HOW NONCOMMUTING ALGEBRA ARISES IN SYSTEMS THEORY

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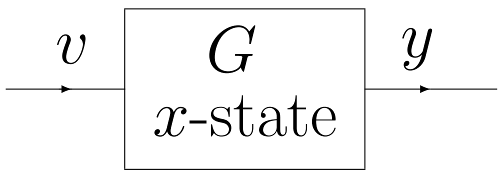


$$\frac{dx(t)}{dt} = Ax(t) + Bv(t)$$

$$y(t) = Cx(t) + Dv(t)$$

A, B, C, D are matrices

x, v, y are vectors

Asymptotically stable

$\mathrm{Re}(\mathrm{eigvals}(A)) \prec 0 \iff$

$A^T\mathbf{E} + \mathbf{E}A \prec 0 \quad \mathbf{E} \succ 0$

Energy dissipating

$G : L^2 \to L^2$

$$\int_0^T |v|^2 dt \geq \int_0^T |Gv|^2 dt$$

$x(0) = 0$

$\exists \quad \mathbf{E} = \mathbf{E}^T \succeq 0$

$$H := A^T\mathbf{E} + \mathbf{E}A + \\ + \mathbf{E}BB^T\mathbf{E} + C^TC = 0$$

$\mathbf{E}$ is called a storage function

Two minimal systems $[A, B, C, D]$ and $[a, b, c, d]$ with the same input to output map.

$\exists$ $\mathbf{M}$ invertible, so that

$$\mathbf{M}A\mathbf{M}^{-1} = a$$

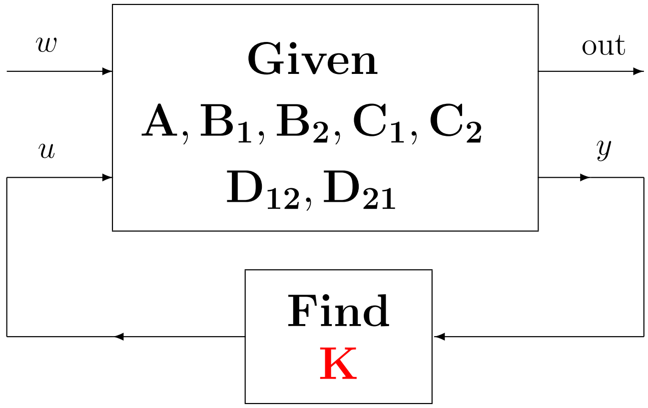
$$\mathbf{M}B = b$$

$$C\mathbf{M}^{-1} = c$$

Every state is reachable from $x = 0$

$(B \ AB \ A^2B \cdots) : \ell^2 \to X$ is onto

H^∞ Control Problem



$$\frac{dx}{dt} = Ax + B_1 w + B_2 u$$
$$\text{out} = C_1 x + D_{12} u + D_{11} w$$
$$y = C_2 x + D_{21} w$$

$D_{21} = I \qquad D_{12} D'_{12} = I \qquad D'_{12} D_{12} = I \qquad D_{11} = 0$

PROBLEM: Find a control law **K** $: y \to u$ which makes the system dissipative over every finite horizon:

