in C$. For such z we have that

$$\begin{aligned} \overline{z}^t\mathcal{A}(e^{j\theta_0})z + & \; Re[z^t\mathcal{B}(e^{j\theta_0})z] \\ &= (1,-1)A(e^{j\theta})\begin{pmatrix} 1 \\ -1 \end{pmatrix}|z_1|^2 + Re[(1,-1)B(e^{j\theta})\begin{pmatrix} 1 \\ -1 \end{pmatrix} z_1^2] \\ &= 72|z_1|^2 + Re[-4z_1^2] \\ &\geq 68|z_1|^2 \\ &= 34|z|^2 \end{aligned} \tag{17.3}$$

Therefore condition (17.2) holds for all θ and all z, and this implies that f_1 is a strict local minimizer.

Now we show that $f_0(e^{j\theta}) = (0,0)$ is not a directional minimizer (in particular, it is not a local minimizer). We claim that the function $h(e^{j\theta}) = (1,-1)^t$ is a descent direction relative to f_0, thus implying that f_0 is not a directional minimizer. Indeed, from

$$\begin{aligned}\Gamma(e^{j\theta}, f_0(e^{j\theta}) & +th(e^{j\theta})) \\ & = \Gamma(e^{j\theta}, t, -t) \\ & = 2t^2 + |100 + te^{j\theta} - 0.1t^2|^2 + |100 - te^{j\theta} - 0.1t^2|^2 \\ & = 20000 - 36t + 0.02t^4\end{aligned}$$

we see that for all small $t > 0$,

$$\sup_{\theta} \Gamma(e^{j\theta}, f_0(e^{j\theta}) + th(e^{j\theta})) < 20000$$

17.3 Existence, smoothness, and uniqueness of solutions

In this section we return to basic qualitative questions which were addressed for $N = 1$ in Section 10.4. Do solutions in H^∞_N exist? Are solutions unique? Are solutions smooth?

Again, $S_\theta(c)$ refers to the sublevel sets

$$S_\theta(c) = \{z : \Gamma(e^{j\theta}, z) \leq c\}$$

The main difference between the $N = 1$ and $N > 1$ case is that in higher dimensions solutions may not be unique. We saw an example of non uniqueness earlier in Section 15.4. Those steeped in the lore of Several Complex Variables will appreciate the fact that Γ in this example is strictly plurisubharmonic in z.

Our standard assumption on the (OPT) problem is:

SA Γ depends smoothly on θ, is real analytic in z (and in $\bar{z}$) and has gradient $\overline{\frac{\partial \Gamma}{\partial z}(e^{i\theta}, z)}$ which never vanishes when $\Gamma(e^{i\theta}, z) = \gamma^*$. The sets $\mathcal{S}_\theta(\gamma^*)$ are connected, simply connected, have nonempty interior, and are uniformly bounded in θ.

While γ^* may not be known in advance in a particular situation, one might verify that all $\mathcal{S}_\theta(\gamma)$ for a wide range of γ satisfy these conditions; this is because the conditions are not very restrictive.

We now give a list of results. Many of these can be found in **[HH]**, **[HMar]**, **[HMer2]**. Definitions of the more specialized terms in the theorems are given below.

When $N \geq 1$ we have

Theorem 24 *If $\mathcal{S}_\theta(\gamma^*)$ is strictly convex (uniformly in θ), then an H_N^∞ solution f^* to* OPT *exists and it is unique. Also $f^*(e^{i\theta}) \in \partial\mathcal{S}_\theta$ for a.e. θ.*

Theorem 25 *Suppose* SA *holds and that each $\mathcal{S}_\theta$ is polynomially convex. Then an H_N^∞ solution f^* to* OPT *exists. Moreover, if a sequence $f_k \in H_N^\infty$ approximately solves* OPT *(in the sense $\sup_\theta \Gamma(e^{i\theta}, f_k(e^{i\theta})) = \gamma^k$ with $\gamma^k \searrow \gamma^*$), then a subsequence which converges in normal family sense has as its limit a function f_∞ in H_N^∞ which satisfies $\Gamma(e^{i\theta}, f_\infty(e^{i\theta})) \leq \gamma^*$ almost everywhere.*

A strictly convex set is a convex set with no line segment contained in its boundary. The unit ball in C^N is strictly convex, while the unit ball in the space M_{mn} of $m \times n$ matrices is not unless $m = 1$ or $n = 1$. Polynomially convex sets are a much broader class of sets than convex sets. While convex sets are intersections of half spaces $\{w : Re\ell(w) \leq 0\}$ where ℓ is a linear function into the complex plane. While *polynomially convex sets* are intersections of sets of the form $\{w : Rep(w) \leq 0\}$ where p is a polynomial.

