DANCING the NC RAG

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Your narrator is Bill Helton UCSD

NCAlgebra[a]

[a] Helton, deOliveira (UCSD), Stankus (CalPoly SanLObispo), Miller

OUTLINE

Noncommutative real algebraic geometry NC RAG

NC polys evaluating and comparing them.

Eg. NC convexity, NC analytic functions

Misc. NC analysis ventures

NC geometry 1970s

Free probability

NC algebraic geometry

NC analytic func thy

NC RAG (Linear systems engineering)

Trace of NC polys (physics, NC geom)

OUTLINE

NC Positivstellensatze and NullStellensatze with and without a Trace.

Linear systems engineering motivation.

LMI representations

Classical

NonCommutative

Numerics for NC RAG

Beyond Convexity

NonCommutative Laplace equation

NonCommutative Plurisubharmonic Polynomials

1. "POSITIVE" NC POLYS

$\mathbf{p \geq 0}$ **Means What ?**

p is a symmetric polynomial in non-commutating variables $x = \{x_1, x_2\}$ with real coefficients, eg.

$$p(x) = {x_1}^2 + ({x_1}^2)^T + {x_2}^T x_2$$

Define MATRIX POSITIVE polynomial

Plug in $n \times n$ matrices X_j for x_j in p

always get

$p(X_1, X_2)$ is a PosSD $n \times n$ matrix.

Function p of noncommutative variables $x := (x_1, x_2)$ is MATRIX CONVEX (geometric def.) $0 \leq \alpha \leq 1$

$$p(\alpha X + (1-\alpha)Y) \preceq \alpha p(X) + (1-\alpha)p(Y)$$

$$\frac{1}{2}p(X) + \frac{1}{2}p(Y) - p(\frac{1}{2}X + \frac{1}{2}Y) \ is \ Pos \ Def?$$

Question: Consider the noncommutative polynomial

$$p(x) := x^4 + (x^4)^T.$$

Is it matrix convex?

NC ANALYSIS VENTURES

NC geometry 1970s, K theory and (cyclic) homology
Free probability
NC algebraic geometry- not very NC, quantum plane

NC analytic func thy Victor V

NC RAG (Linear systems engineering) Igor K, Scott M, Mauricio deO, Mihai P, Markus S, Victor V, Students: Jeremy Greene, Michael Harrison, Joules Nahas, Chris Nelson

Trace of NCpolys - Igor K, Markus S (phys, NC geom)

1. NC RAG and SEMIRAG

IN A FREE $*$-ALGEBRA

NC Positivstellensatz

NC Nullstellensatz

Traces of NC Polynomials

IN AN UN FREE $*$-ALGEBRA

SoS for Differential operators

NC SET

p is a symmetric NC polynomial in g NC variables

$p(0_n) = I_n$

$X = (X_1, \cdots, X_g)$

$\mathcal{D}_p^n$:= closure of

$$\{X \in (\mathbb{SR}^{n\times n})^g : \ p(X) \succ 0\}$$

$\mathcal{D}_p$:= **POSITIVITY DOMAIN** $= \cup_n \mathcal{D}_p^n$.

p is the defining polynomial for the domain $\mathcal{D}_p$.

Example:

$$p = 1 - x_1^4 - x_2^4$$

$\mathcal{D}_p^2 = \{X \in (\mathbb{S}R^{3\times 3})^g : \ I - X_1^4 - X_2^4 \succ 0\}$

q is Positive on a Domain -Definition

$$\mathcal{D}_p := \ clos\ \{X : p_1(X) \succ 0, \dots, p_c(X) \succ 0\}$$

$q(x)$ is **MATRIX POSITIVE on $\mathcal{D}_p$** means: For any matrices X, satisfying

$$p_1(X) \succeq 0, \dots, p_c(X) \succeq 0,$$

we have $q(X)$ is a Positive Semidefinite matrix.

Example $\quad q$ is a weighted sum of squares:

$$q(x) = \sum_{j=1}^{c} \mathcal{L}_j(x)^T p_j(x) \mathcal{L}_j(x)$$

clearly is MATRIX POS on $\mathcal{D}_p$.

NC STRICT POSITIVSTELLENSATZ

THM (McC +H) 2004 follows M. Putinar: q is "strict operator positive" on the bounded Positivity Set IFF

$$\mathcal{D}_p := \{X : p_1(X) \succeq 0, \ldots, p_c(X) \succeq 0\},$$

then
$$q(x) = \sum_{k}^{fnt} \rho_k(x)^T \rho_k(x) +$$

$$\sum_{i,j=1}^{fnt} \mathcal{L}_j^i(x)^T p_j(x) \mathcal{L}_j^i(x) + \sum_{j=1}^{fnt} \mathcal{L}_j^{c+1}(x)^T [bound^2 - |x|^2] \mathcal{L}_j^{c+1}(x)$$

THM Suppose $\mathcal{D}_p$ is convex with NC interior:

1. "matrix positive" will do. 2004
2. (with Klep Sept09)

No $[bound^2 - |x|^2]$ term is needed in PosSS.

If $\mathcal{D}_p$ is everything, then rep is still true $q = SOS$, 2002.

TRACE OF AN NC POLYNOMIAL

Q? Given q a symmetric NC poly in symmetric vars. When is $Trace\ q(X)$ positive for all tuples of symmetric matrices X?

