Note that $f(s)$ has a pole in the L.H.P. and a zero in the R.H.P. Correspondingly, $F(\zeta)$ has a pole outside the unit disk and a zero inside the unit disk.

Instead of A_{RHP} we will use instead its "disk" counterpart that we denote by A. In other words, A is the set of all functions $f(\zeta)$ that are bounded, real and continuous on the closed unit disk, and analytic on the open unit disk. With the change of coordinates, the main optimization problem becomes

OPT Given a performance function $\Gamma(e^{j\theta}, z)$, find $\gamma^* \geq 0$ and f^* in A such that

$$\gamma^* = \inf_{f \in A} \sup_{\theta} \Gamma(e^{j\theta}, f(e^{j\theta})) = \sup_{\theta} \Gamma(e^{j\theta}, f^*(e^{j\theta})).$$

9.4 Performance functions and sublevel sets

With the coordinate change introduced in the previous subsection, the sublevel sets of $\Gamma(e^{j\theta}, z)$ now depend on θ:

$$S_\theta(c) = \{z : \Gamma(e^{j\theta}, z) \leq c\} \tag{9.5}$$

Theory of the OPT problem (see [HMar]) implies that the existence and other properties of solutions to OPT are closely related to properties of the sublevel sets of the performance function. Usually "well behaved" sublevel sets correspond to solutions one can consider "nice". A well behaved sublevel set has the following characteristics:

- bounded: the set is contained in some disk with finite radius
- connected: two points in $S_\theta(c)$ can be joined by an arc that lies totally in $S_\theta(c)$
- simply connected: no "holes", i.e., the set complementary to $S_\theta(c)$ is connected
- The gradient of $\Gamma(e^{j\theta}, z)$ in $z = x + jy$ does not vanish when z is in the boundary of $S_\theta(c)$

Other desirable properties are:

- convexity: two points in $S_\theta(c)$ can be joined by a segment that lies totally in $S_\theta(c)$

- smooth boundary
- smooth dependence on θ

Example 1. For the performance function $\Gamma(e^{j\theta}, z) = |1/(e^{j\theta} + 0.5) - z|^2$ the sublevel set $S_\theta(c)$ is a disk, centered at $1/(e^{j\theta} + 0.5)$ with radius $\sqrt{c}$.

Example 2. If $\Gamma(e^{j\theta}, z) = (2 + \cos(\theta))|1/(e^{j\theta} + 0.5) - z|^2$, then the sublevel set $S_\theta(c)$ is a disk, centered at $1/(e^{j\theta} + 0.5)$ with radius $\sqrt{c/(2 + \cos(\theta))}$.

Example 3. Suppose that two continuous functions $k(e^{j\theta})$ and $w(e^{j\theta})$ are given, and that $w(e^{j\theta}) > 0$ for all θ. With these functions we build the performance

$$\Gamma(e^{j\theta}, z) = w(e^{j\theta}) \mid k(e^{j\theta}) - z \mid^2 \tag{9.6}$$

This performance function appears in control applications and it is quite informative to study. The sublevel sets $S_\theta(c)$ of Γ are disks with center $k(e^{j\theta})$ and radius $(c/w(e^{j\theta}))^{1/2}$. Thus $S_\theta(c)$ is connected and simply connected, and it is bounded since the radius is finite. The sets S_θ are also convex and have smooth boundary. Since $k(e^{j\theta})$ and $w(e^{j\theta})$ are continuous, we have that $S_\theta(c)$ varies continuously with θ. If $k(e^{j\theta})$ and $w(e^{j\theta})$ are smooth, then $S_\theta(c)$ varies smoothly in θ for fixed c. Performance functions with sublevel sets that are disks are called *quasicircular*. The following is an example.

Example 4. Suppose that $w_1(e^{j\theta})$ and $w_2(e^{j\theta})$ are given positive functions, and

$$\Gamma(e^{j\theta}, z) = w_1(e^{j\theta}) \mid z \mid^2 + w_2(e^{j\theta}) \mid 1 - z \mid^2 \tag{9.7}$$

This performance function arises in H^∞ -control literature, where it is called *mixed sensitivity*. It is quasicircular, a fact that we now verify. Use the formula $\mid a - b \mid^2 = \mid a \mid^2 - 2\mathrm{Re}\,(\overline{a}\, b) + \mid b \mid^2$ to rewrite (9.7) as

$$\Gamma(e^{j\theta}, z) = (w_1(e^{j\theta}) + w_2(e^{j\theta})) \mid z \mid^2 - 2\mathrm{Re}\,(w_2(e^{j\theta}) z) + w_2(e^{j\theta}) \tag{9.8}$$

Collecting terms and completing the square we get

$$\Gamma(e^{j\theta}, z) = (w_1(e^{j\theta}) + w_2(e^{j\theta})) \mid z - \frac{w_2}{w_1(e^{j\theta}) + w_2(e^{j\theta})} \mid^2 + \frac{w_1(e^{j\theta})\, w_2(e^{j\theta})}{w_1(e^{j\theta}) + w_2(e^{j\theta})} \tag{9.9}$$

Therefore $S_\theta(c)$ is a disk with center $w_2(e^{j\theta})/(w_1(e^{j\theta}) + w_2(e^{j\theta}))$ and radius

$$\frac{1}{w_1(e^{j\theta}) + w_2(e^{j\theta})}\,(c - \frac{w_1(e^{j\theta})\, w_2(e^{j\theta})}{w_1(e^{j\theta}) + w_2(e^{j\theta})}),$$

and Γ is quasicircular.