17.4 Proof of Theorem 23

The first part of Theorem 23 is a consequence of the follwing lemma:

Lemma 3 *The hypothesis of Theorem 23 implies III with strict inequality.*

Note that since the mapping given by

$$H(e^{j\theta}, z) = \overline{z}^t \mathcal{A}(e^{j\theta}) z + Re[z^t \mathcal{B}(e^{j\theta}) z] \tag{17.4}$$

is continuous in z and $e^{j\theta}$jointly, there exists an arc $J \subset \partial D$ centered at $e^{j\theta}$ such that relation (17.2) holds for all $e^{j\theta} \in J$. If now III does not hold for some nonzero $h \in \mathcal{N}(a)$, then (17.2) implies that this function satisfies $h(e^{j\theta}) = 0$ for all $e^{j\theta} \in J$. Since h is analytic, h is the function 0, contradicting the assumption on the choice of h. ■

To prove the second assertion in Theorem 23, the following two lemmas are needed.

Lemma 4 *Assume the notation and hypotheses of Proposition 23. If $h \in A_n$ is such that $a(e^{j\theta})^t h(e^{j\theta})$ is not the function 0 then*

$$\sup_\theta Re[a(e^{j\theta})^t h(e^{j\theta})] > 0$$

Proof of Lemma 4. Let $F = (F_1, \ldots, F_N)^t$ and $\lambda(e^{j\theta})$ be the functions that appear in statement II. The scalar function $g(\zeta) = \zeta F(\zeta)^t h(\zeta)$ is analytic and also belongs to the Hardy space H^1. Since $g(0) = 0$ and g is an open mapping (g is nonconstant), it maps a neighborhood of 0 onto a neighborhood of 0. Hence for some $\zeta \in D$, we have that $g(\zeta) > 0$. The Poisson representation of g implies that $Re[g(e^{j\theta})] > 0$ on a set $J \subset \partial D$ with positive measure. Then

$$\sup_\theta Re[a(e^{j\theta})h(e^{j\theta})] = \sup_\theta \lambda(e^{j\theta}) Re[g(e^{j\theta})] > 0 \qquad \blacksquare$$

Lemma 5 *Assume the notation of Proposition 23. There exists η_0 such that if $|Re[a(e^{j\theta})^t z| < \eta_0 ||z||^2_{C^N}$, then $H(e^{j\theta}, z) \geq \delta/2$.*

Proof of Lemma 5. This is a consequence of the continuity of H. $\blacksquare$

Assume for the sake of contradiction that $\{h_n\}$ is a sequence in A_N such that $||h_n||_\infty \to 0$ as $n \to \infty$ and

$$\sup_\theta \Gamma(e^{j\theta}, f^*(e^{j\theta}) + h_n(e^{j\theta})) \leq \sup_\theta \Gamma(e^{j\theta}, f^*(e^{j\theta})) \quad \forall n \tag{17.5}$$

For each n choose θ_n to be a point at which $\sup_\theta Re[a(e^{j\theta})^t h_n(e^{j\theta})]$ is attained, that is,

$$Re[a(e^{j\theta_n})h_n(e^{j\theta_n})] = \sup_\theta Re[a(e^{j\theta})^t h_n(e^{j\theta})] \tag{17.6}$$

We now consider three cases.