NATURAL CONJECTURE: If $Trace\ q(X)$ is matrix positive, then $q = \quad SOS \ + \ commutators.$
FALSE (for a certain q of deg 6, 8, etc): Igor K and Marcus S 2008ish.

THM (Igor K and Marcus S, 2008ish)
$Trace\ q(X) \ = \ 0$ **for all** X **IFF**
q **is a sum of commutators.**

Also see extensions from symmetric variables to free vars by Igor K and M. Brešar

TRACE pos on a Ball

Q? **Trace Putinar NC PosSS** **If** $Trace\ q(X)$ **is positive on the unit ball, does**

$$q = SOS + \sum_{j=1}^{c} \mathcal{L}_j(x)^T(1 - [x_1^2 + \cdots x_g^2])(x)\mathcal{L}_j(x)$$

$$+\ commutators.$$

THM: (Igor K and Markus S)

Answer is YES **IFF** **the Connes conjecture is true.**

NC NULLSTELLENSATZ

THM (McC, Putinar, Bergman, H 2004-2007ish)

Suppose polys $p_1(x), \cdots p_c(x)$ depend only on x.
IF $q(x, x^T)$ is zero on the zero set of $p_1(x), \cdots p_c(x)$,
that is, for matrix tuples X and vectors v

$$q(X)v = 0 \quad when \ 0 = p_1(X)v = \cdots = p_c(X)v.$$

THEN there are polynomials $r_1, \cdots, r_c$

$$q = r_1 p_1 + \cdots + r_c p_c$$

There is a counter example where p is not analytic.

NC but not FREE

Let $\mathcal{A}$ be the algebra of differential operators with polynomial coefficients, a member G $: f \to Gf$

$$Gf(x) := q_1(x)\frac{\partial f(x)}{\partial x_1} + \cdots + q_g(x)\frac{\partial f(x)}{\partial x_g}$$

Weyl Algebra: Generators M_j, D_j. Satisfy:

$M_jD_j - D_jM_j = 1 \quad M_jM_k = M_kM_j \quad D_jD_k = D_kD_j$

Konrad Schmüdgen's Thm 2005: Given a "positive" partial differential operator P, there is a Partial Diff Operator $B \neq 0$ such that

B^TPB is a sum of squares of differential operators.

See also Jaka Cimprič

FREE SUMMARY

1. THM **1** **EVERY "POSITIVE" NC POLYNOMIAL IS A SUM OF SQUARES.** $\leftarrow$ **COMPUTABLE.**

2. THM **2** **THMs 1 LOCALIZES,**
eg NC STRICT POSITIVSTELLENSATZ
NichtNirgendNiederNoItAin'tPositivSS ??spelling
I. K & M. S

3. THM **3** **NC NULLSTELLENSATZ**
For $p(x)$ no transposes

$$q(X)v = 0 \quad where \quad p(X)v = 0$$

4. THM **4** **Trace of NC polynomial p is 0** **IFF**
p is a sum of commutators.

QUESTIONS

Q1? There is no all encompassing NC Nullstellensatz.

Q2? In the NC Putinar PosSS:

a. When can one get rid of the bound term?

b. For which unbounded domains $\mathcal{D}_q$ does it work?

TRACE:

Q1? If $Trace\ p(X)$ is positive on the unit ball, does

$$p = SOS + \sum_{j=1}^{c} \mathcal{L}_j(x)^T(1-[x_1^2+\cdots x_g^2])(x)\mathcal{L}_j(x) + commutators?$$

Q2? Is there a nice NC certificate for $Trace\ p(X)$ to be positive for all matrix tuples X?

SYSTEMS ENGINEERING MOTIVATION FOR NONCOMMUTATIVE FORMULAS

See other set of slides

LMI REPRESENTATIONS

I. WHICH SETS $\mathcal{C}$ HAVE AN LMI REPRESENTATION?

II. MATRIX UNKNOWNS

Linear Matrix Inequalities (LMI)

1. Maximize a linear functional $\ell(x) := c_1x_1 + \cdots + c_nx_n$ over a set $\mathcal{C}$ having a LMI representation:

$$\mathcal{C} = \{x \in \mathbb{R}^n : L(x) := A_0 + x_1A_1 + \cdots + x_nA_n \ PosDef \ \}$$

Example:

$$\text{min} \qquad \ell(x) := x_1 + x_2 \qquad x \in \mathcal{C}$$

$$s.t. \quad \mathcal{C} := \begin{bmatrix} 3 - 2x_1 + x_2 & x_1 & -x_1 - x_2 \\ x_1 & 3 + x_1 - 2x_2 & x_2 \\ -x_1 - x_2 & x_2 & 1 + x_1 + x_2 \end{bmatrix} \succeq 0$$

- SDP generalizes LP (Linear Programming)

2. $Min_x \ [Max \ Eigenvalue](L(x))$

USES

Nesterov Nimerovski early 1990's

SDP solvers. Software: SeDuMi, SDPT3, DSDP, ...

1. **Systems and Control**
   - main advance in linear systems in 90's
2. **Combinatorics^D -Approximate solns to integer programming**
3. **Poly = Sums of squares, polynomial inequalities (Semialgebraic geometry SAG $\subset$ RAG)**
4. **Statistics^D -**
5. **Misc-**

D bill knows nothing about

Can (for modest size) compute

1. **If p is a Sum of Squares implies p is positive.**
   (Finding SoS converts to an LMI)

2. **p is positive where g_1 and g_2 are both positive.**
   A "Schmüdgen type Positivstellensatz"

$$p(x) = SoS_1(x) + \sum_{|\alpha| \leq K} SoS_\alpha(x)\ \Pi_j g_j^{\alpha_j}(x)$$

   (Finding SoS_α converts to an LMI)

3. **Putinar type PosSS is more practical**

BACK TO LMI REPRESENTATIONS?