Example 5. If $w_1(e^{j\theta})$ and $w_2(e^{j\theta})$ are given positive functions, and

$$\Gamma(e^{j\theta}, z) = w_1(e^{j\theta}) \ \big(\mathrm{Re}(\ 1/e^{j\theta} - z\ )\big)^2 \ + \ w_2(e^{j\theta}) \ \big(\mathrm{Im}(\ 1/e^{j\theta} - z\ )\big)^2 \tag{9.10}$$

then $S_\theta(c)$ is an ellipse centered at $1/e^{j\theta}$ and semiaxes that vary with θ.

Example 6. Set

$$\Gamma(e^{j\theta}, z) = |\ 1 \ + \ (1/e^{j\theta} \ - \ z)^2\ |^2 \tag{9.11}$$

Then $S_\theta(c)$ is a lemniscate centered at $1/e^{j\theta}$. For small c the sublevel sets are convex, for c slightly smaller than 1 the sublevel sets are non-convex, and for $c > 1$ the sublevel sets are no longer connected sets.

Example 7. Let $w_1(e^{j\theta})$ and $w_2(e^{j\theta})$ be given positive functions, and

$$\Gamma(e^{j\theta}, z) = \max\{\ w_1(e^{j\theta})|\ 1/e^{j\theta} - z\ |\ ,\ w_2(e^{j\theta})|\ \frac{1}{e^{j\theta} - 0.5} - z\ |\ \} \tag{9.12}$$

Then $S_\theta(c)$ is the intersection of two disks centered at $1/e^{j\theta}$ and at $\frac{1}{e^{j\theta} - 0.5}$, with radii $c/w_1(e^{j\theta})$ and $c/w_2(e^{j\theta})$.

Example 8. Let $w_1(e^{j\theta})$ and $w_2(e^{j\theta})$ be given positive functions, and

$$\Gamma(e^{j\theta}, z) = w_1(e^{j\theta})\mathrm{Re}(\ 1/e^{j\theta} - z\ ) \ + \ w_2(e^{j\theta})\mathrm{Im}(\ \frac{1}{e^{j\theta} - 0.5} - z\ ) \tag{9.13}$$

Then $S_\theta(c)$ is a rhombus centered at $1/e^{j\theta}$.

Chapter 10

Solutions to OPT

Suppose that we are handed a candidate f for a solution to an OPT problem and asked if it is a solution. The main result in this chapter (Theorem 1) provides such a test.

10.1 Complex partial derivatives

Since

$$x = \frac{1}{2}(z + \overline{z})$$

$$y = \frac{1}{2j}(z - \overline{z})$$

we can change the variables in a function g of x and y and write g as a function of the variables z and $\overline{z}$. For example,

$$\begin{aligned} g &= x^2 + y^2 && \Rightarrow \quad g = z\overline{z} \\ g &= x^2 - y^2 + 2xyj && \Rightarrow \quad g = z^2 \\ g &= y - x^2 j && \Rightarrow \quad g = \tfrac{1}{2j}(z - \overline{z}) - (\tfrac{z+\overline{z}}{2})^2 j \end{aligned} \tag{10.1}$$

Partial derivatives with respect to the variables z and $\overline{z}$ are defined by the formulas

$$\begin{aligned} \tfrac{\partial}{\partial z} &= \tfrac{1}{2}(\tfrac{\partial}{\partial x} - j\tfrac{\partial}{\partial y}) \\ \tfrac{\partial}{\partial \overline{z}} &= \tfrac{1}{2j}(\tfrac{\partial}{\partial x} + j\tfrac{\partial}{\partial y}) \end{aligned} \tag{10.2}$$

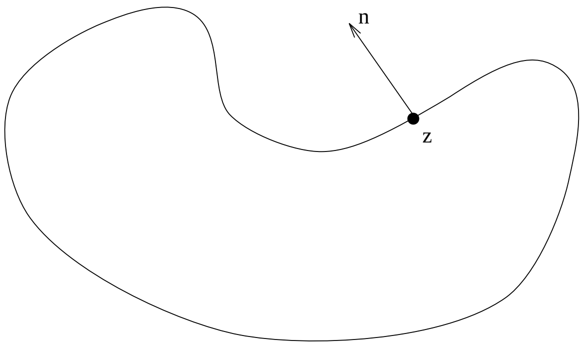

Figure 10.1: The sublevel set $S_\theta(c)$ and the gradient of a real valued function $g(z)$.