Case 1. *The following inequality holds for an infinite number of n's:*

$$0 < Re[a(e^{j\theta_n})^t h_n(e^{j\theta_n})] < \eta_0 ||h_n(e^{j\theta_n})||_{C^N} \tag{17.7}$$

Drop terms if necessary to assume that (17.7) holds for all n. Note that $h_n(e^{j\theta_n}) \neq 0$. We now use the Taylor expansion of Γ in the variable z to define

$$\begin{aligned} x_n := \quad & \Gamma(e^{j\theta_n}, f^*(e^{j\theta_n}) + h_n(e^{j\theta_n})) - \Gamma(e^{j\theta_n}, f^*(e^{j\theta_n})) \\ & -2Re[a(e^{j\theta_n})^t h_n(e^{j\theta_n})] - H(e^{j\theta_n}, z)[h_n(e^{j\theta_n}), h_n(e^{j\theta_n})] \end{aligned}$$

By (17.6) and (17.5) we have that

$$x_n \leq -H(e^{j\theta_n}, h_n(e^{j\theta_n})) \tag{17.8}$$

Note that (17.7) allows us to combine (17.8) with Lemma 5 to conclude that for all n,

$$\frac{x_n}{||h_n(e^{j\theta_n})||^2_{C^N}} \leq -\frac{\delta_0}{2} \tag{17.9}$$

On the other hand, since $h_n(e^{j\theta_n}) \to 0$ as $n \to \infty$, and by the smoothness of Γ, we have that

$$\frac{x_n}{||h_n(e^{j\theta_n})||^2_{C^N}} \to 0 \text{ as } \; n \to \infty \tag{17.10}$$

which contradicts (17.9).
Case 2. *For all n the following relation holds:*

$$Re[a(e^{j\theta_n})^t h_n(e^{j\theta_n})] \geq \eta_0 ||h_n(e^{j\theta_n})||_{C^N} \tag{17.11}$$

Set

$$y_n := \Gamma(e^{j\theta_n}, f^*(e^{j\theta_n}) + h_n(e^{j\theta_n})) - \Gamma(e^{j\theta_n}, f^*(e^{j\theta_n})) - 2Re[a(e^{j\theta_n})^t h_n(e^{j\theta_n})]$$

Since $h_n(e^{j\theta_n}) \to 0$ as $n \to \infty$, and by the smoothness of Γ we have,

$$\frac{y_n}{||h_n(e^{j\theta_n})||_{C^N}} \to 0 \text{ as } \; n \to \infty \tag{17.12}$$

But by relation (17.11) and by the hypothesis (17.5), we have that

$$\frac{y_n}{||h_n(e^{j\theta_n})||_{C^N}} \leq -\frac{\delta_0}{2} \tag{17.13}$$

which contradicts (17.12).
Case 3. *The following equality holds for an infinite number of n's:*

$$Re[a(e^{j\theta_n})^t h_n(e^{j\theta_n})] = 0 \tag{17.14}$$

Drop terms if necessary to assume that this relation holds for all n. Lemma 4 implies that $Re[a(e^{j\theta})^t h_n(e^{j\theta})] = 0$ for all $e^{j\theta}$ and all n. Define x_n as in Case 1 but now θ_n is chosen so that $||h(e^{j\theta})||_{C^N}$ attains its maximum at $\theta = \theta_n$. Just as before, the smoothness of Γ and the fact that $h_n(e^{j\theta_n}) \to 0$ as $n \to \infty$ imply a relation similar to (17.10). On the other hand, the hypotheses (17.14) and (17.2) imply a statement similar to (17.9) but with δ instead of $\delta/2$, thus yielding a contradiction. ■

17.5 References

The main theorem in this section is from **[HMer2]**. More results can be found in **[H3]** and in **[HMar]**. See **[H1]** for connections with other areas of Mathematics. Several Complex Variables connections are discussed in **[HMer1]**, see also **[HH]**, **[HMar]** and **[HV]**.

[H1] J. W. Helton. Operator Theory, Matrices and Electrical Engineering. NSF. 1987

[H2] J. W. Helton. "Optimization over spaces of analytic functions and the Corona problem," J. Operator Theory, 15 (1986), No. 2, 359–375.

[H3] J. W. Helton: Optimal frequency domain design vs. an area of several complex variables, plenary address. MTNS, 1989.

[HH] J. W. Helton and R. Howe. "A bang-bang theorem for optimization over spaces of analytic functions," J. Approx. Theory, 47 (1986), No. 2, 101–121.