DEFINITION: A set $\mathcal{C} \subset \mathrm{R}^m$ has an **Linear Matrix Inequality (LMI) REPRESENTATION** provided there are symmetric matrices $A_0, A_1, A_2, \cdots, A_m$ for which the **Linear Pencil** , $L(x) := A_0 + A_1 x_1 + \cdots + A_m x_m$, has positivity set,

$$\mathcal{D}_L := \{(x_1, \cdots, x_m) : A_0 + A_1 x_1 + \cdots + A_m x_m \quad \text{is PosSD}\}$$

equals the set $\mathcal{C}$; that is, $\mathcal{C} = \mathcal{D}_L$.

Which convex sets have an LMI representation?

EXAMPLE

$$\mathcal{C} := \{(x_1, x_2) : 1+2x_1+3x_2-(3x_1+5x_2)(3x_1+2x_2) \geq 0\}$$

has the LMI Rep

$$\mathcal{C} = \{x : L(x) \succeq 0\} \qquad \text{here } x := (x_1, x_2)$$

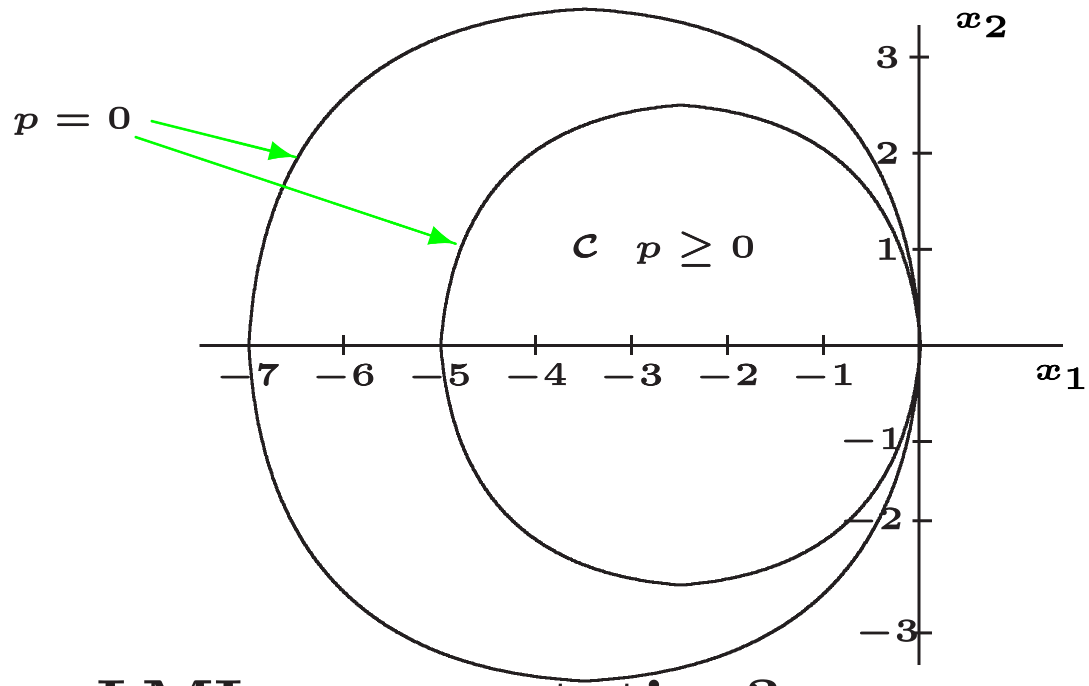
with

$$L(x) = \begin{pmatrix} 1+2x_1+3x_2 & 3x_1+5x_2 \\ 3x_1+2x_2 & 1 \end{pmatrix}$$

Pf: The determinant of $L(x)$ is pos iff $L(x)$ is PosSD.

QUESTION 1

Does this set $\mathcal{C}$ which is the middle component of



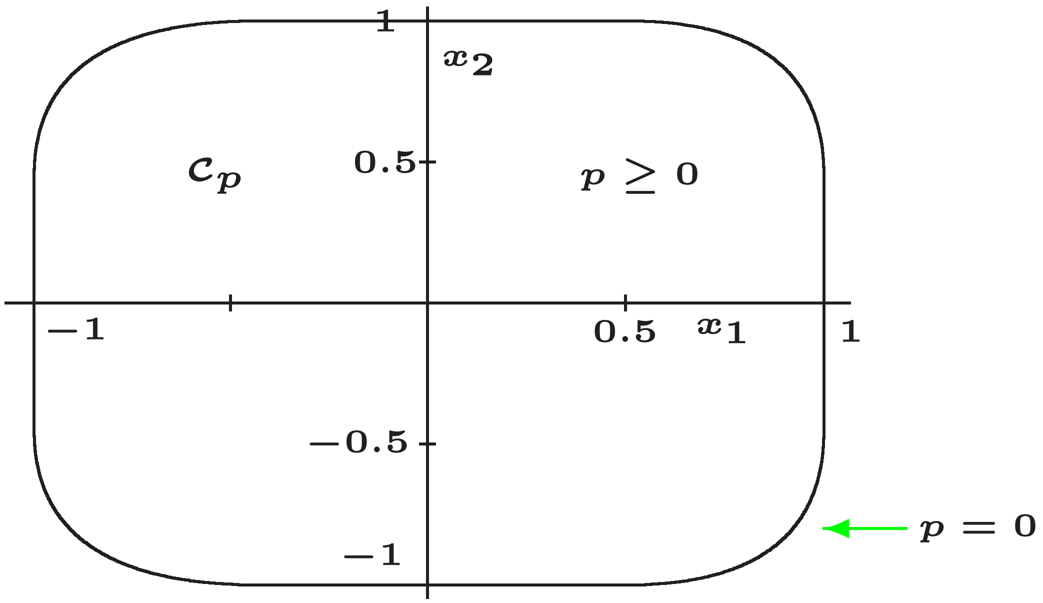
have an LMI representation?