The partial derivatives with respect to z and to $\overline{z}$ satisfy the standard differentiation rules for functions of one real variable. Some examples are

$$\begin{aligned} \frac{\partial}{\partial z}\left(\mid z \mid^2\right) &= \frac{\partial}{\partial z}\left(z\overline{z}\right) = \overline{z} \\ \frac{\partial}{\partial z}\left(z^2\right) &= 2z \\ \frac{\partial}{\partial \overline{z}}\left(z^2\right) &= 0 \\ \frac{\partial}{\partial z}\left(\frac{1}{2j}(z-\overline{z}) - (\frac{z+\overline{z})}{2})^2 j\right) &= \frac{1}{2j} - (z+\overline{z}) \end{aligned} \tag{10.3}$$

It is well known that complex functions $g = u(x,y) + jv(x,y)$ are analytic as a function of $z = x + jy$ if and only if u and v are harmonic functions that satisfy the Cauchy-Riemann equations (c.f. [**A**]). On the other hand it can be shown that the Cauchy-Riemann equations can be written as $\frac{\partial}{\partial \overline{z}} g = 0$. Thus the latter equation is equivalent to saying that g is analytic.

A consequence of the definition of $\partial/\partial\overline{z}$ is that if g is a *real valued* function of z then $\frac{\partial g}{\partial \overline{z}} g$ is precisely the gradient of g in the real variables x and y. Thus we have that (abusing notation),

$$\frac{\partial g}{\partial \bar{z}}(z) = \overline{\frac{\partial g}{\partial z}}(z) = \nabla_{x,y} g(x+iy) = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})$$

See Fig. 10.1.

Although we write functions $\Gamma(e^{j\theta}, z)$ as dependent on z only, actually they also depend on $\overline{z}$. For convenience the variable $\overline{z}$ is not made explicit. For example,

$$\Gamma(e^{j\theta}, z) = \left|\ 1/e^{j\theta} - z\right|^2 \tag{10.4}$$

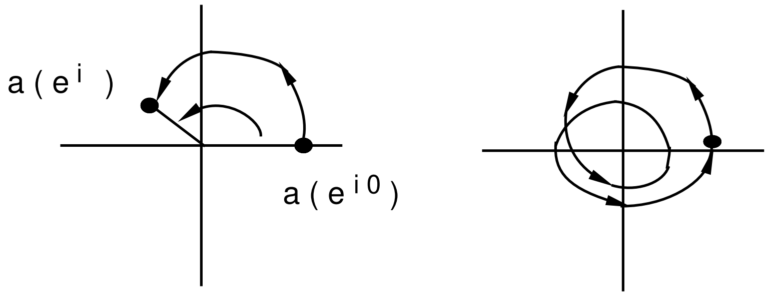

Figure 10.2: The winding number of a function about 0.

is a function of *both* z and $\overline{z}$:

$$\Gamma(e^{j\theta}, z) = \left(1/e^{j\theta} - z \right) \overline{\left(1/e^{j\theta} - z \right)} = \left(1/e^{j\theta} - z \right) \left(e^{j\theta} - \overline{z} \right) \tag{10.5}$$

In particular we have that

$$\frac{\partial \Gamma}{\partial z}(e^{j\theta}, z) = (-1)\, \overline{\left(1/e^{j\theta} - z \right)} \tag{10.6}$$

10.2 Winding number

Another concept we need to introduce is *winding number* of a curve with respect to a point. If $a : \partial D \rightarrow C$ is continuous and never 0, then as θ varies from 0 to 2π there is always a well defined *phase* (or angle) determined by $a(e^{j\theta})$ and the positive semiaxis. Furthermore we can choose the phase to vary continuously with θ, provided we allow for angles to take any real value. This phase function is called a *continuous argument* (c.f. [**A**]).

Since $a(e^0) = a(e^{2\pi i})$, the increment in the value of any continuous argument from $\theta = 0$ to $\theta = 2\pi$ is an integer multiple of 2π, say $2\pi n$ (see Fig. 10.2). The number n is called the "winding number of $a(\cdot)$ with respect to 0, and is denoted wind (a ; 0).

10.3 Main result on the characterization of solutions

The characterization of scalar valued solutions to OPT is

Theorem 13 *Suppose a given performance function* $\Gamma(e^{j\theta}, z)$ *is of class* C^3. *Let* f^* *be a function in* A *for which the function* $a(e^{j\theta}) \stackrel{\Delta}{=} \frac{\partial \Gamma}{\partial z}(e^{j\theta}, f^*(e^{j\theta}))$ *never vanishes. Then* f^* *is a strict local minimizer to OPT if and only if* *(i)* $\Gamma(e^{j\theta}, f^*(e^{j\theta}))$ *is constant in* θ.

(ii) $\text{wind}\ (\ a\ ;\ 0\ ) > 0$.