[HMar] J. W. Helton and D. Marshall: "Frequency domain design and analytic selections," Indiana Univ. Math. Journal, 39 (Spring 1990), No. 1, 157–184.

[HMer1] J. W. Helton and O. Merino: "Optimal Analytic Disks," Collection: Several complex variables and complex geometry, Part 2, (Santa Cruz, CA, 1991) Proc. Sympos. Pure Math., 5 251–262.

[HMer2] J. W. Helton and O. Merino. Conditions for Optimality over $\mathcal{H}^\infty$. SIAM J. Cont. Opt., Vol. 31, No. 6, November 1993.

[HV] J. W. Helton, and A. Vityaev. Preprint

Part V

Appendices

Appendix A

History and Perspective

While historians trace control to Archimedes time and the theory of control to a paper of James Clerk Maxwell on governors, the subject called classical control began in the second world war in labs in England and the U.S. which designed radar-driven antiaircraft guns. The beginnings of classical control were summarized in a book **[JNP]** by James, Nichols, and Phillips. According to Ralph Phillips, a book which was very influential in their lab was Bode's famous book on amplifiers. However, this book was not referenced in **[JNP]**. The primary technique which emerged was adjusting parameters in low order (e.g., degree two) rational functions and checking that the graphs lie in certain regions. Classical control dominated industrial practice for many years, even though it could only be taught by example and its main technique was trial and error.

Much of the theory of control in the 1960's and 70's focused on achieving desired frequency domain performance as closely as possible in a mean-square-error sense. We have nothing to add to the literature here so do not give a historical treatment.

The subject of optimizing worst case error in the frequency domain along its present lines started not with control but with circuits. One issue was to design amplifiers with maximum gain over a given frequency band. Another was the design of circuits with minimum broadband power loss. Indeed H^∞ control is a subset of a broader subject, worst case design in the frequency domain. In paradigm engineering problems this produces what a mathematician calls an 'interpolation problem' for analytic functions. The techniques of Nevanlinna-Pick interpolation had their first serious introduction into engineering in a SISO circuits paper of Youla and Saito **[YS]** in the middle 1960's. Further development waited until the mid-seventies, when Helton **[H1–H3]** applied them and more general techniques from operator theory to amplifier problems. The methods of commutant lifting **[A, NF, S]** and of Admajan-Arov-Krein **[AAK1–AAK3]** were used to solve MIMO optimization problems. The disk method used here was first described in an engineering article on gain equalization **[H3]**, Tnd followed the foundational theory which was published in the mathematics article **[H2]** two years earlier.

In the late 1970s G. Zames **[Z]** began to marshal arguments indicating that H^∞ rather than H^2 was the physically proper setting for control. Zames suggested on several occasions that these methods were the appropriate ones for codifying classical control. These efforts yielded a mathematical problem which Helton identified as an interpolation problem solvable by existing means (see acknowledgement in **[ZF1]**). Zames and Francis **[ZF]** used this to solve the resulting problem. Soon thereafter Chang-Pearson **[CP]** and Francis-Helton-Zames solved it for many-input, many-output MIMO systems **[FHZ]**.

The pioneering work of Zames and Francis treated only sensitivity optimization. In 1983 three independent efforts emphasized bandwidth constraints, formulated the problem as a precise mathematics problem and indicated effective numerical methods for their solution: Doyle **[D1]**, Helton **[H4]**, and Kwakernaak **[K1]**. Helton treated the problem graphically, just as it is done in Part I of this book, by approximating multiple disk constraints with a single moving disk. Kwakernaak gives an equivalent method, but it is expressed algebraically in terms of two weight functions W_1 and W_2 on the sensitivity function $I - T$ and on T. Also all of these papers described quantitative methods which were soon implemented on computers. It was these papers actually laid out precisely the tradeoff in control between performance at low frequency and roll off at higher frequency and how one solves the resulting mathematics problem. This is in perfect analogy with amplifier design where one wants large gain over as wide a band as possible, producing the famous gain bandwidth trade off. Rather remarkable in textbook treatments of control through the 1970's was the absence (or cursory treatment) of bandwidth constraints. On the other hand, many control practitioners of that period would say that bandwidth constraints were important, but when they went to formulate them as a mathematical optimization problem would provide no high frequency constraint. Thus they made the fundamental mistake of H^∞ control (see Section 11.4), and the concomitant confusion cost the methods much credibility. As of the 1990's the control community's views have become more realistic. This conceptual shift is one of the main accomplishments of H^∞ control. The moral of the story is that nothing stabilizes a nebulous philosophical discussion like the the ability to solve the math problems which arise exactly. Mush dries up quickly under this torch.