$$p(x_1, x_2) = (x_1{}^2 + x_2{}^2)(x_1{}^2 + x_2{}^2 + 12x_1 - 1) + 36x_1{}^2$$

$$\mathcal{C} := \{x \in R^2 : p(x) \geq 0\} \text{ component of (-1, 0)}$$

QUESTION 2

Does this set have an LMI representation?



$$p(x_1, x_2) = 1 - {x_1}^4 - {x_2}^4$$

$\mathcal{C}_p := \{x \in R^2 : p(x) \geq 0\}$ **has degree 4.**

A REPRESENTABLE SET $\mathcal{C}$ IS SEMIALGEBRAIC

Suppose

$$L(x) := A_0 + A_1 x_1 + \cdots + A_m x_m$$

represents a set $\mathcal{C}$, $\mathcal{C} = \mathcal{D}_L$. Define a polynomial $\check{p}$ by

$$\check{p}(x) := \det[A_0 + A_1 x_1 + \cdots + A_m x_m]. \qquad (1)$$

$$\text{Boundary of } \mathcal{C} \subset \{x : \check{p}(x) = 0\} =: Z_p.$$

NECC for LMI Rep:

Boundary of $\mathcal{C}$ is contained in the zero set of some polynomial, i.e. is an algebraic curve. The minimal degree defining polynomial p is unique; of course p has some degree d and we say that $\mathcal{C}$

$$\mathcal{C} \text{ is a Semialgebraic Set of Degree } d.$$

Rigid Convexity

DEFINE: A convex set $\mathcal{C}$ in R^m with minimal degree (denote degree by d) defining polynomial p to be **rigidly convex** provided

> for every point x^0 in $\mathcal{C}$ and every line ℓ through x^0 (except for maybe a finite number of lines), the line ℓ intersects the the zero set $\{x \in R^m : p(x) = 0\}$ of p in exactly d points [a].

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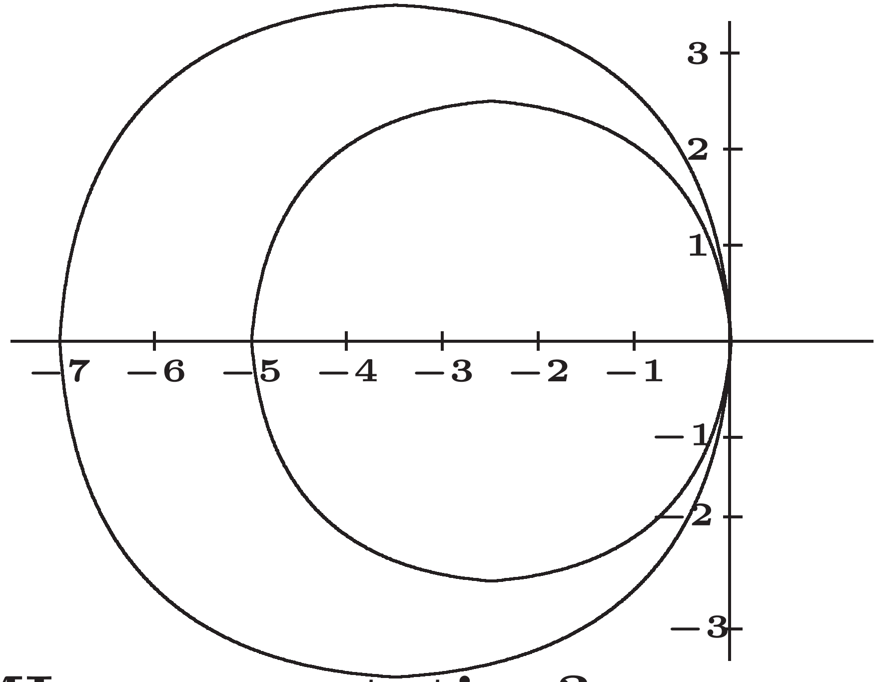
[a] In this counting one ignores lines which go thru x^0 and hit the boundary of $\mathcal{C}$ at ∞.

REVISIT QUEST 1

Does this set $\mathcal{C}$ which is the middle component of



have an LMI representation?