The following is a very nice geometric interpretation of (i) and (ii) of Theorem 13. Suppose that γ^* is the optimal value of OPT. The sublevel sets

$$S_\theta(c) = \{z \in C : \Gamma(e^{j\theta}, z) \leq c\}$$

at $c = \gamma^*$ can be represented as a solid in 3D as shown in Fig. 10.3. In this figure the variable θ is represented in one of the axes, and at each constant θ the plane perpendicular to the θ axis represents the complex plane containing the sublevel set S_θ.

Condition (i) in Theorem 13 implies that $f^*(e^{j\theta})$ is always on the boundary of the set $S_\theta(\gamma^*)$. Thus as θ varies from 0 to 2π, the pair $(\theta, f(e^{j\theta}))$ describes a curve on the boundary of the solid shown in Fig. 10.3. In other words, if the curve $(e^{j\theta}, f^*(e^{j\theta}))$ dips inside the solid, or if it goes out of the solid, then f^* does not satisfy condition (i).

To see what (ii) means, consider the function $\overline{a(e^{j\theta})} = \overline{\frac{\partial \Gamma}{\partial z}(e^{j\theta}, f^*(e^{j\theta}))}$. At each θ this represents an exterior normal to the set $S_\theta(\gamma^*) \subset C$ at the point $f^*(e^{j\theta})$. As θ varies from 0 to 2π, the function $\overline{a}$ winds about the θ axis. Condition (ii) of the theorem says that if f^* is an optimizer, then $\bar{a}$ winds at least once in the clockwise sense.

Example 1. The constant function $f(e^{j\theta}) = 0$ is a solution to OPT when the performance is $\Gamma(e^{j\theta}, z) = |1/e^{j\theta} - z|^2$. Indeed, $\Gamma(e^{j\theta}, 0) = 1$ and $\frac{\partial \Gamma}{\partial z}(e^{j\theta}, 0) = -e^{j\theta}$, so conditions (i) and (ii) in Theorem 1 are satisfied.

Exercise. Let α and β be positive constants. Consider the performance

$$\Gamma(e^{j\theta}, z) = \alpha\ \left(\text{Re}(\ 1/e^{j\theta} - z\ )\right)^2\ +\ \beta\ \left(\text{Im}(\ 1/e^{j\theta} - z\ )\right)^2 \qquad (10.7)$$

Prove that for a suitable c_0, the function $f(e^{j\theta}) = c_0 e^{j\theta}$ is a solution to OPT.

10.4 Properties of solutions

A knowledge of the fundamental qualitative properties of solutions is very useful to someone person who is using a computer program to solve problems like OPT. Then when the program produces odd answers the user can have an idea of what is happening or at least eliminate some possibilities.

Three important questions from the qualitative point of view are:

- Does OPT have a solution or merely an optimizing sequence?

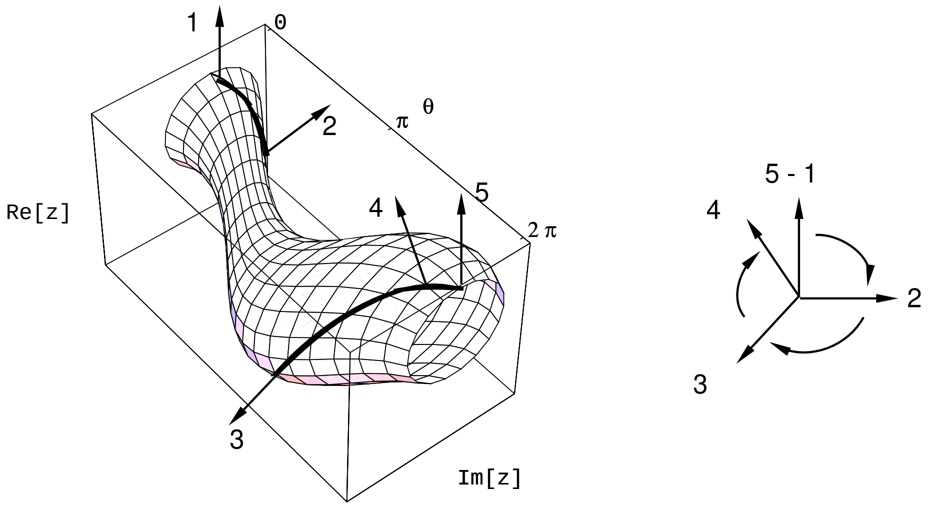

Figure 10.3: The optimal performance surface. In (a), the curve $(e^{j\theta}, f^*(e^{j\theta}))$ lies on the boundary of the solid. As θ increases from 0 to 2π, the arrow representing a normal to the sublevel set makes a complete clockwise revolution with respect to the θ axis, as indicated in (b).

- If a solution exists is it unique?
- If a solution exists is it smooth?