All of the traditional methods in H^∞ solved only optimization problems where the specification sets $\mathcal{S}_\theta$ are disks. There is a beautiful and extensive theory of this. Much of the mathematical theory came directly from existing work in the study of operators on Hilbert Space. However, an important accomplishment of theoretical engineers was to formulate the problem in a new set of coordinates (called state space coordinates) which are well suited to many problems. This is an extremely elegant theory (cf **[BGR]**, **[F]**, **[GL]**, **[GM]**).

As basic as it is, the work on internal stability treated in Part I was done only recently. In the famous article of Youla, Jabr, Bongiorno **[YJB]**, the concept of internal stability was introduced and reduced to a mathematical problem in which all H^∞ functions meet a given interpolation constraints. The authors then solved the H^2 control problem, (rather than optimizing over H^∞ as is done in this book).

Independent of the rise of H^∞ control was the Horowitz approach **[Ho]**. The relationship between his approach and ours is described in **[BHMer]**. Another independent development was Tannenbaum's **[T]** very clever use of Nevanlinna-Pick interpolation in a control problem in 1980. Also early on the H^∞ stage was Kwakernaak's polynomial theory cf.**[K2]**. Another major development that dovetailed closely with the invention of H^∞ control was a tractable theory of plant uncertainty. A good historical treatment appears in **[DFT]**. Another application of these techniques is to robust stabilization of systems (**[Kim]**)

The mathematical theory in this book (Part IV) is quite different and more general than the classical mathematical theory (Nevanlinna-Pick, Nehari) which applies only to disk problems. In Chapter 16 we showed how the basic result of Nehari's for disks follows from our approach. Surprisingly, it is not much more difficult to derive by our approach than it is classically. The reader interested in connections of the theory developed here with other areas may want to see **[Dy]**, **[FF]** and **[H8]**. Work on the general theory for solving the optimization problem OPT described in Part III and Part IV started in 1981 by Helton and Howe, who worked on convex problems **[HH]**. Since then theoretical work has been done by J. W. Helton, S.Hui, D.Marshall, O. Merino, Z.Slodkowski, A. Vityaev K. Lenz and E. Wegert in papers listed in the bibliography. Theoretical work pertaining to numerical issues has been continued by Helton, Merino, and T. Walker, **[HMer2]**, **[HMer3]**, **[HMW]**, and numerical testing and more theoretical work has been done by J. Bence, J. W. Helton, O. Merino, J. Myers, D. Schwartz, T. Walker. Interestingly, the theory developed to solve the engineering problem OPT has strong ties with ongoing work in the study of functions of several complex variables. For an account of this see **[H5]**, **[HMer1]**. The main result used here (Theorem **[OPT** $N > 1$**]**)

is taken from **[HMer2]**. The paper **[HMW]** introduced the Newton computer algorithm of Chapter 7. The reason Newton's method had not been succesfull before on such an old problem is probably because sup norm is not a smooth performance function. Thus one needs a special approach to discover it.

The SISO winding number results PART IV are from **[H6]**.

References

[AAK1] V. M. Adamjam, D. Z. Arov and M. G. Krein. Infinite Hankel matrices and generalized problems of caratheodory-Fejer and F. Riesz, Functional Anal. Appl. 2 (1968), pp. 1-18.

[AAK2] V. M. Adamjam, D. Z. Arov and M. G. Krein. Analytic properties of Schmidt pairs for Hankel operator and the generalized Schur-Takagi problem. Math. USSR-Sb. 15 (1972), pp. 15-78.