$$\int_0^T |out(t)|^2 dt \leq \int_0^T |w(t)|^2 dt$$

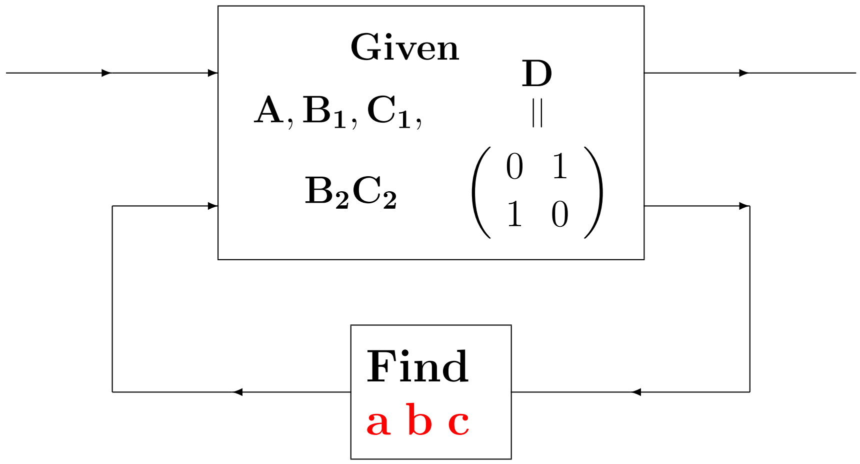
The unknown **K** is the system

$$\frac{d\xi}{dt} = \mathbf{a}\xi + \mathbf{b} \qquad\qquad u = \mathbf{c}\xi$$

So **a**, **b**, **c** are the critical unknowns.

CONVERSION TO ALGEBRA

Engineering Problem: Make a given system dissipative by designing a feedback law.



DYNAMICS of "closed loop" system: BLOCK matrices

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

ENERGY DISSIPATION:

$$H := \mathcal{A}^T \mathbf{E} + \mathbf{E}\mathcal{A} + \mathbf{E}\mathcal{B}\mathcal{B}^T\mathbf{E} + \mathcal{C}^T\mathcal{C} = 0$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E_{11}} & \mathbf{E_{12}} \\ \mathbf{E_{21}} & \mathbf{E_{22}} \end{pmatrix} \qquad \mathbf{E_{12}} = \mathbf{E_{21}}^T$$

$$H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & E_{yy} \end{pmatrix} \qquad H_{xy} = H_{yx}^T$$

H^∞ Control Problem

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = \mathbf{E_{11}}\,A + A^T\,\mathbf{E_{11}} + C_1^T\,C_1 + \mathbf{E_{12}}^T\,\mathbf{b}\,C_2 + C_2^T\,\mathbf{b}^T\,\mathbf{E_{12}}^T + \mathbf{E_{11}}\,B_1\,\mathbf{b}^T\,\mathbf{E_{12}}^T + \mathbf{E_{11}}\,B_1\,B_1^T\,\mathbf{E_{11}} + \mathbf{E_{12}}\,\mathbf{b}\,\mathbf{b}^T\,\mathbf{E_{12}}^T + \mathbf{E_{12}}\,\mathbf{b}\,B_1^T\,\mathbf{E_{11}}$$

$$H_{xz} = \mathbf{E_{21}}\,A + \frac{a^T\,(\mathbf{E_{21}}+\mathbf{E_{12}}^T)}{2} + \mathbf{c}^T\,C_1 + \mathbf{E_{22}}\,\mathbf{b}\,C_2 + \mathbf{c}^T\,B_2^T\,\mathbf{E_{11}}^T + \frac{\mathbf{E_{21}}\,B_1\,\mathbf{b}^T\,(\mathbf{E_{21}}+\mathbf{E_{12}}^T)}{2} + \mathbf{E_{21}}\,B_1\,B_1^T\,\mathbf{E_{11}}^T + \frac{\mathbf{E_{22}}\,\mathbf{b}\,\mathbf{b}^T\,(\mathbf{E_{21}}+\mathbf{E_{12}}^T)}{2} + \mathbf{E_{22}}\,\mathbf{b}\,B_1^T\,\mathbf{E_{11}}^T$$