In the present case of scalar valued functions f, the OPT problem is incredibly well behaved (even when the performance function Γ is not convex in z). Solutions exist in A and are unique! Solutions are smooth (infinitely differentiable) if the performance function Γ is smooth. A discussion of existence and uniqueness of solutions in the scalar case can be found in **[HMar]**.

The sublevel sets

$$S_\theta(c) = \{z : \Gamma(e^{j\theta}, z) \leq c\}$$

are basic to the study of qualitative properties of OPT. The *standard assumption* that we make on sublevel sets is

Standard Assumption (SA): $S_\theta(c)$ is connected and simply connected, and as θ varies it is uniformly bounded and has area uniformly bounded away from 0. Furthermore, the gradient of $\Gamma(e^{j\theta}, z)$ in $z = x + jy$ is not zero for all z in the boundary of $S_\theta(c)$.

Theorem 14 *Let $\Gamma(e^{j\theta}, z)$ be at least once continuously differentiable with sublevel sets that satisfy (SA). Then solutions $f^* \in A$ to OPT exist and are unique. If Γ is infinitely differentiable, then f^* is infinitely differentiable on the circle.*

One consequence of Theorem 14 is that *global minimizers* and *local minimizers* to OPT are the same. In particular, if one calculates a solution using a method for finding local solutions, then the solution is in fact the global solution.

Also one should not take smoothness for granted. In the very natural and important problem of optimizing H^∞ performance over stable compensators, one finds that the optimal closed loop transfer function is almost never continuous [??]. This notion originated in circuit theory and there it is called the *spiked gain principle* **[H3]**.

The results cited in this section extend to optimization problems where the performance is not necessarily real symmetric, and the optimization is performed over analytic functions that are not necessarily ' “real”, i.e., satisfy $f(e^{-j\theta}) = \overline{f(e^{j\theta})}$.

Chapter 11

Facts about analytic functions

Some basic properties of analytic functions are presented in this section. Section 11.1 presents a result on approximation of functions on an arc of the circle. Spectral factorization is discussed in Section 11.2, and an analog result on phase of functions is presented in Section 11.3. The chapter is closed with a discussion in Section 11.4 of an important design principle.

11.1 Any function on an arc is a small perturbation of a function analytic on the Disk.

Theorem 15 *Let P be a closed subset of the unit circle ∂D which does not equal the whole unit circle. If g is any continuous function on P, then there is a sequence of rational functions $r_k \in A$ which approximate g on P. That is,*

$$\sup_{e^{j\theta} \in P} |r_k(e^{j\theta}) - g(e^{j\theta})| \to 0 \qquad \text{as } k \to \infty$$

Moreover (Runge), if $\mathcal{R}$ is any subset of the complex plane C which does not intersect the closed unit disk, then one can choose each r_k to have its poles in $\mathcal{R}$.

Proof: One can use a suitable (analytic) linear fractional transformation $v = l(\zeta)$ to map the unit disk to the unit disk, and the set P to a set P_1 that is a subset of the open disk $D(1,1)$ centered at 1 with radius 1. Then to approximate g on P by analytic functions is equivalent to approximating the function $g_1 = g \circ l^{-1}$ on P_1 by analytic functions of v: if f is a function analytic on the disk such that

$$|g_1(e^{j\theta}) - f(e^{j\theta})| < \epsilon \qquad \text{for } e^{j\theta} \in P_1$$

then clearly $f \circ l^{-1}$ is analytic and

$$|g(e^{j\theta}) - f(l^{-1}(e^{j\theta}))| < \epsilon \qquad \text{for } e^{j\theta} \in P$$

Thus we may assume that P is a set contained in the disk $D(1,1)$, and we make this assumption in the rest of the proof.

As a first case consider a function $g(\zeta) = \sum_{\ell=-n}^{n} c_\ell \zeta^\ell$ where $n > 0$ is fixed. The function $g(\zeta)$ can be expanded in a power series of $\zeta - 1$ that is uniformly and absolutely convergent on compact subsets of $D(1,1)$. By taking as many terms as necessary in this power series, we can approximate $g(e^{j\theta})$ by analytic polynomials of $e^{j\theta}$.

The same holds when g is a continuous function on P, since such functions are approximable by expressions of the form $\sum_{\ell=-n}^{n} c_\ell (e^{j\theta})^\ell$ (Weierstrass approximation Theorem).

Therefore g is approximable by analytic functions on P. Finally, Runge's Theorem **[R]** guarantees that any analytic function on the disk that is continuous on the closed unit disk can be uniformly approximated on the circle by rational functions whose poles are in any specified set that does not intersect the closed disk. ∎

How badly behaved are the functions r_k in Theorem 15 *outside* the arc R? The answer is given by the following theorem:

Theorem 16 *Let P be a closed subset of the unit circle ∂D which does not equal the whole unit circle. If g is any continuous function on P, then either g extends to a bounded analytic function on the disk D, or, for any sequence $\{f_n\} \subset A$ such that*

$$\sup_{e^{j\theta} \in P} |f_n(e^{j\theta}) - g(e^{j\theta})| \to 0 \qquad \text{as } n \to \infty,$$

the sequence must satisfy $\sup_\theta |f_n(e^{j\theta})| \to \infty$ as $n \to \infty$.