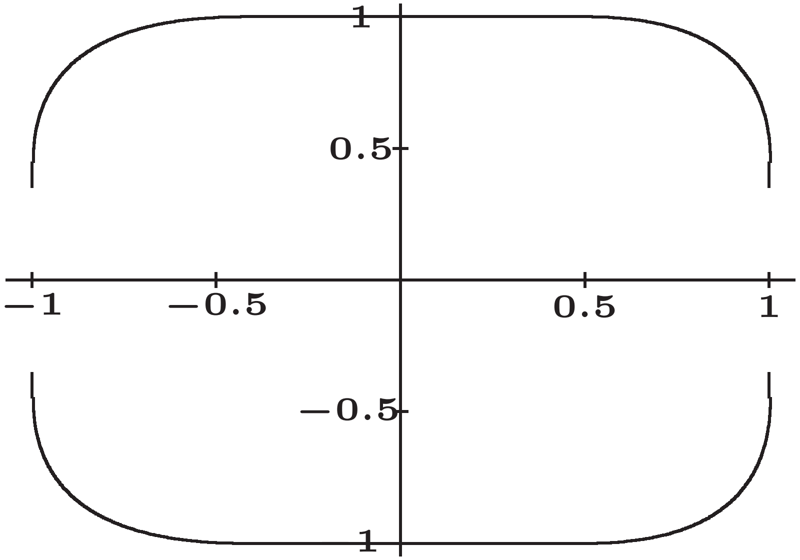
$$p(x_1, x_2) = (x_1{}^2 + x_2{}^2)(x_1{}^2 + x_2{}^2 + 12x_1 - 1) + 36x_1{}^2 \geq 0$$

$$\mathcal{C} := \{x \in R^2 : p(x) \geq 0\} \text{ connecteed to -1}$$

QUESTION 2 REVISITED

$$p(x_1, x_2) = 1 - {x_1}^4 - {x_2}^4$$

$$\mathcal{C}_p := \{x \in R^2 : p(x) \geq 0\} \text{ has degree 4.}$$



All lines through 0 intersect the set $p = 0$ in R^2 exactly two times. Thus $\mathcal{C}_p$ is not rigidly convex, and has no LMI representation.

IN R^2 RIGID CONVEXITY RULES

THM (Vinnikov + H).

If $\mathcal{C}$ is a closed convex set in R^m with an LMI representation, then $\mathcal{C}$ is rigidly convex.

When $m = 2$, the converse is true, namely, a rigidly convex degree d set has a LMI representation with symmetric matrices $A_j \in R^{d \times d}$.

The Proof of necessity is trivial. The Proof of sufficiency ($m = 2$) is not at all elementary. Uses algebraic geometry methods of Vinnikov. Proof was used in soln to 1958 Lax Conjecture by Lewis Parrilo Ramana.

HOW DOES NC CONVEXITY compare to classical CONVEXITY?

Q? Can we treat many more problems with convex techniques than LMI techniques?

Scott McCullough + H

Ultimately we want Symbolic Algorithms

convSetHB.tex

NC BASIC SEMIALGEBRAIC SET

EXAMPLE: p is a symmetric NC polynomial in g variables $\quad p(0_n) = I_n$

$\mathcal{D}_p^n :=$ closure of component of 0 of

$$\{X \in (\mathbb{SR}^{n\times n})^g : \ p(X) \succ 0\}$$

$\mathcal{D}_p :=$ Positivity Domain $= \cup_n \mathcal{D}_p^n$.

p is a defining polynomial for the domain $\mathcal{D}_p$.

Example:

$$p = 1 - x_1^4 - x_2^4$$

$\mathcal{D}_p^2 = \{X \in (\mathbb{SR}^{2\times 2})^2 : \ I - X_1^4 - X_2^4 \succ 0\}$

NC BASIC SEMIALGEBRAIC SET

p is a symmetric $\delta \times \delta-$ matrix of NC polynomials
$p(0_n)$ is invertible. Eg.

$$p = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}$$

$\mathcal{D}_p^n$:= closure of component of 0 of

$$\{X \in (\mathbb{SR}^{n \times n})^g : \ p(X) \ invertible\}$$

$\mathcal{D}_p$:= Invertibility Domain $= \cup_n \mathcal{D}_p^n$.

A NC basic semialgebraic set is one of the form $\mathcal{D}_p$,
p is called a defining polynomial for $\mathcal{D}_p$.

LINEAR MATRIX INEQUALITIES LMIs

GIVEN a linear pencil $L(x) := A_0 + A_1 x_1 + \cdots + A_g x_g$ symmetric matrices

FIND matrices $X := \{X_1, X_2, \cdots, X_g\}$ making $L(X)$ Pos SemiDef.

NUM SOLN: Interior Pt Method uses barrier $b_\epsilon(X) := -\epsilon \ell n \det L(X)$

CONVEX MATRIX INEQUALITIES CMIs

QUESTION: How much more general are Convex Matrix Inequalities than Linear Matrix Inequalities?

LMI REPS for NC CONVEX SETS

THM: McC-H

SUPPOSE p is an NC symmetric polynomial, $p(0)$ invertible.

IF $\mathcal{D}_p$ is "convex" meaning:

$\mathcal{D}_p$ in each dimension n, is a convex set of $n \times n$ matrix tuples.

THEN there is a monic linear pencil $L(x)$ which "represents" $\mathcal{D}_p$ as

$$\mathcal{D}_p = \mathcal{D}_L.$$

Q? Represent convex $\mathcal{D}_r$ where r is NC rational? (Maybe not hard).