$$H_{zx} = A^T\,\mathbf{E_{21}}^T + C_1^T\,\mathbf{c} + \frac{(\mathbf{E_{12}}+\mathbf{E_{21}}^T)\,\mathbf{a}}{2} + \mathbf{E_{11}}\,B_2\,\mathbf{c} + C_2^T\,\mathbf{b}^T\,\mathbf{E_{22}}^T + \mathbf{E_{11}}\,B_1\,\mathbf{b}^T\,\mathbf{E_{22}}^T + \mathbf{E_{11}}\,B_1\,B_1^T\,\mathbf{E_{21}}^T + \frac{(\mathbf{E_{12}}+\mathbf{E_{21}}^T)\,\mathbf{b}\,\mathbf{b}^T\,\mathbf{E_{22}}^T}{2} + \frac{(\mathbf{E_{12}}+\mathbf{E_{21}}^T)\,\mathbf{b}\,B_1^T\,\mathbf{E_{21}}^T}{2}$$

$$H_{zz} = \mathbf{E_{22}}\,\mathbf{a} + \mathbf{a}^T\,\mathbf{E_{22}}^T + \mathbf{c}^T\,\mathbf{c} + \mathbf{E_{21}}\,B_2\,\mathbf{c} + \mathbf{c}^T\,B_2^T\,\mathbf{E_{21}}^T + \mathbf{E_{21}}\,B_1\,\mathbf{b}^T\,\mathbf{E_{22}}^T + \mathbf{E_{21}}\,B_1\,B_1^T\,\mathbf{E_{21}}^T + \mathbf{E_{22}}\,\mathbf{b}\,\mathbf{b}^T\,\mathbf{E_{22}}^T + \mathbf{E_{22}}\,\mathbf{b}\,B_1^T\,\mathbf{E_{21}}^T$$

(HGRAIL) A, B_1, B_2, C_1, C_2 are knowns.

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \succeq 0$ for unknowns

a, **b**, **c** and for $\mathbf{E_{11}}$, $\mathbf{E_{12}}$, $\mathbf{E_{21}}$ and $\mathbf{E_{22}}$

When can they be solved?

If these equations can be solved, find formulas for the solution.

TEXTBOOK SOLUTION TO THE H^∞ PROB

DGKF = Doyle-Glover Kargonekar - Francis 1989 ish

KEY Riccatis

$$DGKF_X := (A - B_2C_1)'\mathbf{X} + \mathbf{X}(A - B_2C_1)$$
$$+\mathbf{X}(\gamma^{-2}B_1B_1' - B_2^{-1}B_2')\mathbf{X}$$

$$DGKF_Y := A^\times\mathbf{Y} + \mathbf{Y}A^{\times\prime} + \mathbf{Y}(\gamma^{-2}C_1'C_1 - C_2'C_2)\mathbf{Y}$$

here $A^\times := A - B_1C_2$.

THM DGKF There is a system **K** solving the control problem if there exist solutions

$$\mathbf{X} \succeq 0 \qquad and \qquad \mathbf{Y} \succ 0$$

to inequalities the

$$\mathbf{DGKF_Y \preceq 0} \text{ and } \mathbf{DGKF_X \preceq 0}$$

which satisfy the coupling condition

$$\mathbf{X} - \mathbf{Y}^{-1} \prec 0.$$

This is iff provided $\mathbf{Y} \succeq 0$ and $\mathbf{Y}^{-1}$ is interpreted correctly.

ALL THE RAGE

Riccati Inequalities

$$A_1'\mathbf{X} + \mathbf{X}A_1 + \mathbf{X}Q_1\mathbf{X} + R_1 \preceq 0$$
$$A_2'\mathbf{X} + \mathbf{X}A_2 + \mathbf{X}Q_2\mathbf{X} + R_2 \preceq 0$$
$$\mathbf{X} \succeq 0$$

These are "matrix convex" in the unknown $\mathbf{X}$ provided Q_1, Q_2 are positive semidefinite matrices. If such an $\mathbf{X}$ exists, then can simultaneously control or stablize several systems.

Numerical Solution Can solve convex (especially linear) matrix inequalities numerically with $\mathbf{X}$ smaller than 150×150 matrices using interior point optimization methods - called **semidefinite programming**.

Main Algebra Problem **"Convert" your engineering problem to a set of equivalent'convex matrix inequalities"** .