Since the proof of this theorem requires more advanced mathematics, we provide only a sketch here. If $\sup_\theta |f_n(e^{j\theta})| \leq k_0$ for some constant k_0 and all n, then $\{f_n\}$ is a normal family, c.f. **[A]**. Therefore it has a subsequence which is convergent uniformly on compact subsets of the disk to a function $f_0 \in H^\infty$. Moreover, $|f_n(e^{j\theta}) - f_0(e^{j\theta})| \to 0$ as $n \to \infty$, for all $e^{j\theta} \in P$. Then $g(e^{j\theta}) = f_0(e^{j\theta})$ for $e^{j\theta} \in P$, so f_0 is an extension of g analytic on the disk.

11.2 Spectral factorization: analytic functions with prescribed amplitude

Theorem 17 *Given a differentiable function* $r(e^{j\theta}) > 0$*, there exists a unique function* $f \in A$ *that is never 0 on the closed disk* D^- *and such that factors* r *in the sense*

$$r(e^{j\theta}) = f(e^{j\theta})\overline{f(e^{j\theta})}$$

Moreover, if r *is a polynomial or rational, then so is* f*.*

Proof: We give a proof for the case when r is a polynomial. Suppose that r is given by

$$r(e^{j\theta}) = \sum_{n=-N}^{n=N} c_n (e^{j\theta})^n.$$

Since r is real valued, we have that $r(e^{j\theta}) = \overline{r(e^{j\theta})}$, and from this it follows that $c_{-n} = \overline{c_n}$. Now $r(\zeta)$ is a rational function of $\zeta \in C$, with poles located at $\zeta = 0$, and no zeros on the unit circle. If $\zeta_0 \in C$ is such that $r(\zeta_0) = 0$, we have that

$$\begin{aligned} 0 &= \overline{r(\zeta_0)} \\ &= \textstyle\sum_{n=-N}^{n=N} \overline{c_n}(\overline{\zeta_0})^n \\ &= \textstyle\sum_{n=-N}^{n=N} c_{-n}(\overline{\zeta_0})^n \\ &= \textstyle\sum_{n=-N}^{n=N} c_{-n}(\frac{\zeta_0}{|\zeta_0|^2})^{-n} \\ &= r(\tfrac{\zeta_0}{|\zeta_0|^2}) \end{aligned} \tag{11.1}$$

We have proved that if ζ is a zero of r, then $\zeta_0^* = \frac{\zeta_0}{|\zeta_0|^2}$ is also a zero. Thus the polynomial $(\zeta - \zeta_0)(\zeta - \zeta_0^*)$ is a factor of the numerator of

$$r(\zeta) = \frac{c_{-N} + c_{-N+1}\zeta + \cdots + c_{N-1}\zeta^{2N-1} + c_N\zeta^{2N}}{\zeta^N}$$

which (as the reader can easily prove), upon division, yields a polynomial exhibiting the same type of symmetric arrangement of the coefficients. One can carry this process until r is completely factored in the following form:

$$r(\zeta) = \frac{(\zeta - \zeta_1)\cdots(\zeta - \zeta_N)(\zeta - \zeta_1^*)\cdots(\zeta - \zeta_N^*)}{\zeta^N} \tag{11.2}$$

Since $|\zeta| < 1$ if and only if $|\zeta^*| > 1$, we may label $\zeta_1, \ldots, \zeta_N$ as the roots of $r(\zeta)$ located in the unit disk so that $\zeta_\ell^*, \ldots, \zeta_N^*$ are outside the unit disk. Now set

$$f(\zeta) = (\zeta - \zeta_1) \cdots (\zeta - \zeta_N).$$

Note that the relation

$$e^{j\theta} - \zeta_0^* = e^{j\theta} - \frac{\zeta_0}{|\zeta_0|^2} = \frac{\zeta_0}{|\zeta_0|^2}(-e^{j\theta})(\overline{e^{j\theta}} - \overline{\zeta})$$

implies that $r(e^{j\theta}) = f(e^{j\theta})\overline{f(e^{j\theta})}$. This proves the theorem in the case when r is a polynomial.