MINIMUM DEGREE

p is a

Minimum degree defining polynomial for $\mathcal{D}_p$ (MDDP)

if any list $q = (q_1, \cdots, q_\delta)$ of (not necessarily symmetric and nonzero) polynomials which satisfies for X *in* (∂D_p)

$$p(X)v = 0 \qquad \Longrightarrow \qquad q(X)v = 0.$$

must have $\deg(q) \geq \deg(p)$.

Comm Case: Weaker than irreducibility. It is: $\mathcal{D}_p$ is bounded by pieces of curves $Z_{p_1}, \cdots, Z_{p_k}$ with each p_j irreducible and $p = p_1 \cdots p_k$

THM (Dym, McCullough +H):

SUPPOSE $\mathcal{D}_p = \{X : p(X) \succ 0\}$ **is convex, bounded with non trivial interior.**

IF p **is a minimal degree defining polynomial for** $\mathcal{D}_p$**,**

THEN p **has degree** ≤ 2**.**

NC CONVEX SETS

THM: (Winkler + Effros)

GIVEN $p_1, p_2, \ldots, p_k$ is a collection of symmetric NC polynomials in symmetric variables.

SUPPOSE $\bigcap_{\{j=1,\ldots,k\}} \mathcal{D}_{p_j}$ is a bounded convex positivity domain.

THEN there exists a collection $\mathcal{P}_1$ of symmetric NC polynomials of degree at most one such that $\mathcal{D}_{\mathcal{P}_1} = \mathcal{D}_{\mathcal{P}}$. One can take direct sums of these L to produce one "infinite" NON COMMUTATIVE LMI.

Pf: Complete Positivity - Straight forward Arvesonism.

THE HARD PROBLEM IS: find a finite set $\mathcal{P}_1$. LONG STORY $\square$

NC SETS and VARIETIES

WHAT DO THESE MEAN?

CURVATURE of a VARIETY

IRREDUCIBILITY

REGULARITY - SMOOTHNESS

A GOOD GUIDE IS THE LMI REP PROBLEM - FOCUSING ON BOUNDARIES OF NC CONVEX SETS see two DYM, McCullough, H papers

QUESTIONS ON NC CONVEX SETS

Q1? LMI Representation for a convex positivity domain $\mathcal{D}_r$ where r is NC rational? (Maybe not hard).

Q2? Main Open question: Given an NC set $\mathcal{D}$, when can you make an NC change of variables ϕ which makes $\mathcal{D} = \phi(\mathcal{C})$ for some $\mathcal{C}$ an NC convex set?

Give and analyze an algorithm.

A body of work is building. Nick Slinglend, UCSD student thesis. Igor Klep, Scott McCullough; Jeremy Greene and Victor Vinnikov and Bill, are struggling away.

NC CONVEX POLYNOMIALS

Function p of noncommutative variables $x := (x_1, x_2)$ is MATRIX CONVEX (geometric def.) $0 \le \alpha \le 1$

$$p(\alpha X + (1-\alpha)Y) \preceq \alpha p(X) + (1-\alpha)p(Y)$$

$$\frac{1}{2}p(X) + \frac{1}{2}p(Y) - p(\frac{1}{2}X + \frac{1}{2}Y) is\ Pos\ Def?$$

Question: Consider the noncommutative polynomial

$$p(x) := x^4 + (x^4)^T.$$

Is it matrix convex?

CONVEX POLYNOMIALS ARE TRIVIAL

THM: (McC + H)

Every symmetric polynomial in noncommutating variables $x_1, \cdots, x_g$ which is matrix convex (on any NC open set) has degree 2 or less.

One Var. Convex & Monotone:

. F. Hansen, M. Uchiyama, Jun Tomiyama

COR A Convex NC Polys is the Schur complement of some linear pencil. **Proof if convex everywhere:**

1. **NC Positive Polynomials**
2. **NC Second Derivatives**
3. **Put the two together**

Second NC Derivatives

Non-commutative rational function

$$\Gamma(x, y) := x^T A x + D y D^T + x y x^T, \quad y = y^T, A = A^T$$

Directional derivative wrt x, $D_x \Gamma(x, y)[h] :=$

$$= \lim_{t \to 0} \frac{1}{t} \left(\Gamma(x + th, y) - \Gamma(x, y) \right) = \frac{d}{dt} \Gamma(x + th, y) \Big|_{t=0}$$

Examples:

- ✓ **First derivative of $\Gamma(x, y)$ in x**

$$D_x \Gamma(x, y)[h] = h^T A x + x^T A h + h y x^T + x y h^T$$

- ✓ **Second derivative of $\Gamma(x, y)$ in x**

$$\mathcal{H}_x \Gamma(x, y)[h] = 2 h y h^T + 2 h^T A h$$

Convexity- Two Definitions

Geometrically matrix convex $\quad 0 \leq \alpha \leq 1$

$$\Gamma(\alpha X^1 + (1-\alpha)X^2) \preceq \alpha\Gamma(X^1) + (1-\alpha)\Gamma(X^2)$$

Matrix Convexity: Plugging any matrices into Hessian $\mathcal{H}\Gamma(x)[h]$ of Γ wrt x gives a PosSD matrix.