[AAK3] V. M. Adamjam, D. Z. Arov and M. G. Krein. Infinite block Hankel matrices and related extension problems, Amer. Math. Soc. Translations 111 (1978), 133-156.

[A] T. Ando. On a pair of commuting contractions. Acta Sci. Math. (Szeged) 24, 1963, 88-90.

[BGR] J. A. Ball, I. Gohberg and L. Rodman. Interpolation of rational matrix functions. Birkhauser-Verlag, Basel, 1990.

[BHM] J. Bence, J. W. Helton and D. E. Marshall. "Optimization over H-infinity," Proc. Conference on Decision and Control, Athens, Greece, December 1986.

[BHMer] F. N. Bailey, J. W. Helton and O. Merino: "Alternative approaches in frequency domain design of single loop feedback systems with plant uncertainty," to appear (1994).

[CP] B. C. Chang and B. Pearson Jr., Optimal disturbance reduction in linear multivariable systems. IEEE Trans. Automat. Contr., vol AC-29, pp. 880-887, 1984.

[D1] J. C. Doyle. Synthesis of robust controllers and filters. Proceedings of 22nd. IEEE Conference on Decision and Control, San Antonio, Texas 1983.

[D2] J. C. Doyle. Robust stability with structured perturbations. Proc. Sym. on Mathematical Theory of Networks and Systems. Stockholm, 1985.

[DFT] Doyle, J. , Francis, B. A., and Tannenbaum, A. Feedback Control Theory. Macmillan Pub. Co., New York, 1992.

[DGKF] J. C. Doyle, K Glover, P. P). Khargonekar, B.A. Francis. State-space solutions to standard H^2 and H^∞ control problems. IEEE Trans. on Automat. Contr. vol 34, pp. 831-847, 1989.

[DS] Doyle, J. and G. Stein, Multivariable feedback design: Concepts for a classical/modern synthesis. IEEE Trans. Automat. Contr., vol. AC-26, pp. 4-16, 1981.

[Dy] H. Dym. J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation. CBMS Reg. Conf. Ser. in Math., vol 71, Amer. Math. Soc., Providence, RI, 1989

[F] B. Francis. First course in H^∞-control. Springer Verlag. 1987

[FHZ] Francis, B. A., Helton, J. W. and Zames, G. H^∞-optimal feedback controller for linear multivariable systems. IEEE Trans. Auto Comnt. , vol AC-29, pp. 888-900.

[FFHKP] C. Foias, B. Francis, J. W. Helton, H. Kwakernaak, J.B.Pearson. H^∞ -control Theory. Lecture Notes in Mathematics No. 1496. E. Mosca and L. Pandolfi, editors. Springer Verlag, Berlin. 1991.

[FF] C. Foias and Frazho. The commutant lifting approach to interpolation problems. Birkhauser-Verlag, Basel, 1991.

[GL] K. Glover and D. C. McFarlane. Robust Controller Design Using Normalized Coprime Factor Pland Descriptions. Vol. 138 in Lecture Notes in Control and Information Sciences. Springer Verlag, New York. 1990.

[GM] M. Green and D. J. N. Limebeer. Book

[H1] J. W. Helton. "Operator theory and broadband matching," Proceedings of Eleventh Annual Allerton Conference on Circuits and Systems Theory, 1976.

[H2] J. W. Helton. "A mathematical view of broadband matching," IEEE International Conference on Circuits and Systems Theory, New York, 1978.

[H3] J. W. Helton. "Broadbanding gain equalization directly from data," IEEE Trans. Circuits and Systems, CAS-28 (1981), No. 12, 1125–1137.

[H4] J. W. Helton. "An H-infinity approach to control," IEEE Conference on Decision Control, San Antonio, Texas, December 1983

[H5] J. W. Helton. "Worst case analysis in the frequency domain: The H-infinity approach to control," IEEE Trans. Auto. Cont., AC-30 (December 1985), No. 12, 1154–1170.

[H6] J. W. Helton. "Optimization over spaces of analytic functions and the Corona problem," J. Operator Theory, 15 (1986), No. 2, 359–375.

[H7] J. W. Helton: Optimal frequency domain design vs. an area of several complex variables, plenary address. MTNS, 1989.