11.3 Analytic functions with prescribed phase

Theorem 18 *Let b be a complex valued, differentiable function of $e^{j\theta}$ that never equals 0. Then, there is a smooth function $f \in A$ whose phase equals the phase of b at every $e^{j\theta}$ if and only if* wind $(b; 0) = n \geq 0$. *Furthermore, the function f has no zeros in the disk if and only if $n = 0$, and in this case f is unique up to a constant. We call this function f the phase (analytic) factor of b.*

Proof: This may be beyond many students so not everyone will want to spend much time on it. Suppose that b is a given differentiable function on the circle with wind $(b; 0) = n = 0$, and let $\log(b) = w + iv$. Since v is a continuous function on the circle, it has an associated Fourier series expansion

$$v(e^{j\theta}) = \sum_{\ell=-\infty}^{\infty} \hat{v}_\ell e^{j\ell\theta} \tag{11.3}$$

where (since v is real valued) $\overline{\hat{v}_\ell} = \hat{v}_{-\ell}$. Now let u be the function on the circle whose Fourier coefficients are given by $\hat{u}_\ell = -i\hat{v}_\ell$ if $\ell < 0$, $\hat{u}_\ell = i\hat{v}_\ell$ if $\ell > 0$, and $\hat{u}_0 = 0$. Set $f(e^{j\theta}) = u(e^{j\theta}) + iv(e^{j\theta})$. A calculation yields

$$g(e^{j\theta}) := u(e^{j\theta}) + iv(e^{j\theta}) = \hat{v_0} + \sum_{\ell=1}^{\infty} 2\hat{v}_\ell e^{j\ell\theta} \tag{11.4}$$

Since b is differentiable, then v is differentiable too and then **[Hof]** the function u is a continuous function of $e^{j\theta}$. Hence g is a continuous function on the circle. In the theory of Hardy Spaces it is shown that (11.4) can be used to define a continuous extension $\tilde{g}$ of g to the closed unit disk, and analytic inside. The Taylor coefficients in the expansion of $\tilde{g}$ about 0 are precisely the Fourier coefficients of g:

$$g(\zeta) = \hat{v_0} + \sum_{\ell=1}^{\infty} 2\hat{v_\ell}(\zeta)^{\ell} \tag{11.5}$$

Finally[1], set $f(\zeta) = e^{g(\zeta)}$. Clearly f has no zeros in D.

To prove uniqueness, note that if $\tilde{f}$ is another analytic function whose phase on the circle is the phase of b, then $\tilde{f}/f$ is analytic on D since f has no zeros there. Also, this ratio is real valued. But this is not possible unless the ratio is a constant.

Now, if $n > 0$ then consider the function $b_1(e^{j\theta}) = b(e^{j\theta})/(e^{j\theta})^n$. By the above discussion, there is a function f analytic on the disk whose phase on the circle is the phase of b_1. But then the phase of $b(e^{j\theta})$ is equal to the phase of $f(e^{j\theta})(e^{j\theta})^n$, which is analytic on the disk D.

We just proved one direction of the Theorem. The other direction is easy: if f is an analytic function with the same phase as the phase of b, then n = number of zeros of f inside the disk ≥ 0. In particular, if f has no zeros, then $n = 0$. ∎

We finish the subsection by mentioning a basic result about rational functions (see **[A]**)

Theorem 19 *If $r(\zeta)$ is a rational function that has no poles or zeros on the unit circle $|\zeta| = 1$, then*

$$\text{wind}\ (\ r\ ;\ 0\ ) = z_r - p_r$$

where z_r = number of zeros of r inside the unit disk, and p_r = number of poles of r inside the unit disk.

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[1] Another approach to proving the Theorem is to construct the solution to Laplace's equation

$$\begin{aligned} \Delta v(x,y) &= 0, \qquad (x,y) \in D \\ v|_{\partial D} &= v_0 \end{aligned} \tag{11.6}$$

for example using Poisson's formula. Now a harmonic function $v^*(x,y)$ can be determined so that $g(x+iy) = v^*(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations on D. The function $f = e^g$ is analytic on the disk D, and its phase on the circle is v, i.e., the phase of b.

11.4 The fundamental mistake of H^∞ control and other stories.

This section is not essential to the proofs which follow in the rest of the book, however, the discussion can help the student reinforce the ideas in section (11.1). It also states a simple principle which every control designer should be familiar with.

Any frequency domain control problem which has been correctly translated to a math problem suitable for computer optimization has the property that at each frequency there is a constraint imposed by the specifications. People who violate this make

The fundamental mistake of H^∞-Control

> *"I am only interested in a frequency band 'B', so I will not put specs outside of the band B."*

The price of making this mistake is high because, upon running an optimization program with such specs as input, one finds that the closed loop transfer function T has huge magnitude off of B. Amusingly enough T will meet the specs perfectly on B. In short, one obtains a solution so ridiculous that it is not even a good basis for trail and error.

Why are the remarks we just made true? Well, they are an immediate consequence of Theorem 15 and of Theorem 16; indeed the remarks are just a rephrasing of the mentioned results only now in the context of control.