> **Geometric and matrix convexity are the same, provided $\Gamma(x)$ is smooth and there are no constraints on x. (cf. Merino+H, CDC1997)**

Notation: $\quad \mathcal{H}\Gamma(x_1, \ldots, x_m)[h_1, \ldots, h_m]$

Eg. $p = X^4 + X^{T^4}$ IS NOT CONVEX

Idea of Proof: $\quad p = XXXX + (XXXX)^T$

$D_X p(X)[H] =$

$$HXXX + XHXX + XXHX + XXXH + (same)^T$$

$\mathcal{H}_X p(X)[H] = \quad 2HHXX + 2HXHX + etc$

Convex p says its Hessian is SoS:

$$\mathcal{H}_X p(X)[H] = \Sigma_j \; \mathcal{L}_j(X)[H]^T \mathcal{L}_j(X)[H].$$

The highest order terms of $\mathcal{L}_j(X)[H]$ are linear in H. So HXX is a term of $\mathcal{L}_j(X)[H]$. Thus the degree 6 term $X^T X^T H^T H X X$ is a term of the degree 4 polynomial $\mathcal{H}_X p(X)[H]$. **Contradiction.** $\square$

Convexity Algorithm

$$\textbf{In}[1] := \quad << \texttt{NCAlgebra.m};$$

$$\textbf{In}[2] := \quad << \texttt{NCConvexity.m};$$

$$\textbf{In}[3] := \quad \texttt{SetNonCommutative}[\texttt{X}, \texttt{Y}];$$

$$\textbf{In}[4] := \quad \texttt{F} \ = \ \texttt{inv}[\texttt{X} - \texttt{inv}[\texttt{Y}]]$$

$$\textbf{Out}[5] := \quad \textbf{inv}[\textbf{X} - \textbf{inv}[\textbf{Y}]]$$

$$\textbf{In}[6] := \quad \texttt{NCConvexityRegion}[\texttt{F},\{\texttt{X}, \texttt{Y}\}]$$

$$\textbf{Out}[7] := \quad \{\{\texttt{2inv}[\texttt{X} - \texttt{inv}[\texttt{Y}]], \ \texttt{2inv}[\texttt{Y}]\}\}$$

Download NCAlgebra : www.math.ucsd / $\sim$ ncalg

NC RATIONAL

THEOREM (McCullough Vinnikov +H)

Suppose r is a symmetric noncommutative rational function in symmetric variables x which is matrix convex near 0 THEN

r is matrix convex on all of (the 0 component of) its domain.

AND

r has a representation in terms of an an LMI.

MORAL:

r CONVEX IN AN NC OPEN SET

OFTEN IMPLIES

r CONVEX EVERYWHERE

LOCAL STRUCTURE implies GLOBAL

NC FUNCTION F HAS

STRUCTURE IN AN NC OPEN SET

OFTEN IMPLIES

F HAS THE STRUCTURE EVERYWHERE

EXAMPLES:

Polynomials with "curvature" of given NC signature.

Convex Rational Functions

SUM: Convex $\mathcal{C}$ vs LMI Rep?

I. $\mathcal{C} \subset R^g$ **has a LMI rep.**

NO: at least requires a line test.

(Pf: Riemann surface techniques)

II. “Dimension Free”:

YES NC bounded convex basic semialgebraic sets with interior have an LMI Rep.

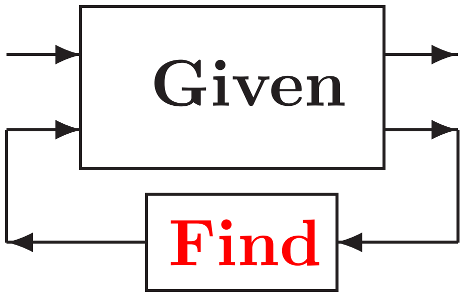
III. $\mathcal{C} \subset R^n$ **has a lift** $\hat{\mathcal{C}}$ **with LMI rep.**

YES: when $\mathcal{C}$ **is nonsingular strictly convex.**

IV. There is computer algebra for manipulation of “whole matrices”. Try NCAlgebra

NUMERICS WITH MATRIX UNKNOWNS: MAIN ISSUES

Systems problems $\longrightarrow$ Matrix Ineq



Many such problems Eg. H^∞ control

Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR Linear Matrix Inequalities (LMI) like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

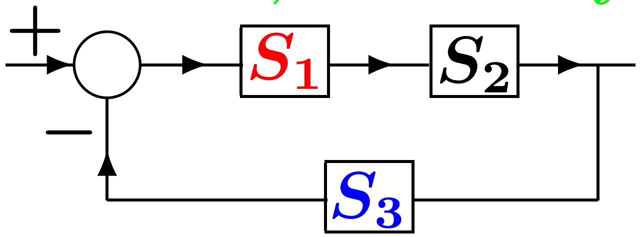
which is equivalent to the Riccati inequality.

ncDimlessPart

DIMENSIONFREE FORMULAS

1. **DIMENSIONFREE FORMULAS – THIS TALK**

   Topology is fixed; but many systems . E.g.



WANT FORMULAS: which hold regardless of the dimension of system S_1, S_2, S_3. Then unknowns are matrices and formulas respect matrix multiplication.

Eg. Most classical control text problems:

Control pre 1990: Zhou,Doyle,Glover.

LMIs in Control: Skelton, Iwasaki,

. Grigoriadis.

– Get noncommutative formulas

Keeping Matrices Whole

Matrices Whole

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{2}$$

Looks the same regardless of system size.

Matrices Entry by Entry – "Disaggregated"

If dimensions of the matrices A, B, C, X are specified, we can write formula (2) with matrices $L_0, \ldots, L_m$ as

$$\sum_{j=0}^{m} L_j \, X_j \succeq 0$$

with the unknown numbers X_j taken as entries of X.