[H8] J. W. Helton. Operator Theory, Analytic Functions, Matrices and Electrical Engineering. CBMS regional conference series in Mathematics. No. 68, Amer. Math. Soc., Providence, RI. 1987.

[HH] J. W. Helton and R. Howe. "A bang-bang theorem for optimization over spaces of analytic functions," J. Approx. Theory, 47 (1986), No. 2, 101–121.

[HMar] J. W. Helton and D. Marshall: "Frequency domain design and analytic selections," Indiana Univ. Math. Journal, 39 (Spring 1990), No. 1, 157–184.

[HMer1] J. W. Helton and O. Merino: "Optimal Analytic Disks," Collection: Several complex variables and complex geometry, Part 2, (Santa Cruz, CA, 1991) Proc. Sympos. Pure Math., 5 251–262.

[HMer2] J. W. Helton and O. Merino. Conditions for Optimality over $\mathcal{H}^\infty$. SIAM J. Cont. Opt., Vol. 31, No. 6, November 1993.

[HMer3] A Novel Approach to Accelerating Newton's Method for sup-norm optimization arising in H^∞-control. J. Optim. Theory Apps. 1993

[HMW] J. W. Helton, O. Merino and T. E. Walker. Algorithms for Optimizing over Analytic Functions. Indiana U. Math. J., Vol 42, No.3 (1993) .

[HV] J. W. Helton, and A. Vityaev. Preprint

[Ho] Horowitz, I. M. Synthesis of Feedback Systems. Academic Press, NY. 1963.

[Hui] S. Hui. Qualitative Properties of Solutions to H^∞-Optimization. Problems. J. of Functional Analysis **75** (1987) 323,348.

[JNP] James, H. M., Nichols N. B. and Phillips, R. P. Theory of Servomechanisms, Radiation Lab. Series, Vol 25, McGraw-Hill, New York, 1947.

[Kim] H. Kimura Robust stabilizability for a class of transfer functions, IEEE Trans Auto. Control AC-29 1984 788-793

[K1] H. Kwakernaak Robustness Optimization of Linear Feeedback Systems. IEEE Conf. Decision Control, San Antonio, Texas, December 1983

[K2] H. Kwakernaak. EEC 1993

[L] K. Lenz. Properties of certain optimal weighted mixed sensitivity designs. To appear.

[NF] Sz.-Nagy, B. and Foias, C. Harmonic Analysis of Operators on Hilbert Space. North Holland, 1970.

[S] Sarason, D. Generalized interpolation in H^∞. Trans. Amer. Math. Soc. vol 127, pp. 179-203. 1967.

[T] A. Tannenbaum. Feedback stabilization of plants with uncertainty in the gain factor. Int. J. of Control, vol. 32, pp1-16, 1980.

[W] E. Wegert Nonlinear Boundary value Problems for Holomorphic Functions and Singular Integral Equations, Academie Verlag, Berlin 1992

[YJB1] D.C. Youla, H. A. Jabr, J.J. Bongiorno. Modern Wiener-Hopf design of optimal controllers -Part I: The single input - single output case. IEEE Trans. Automat. Contr., vol AC-21, pp. 319-338, 1976.

[YJB2] D.C. Youla, H. A. Jabr, J.J. Bongiorno. Modern Wiener-Hopf design of optimal controllers -Part II: The multivariable case. IEEE Trans. Automat. Contr., vol AC-21, pp. 3-13, 1976.

[YS] D.C. Youla and M. Saito. Interpolation with positive real functions. H. Franklin Inst., vol 284, pp. 77-108, 1967.

[Z] Optimal sensitivity and feedback: weighted seminorms, approximate inverses, and plant invariant schemes. Proc. Allerton Conf. 1979.

[ZF1] G. Zames and B. Francis. Feedback and minimax sensitivity. Advanced Group for Aerospace Research and Development, NATO Lecture Notes, No. 117, Multivariable Analysis and Design Techniques

[ZF2] G. Zames and B. Francis. Feedback, minimax sensitivity, and optimal robustness, IEEE Trans. Automat. Contr., vol AC-28, PP. 585-601, 1983

Appendix B

Other Software