A wholly different consequence of Runge's Theorem is that *pole placement*, a valuable technique in control design, has a shaky foundation which (in the authors opinion) has not been carefully analyzed. The basic idea behind pole placement is to set the poles of the closed loop transfer function at some prescribed distance from the $j\omega$ axis to achieve an objective, like stabilizing the system without making it too sluggish. While the technique is effective in particular situations where there is a lot of human intervention and time for trial and error, we maintain that the method is badly codified so it cannot be used algorithmically. This is a consequence of Runge's Theorem (though not of Theorem 15). Runge's Theorem implies that any function in A_{RHP} and smooth on the $j\omega$ axis (even at $\omega = \infty$) can be approximated by r_k's exactly as in Theorem 15. That is, the hypotheses of this theorem are different, but the conclusion is the same. This of course implies that if we have a proposed design producing a closed loop transfer function T_0 that is rational with poles at locations $\{p_j\}$, then we can find T_ε such that $||T - T_\varepsilon||_\infty \leq \varepsilon$ and with poles about any place we want. The number of poles of T_ε might be greater than that of T_0. This and other constraints that one might impose in order to make pole placement a systematic rather than trial and error subject have not been analyzed.

The moral of the pole placement story is different from the tale of the fundamental mistake. Here we are saying that pole placement can be a viable method, but it is a ripe area for more research. To the contrary the fundamental mistake should never be made.

Chapter 12

Proof of the main result

An important tool used in the proof of Therorem 13 is the Taylor expansion of smooth real valued functions of $z \in C$ in powers of z. This is applied to performance functions in Section 12.1. The actual proof of the theorem begins with the necessity of conditions (i) and (ii) of the Theorem. This is the contents of Section 12.2 and Section 12.3 respectively. The proof of the theorem is completed with the discussion of sufficiency of (i) and (ii) in Section 12.4.

12.1 Taylor expansions and performance functions

By using relation (10.2) the reader can verify that if $G(z)$ is a real valued and smooth function, then the following expansion holds:

$$G(z) = G(0) + 2\mathrm{Re}\,(\frac{\partial G}{\partial z}(0)z) + \mathcal{O}(|z|^2) \tag{12.1}$$

To use this in the proof of the Theorem, we consider the function $G(z) = \Gamma(e^{j\theta}, f^*(e^{j\theta}) + z)$ (for now $e^{j\theta}$ is fixed), and for ease of notation we write

$$a(e^{j\theta}) = \frac{\partial \Gamma}{\partial z}(e^{j\theta}, f^*(e^{j\theta})) \tag{12.2}$$

The Taylor expansion can now be restated as

$$\Gamma(e^{j\theta}, f^*(e^{j\theta}) + z) = \Gamma(e^{j\theta}, f^*(e^{j\theta})) + 2\mathrm{Re}a(e^{j\theta})z + \mathcal{O}(e^{j\theta}, |z|^2) \tag{12.3}$$

Figure 12.1: The function $g(e^{j\theta})$ dips below $\gamma - \delta$.

The $\mathcal{O}$ -term in (12.3) represents a function that goes to 0 as least as fast as a constant multiple of $|z|^2$ when $z \longrightarrow 0$. In other words when $|z|$ is small,

$$\Gamma(e^{j\theta}, f^*(e^{j\theta}) + z) \approx \Gamma(e^{j\theta}, f^*(e^{j\theta}) + 2\mathrm{Re}(a(e^{j\theta})z) \tag{12.4}$$

Of course the $\mathcal{O}$ -term in equation (12.3) depends on θ, but it is possible to obtain an inequality that is valid for all $e^{j\theta}$: the hypotheses on smoothness of Γ imply that there exist positive constants c_0 and δ_0 such that the following relation holds:

$$-c_0|z|^2 \leq \mathcal{O}(e^{j\theta}, |z|^2) \leq c_0|z|^2, \quad \forall\ e^{j\theta}, \ |z| < \delta_0 \tag{12.5}$$

We leave the proof of (12.5) to the reader.

12.2 Solutions make the performance flat

Suppose that f^* is a solution to OPT for which $\Gamma(e^{j\theta}, f^*(e^{j\theta}))$ is not constant. By the assumptions, there is a θ-interval τ on which $\Gamma(e^{j\theta}, f^*(e^{j\theta}))$ dips below $\gamma^* - \delta_1$ (see Fig. 12.1). To obtain a contradiction we will produce a function $f \in A$ with formula $f = f^* + \epsilon h$ and with the property that $\sup_\theta \Gamma(e^{j\theta}, f(e^{j\theta})) < \sup_\theta \Gamma(e^{j\theta}, f^*(e^{j\theta}))$. To construct and analyze f we will use the set τ.

By Theorem 15 we can select $h \in A$ with phase that approximates the phase of $-\overline{a(e^{j\theta})}$ on τ^c, the complement of τ. That is,

$$\begin{aligned} \text{phase}\,(a(e^{j\theta})h(e^{j\theta})) \quad &\approx \quad \text{phase}\,(a(e^{j\theta})(-\overline{a(e^{j\theta})})) \\ &= \quad \text{phase}\,(-|a(e^{j\theta})|^2) \quad = \quad -180^\circ \quad \forall \theta \in \tau^c \end{aligned}$$

This uses only a small part of the strength of Theorem 15 in that we could have chosen h to approximate the whole function $-\overline{a(e^{j\theta})}$ on τ^c. Thus,