Disaggregated Matrices

Example: If $A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{2\times 1}, C \in \mathbb{R}^{1\times 2}$, then $X \in \mathbb{S}^2$ and we would take

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix} \quad \text{and the LMI becomes} \quad \sum_{j=0}^{3} L_j \; X_j \succeq 0$$

where the 4×4 symmetric matrices L_0, L_1, L_2, L_3 are:

$$L_0 := \begin{pmatrix} C^T C & 0 \\ 0 & I \end{pmatrix} \qquad L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11} + a_{22} & b_{21} & b_{22} \\ a_{22} + a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix} \qquad L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}$$

Down with vec

+ and − of Keeping Matrices Whole

+ not many variables

+ SHORT FORMULAS

− Trouble is formulas are noncommutative.

+ NCAlgebra package does symbolic noncommutative algebra.

SDP for a class of self-adjoint matrix functions

Given NC rational function F and matrices $A, B, \ldots$. Find

$$\min_{X} \left\{ trace[X_1] : \ X \in \ closure\, \mathcal{D}_{F(\cdot,A,B)} \right\}$$

$$\mathcal{D}_{F(\cdot,A,B)} := \left\{ X \in \mathcal{C} : \ F(X, A, B, \ldots) > 0 \right\}$$

☞ **$F(\cdot)$ is a symmetric matrix concave NC rational function**

For example:

Interior Point method: Method of Centers

Primal-Dual with Camino de Oliveria and Skelton

Primal Dual with de Oliveira

We compute first second NC Directional derivatives symbolically.

The linear Subproblem always has the same form:

LINEAR SUBPROBLEM

Keeping unknowns aggregated as matrices in a Semi Definite Program always leads to Sylvester type linear equations in unknown H:

$$\sum_{j}^{K} A_j H B_j + B_j^T H A_j^T + tr(H) = Q \qquad\qquad H^T = H$$

with coefficients $A_j(x), B_j(x)$ which are computed symbolically (long story).

At k^{th} iteration of opt algorithm plug in current X_k, then $A_j := A_j(X_k)$ $B_j := B_j(X_k)$ are matrices and we solve for H.

Big Issue is: develop a numerical linear solver?

sliCjGradPart

$$L(H) := \sum_{j}^{K} A_j H B_j + B_j^T H A_j^T + tr(H) = Q \qquad H^T = H$$

Develop a Numerical linear solver.

Never vectorize (it has huge cost).

$H \to L(H)$ is computationally cheap, $2Kn^3$.

Mde Oliveira has conjugate gradient algorithms based on this.

Note this representation is not unique.
Finding reps with K small matters (we have symbolic algorithms.) Still somewhat open.

BEYOND CONVEXITY

WHEN PARTS OF NC HESSIANS ARE POSITIVE

THE TRACE: Laplaces equation and inequality, harmonic and subharmonic polynomials

BIG PRINCIPAL BLOCKS: The complex Hessian is positive,
Plurisubharmonic polynomials

NONCOMMUTATIVE HARMONIC POLYNOMIALS

NONCOMMUTATIVE SUBHARMONIC POLYNOMIALS

Chris Nelson UCSD

NC LAPLACIAN

NC Laplacian $\mathrm{Lap}[p, h]$ defined: For x_1, x_2 is

$$\frac{d^2}{dt^2}[p((x_1 + th), x_2)]|_{t=0} + \frac{d^2}{dt^2}[p(x_1, (x_2 + th))]|_{t=0}$$

That is,

$$\mathrm{Lap}[p, h] = NCHess[p, \{x_1, h\}, \{x_2, h\}]$$

The NC Laplacian is the formal trace of the NC Hessian.

Harmonic NC polynomial p means

$$\mathrm{Lap}[p, h] = 0 \text{ for all } x, h.$$

Subharmonic NC polynomial p means
$\mathrm{Lap}[p, H]$ is a pos semi def matrix for all X, H in $(\mathbb{SR}^{n \times n})^g$.

THM Chris Nelson There is an explicit classification of all NC harmonic polynomials (solutions to the NCLaplacian), roughly:

Harmonic NC Extenders: IF NC p is unchanged by permutation of its variables and if its commutative collapse satisfies the commutative Laplace equation, THEN p satisfies the NC Laplace equation.

All NC harmonic polynomials are built from harmonic NC extenders; eg. p_1, p_2, p_3 have disjoint variables

$$\sum_{\{\sigma:\sigma \text{ permutes variablesinthe} p_j\}} \sigma[\, p_1 p_2 p_3 \,]$$

is harmonic.

NC SUBHARMONICS

THM (McAllister-Hernandez- Helton) **If p is subharmonic, then**

$$p = \sum_j^{finite} c_j f_j^T f_j$$

where f_j are NC harmonic and c_j are real numbers. If the c_j are positive, then p is subharmonic.

The converse is false, some homogeneous subharmonics must have a c_j which is negative. Counter example by Chris Nelson (3 vars).

So Nelson's Thm tells us what to use for the f_j.

NC PLURISUBHARMONIC POLYNOMIALS

Jeremy Greene UCSD

Victor Vinnikov Beer Sheva

Bill UCSD

sliPlush.tex

NOW: Polynomials which are not necessarily hereditary

In (commutative) SCV the bianalytic transform (locally) of a convex function is plurisubharmonic.

What does NC Plurisubharmonic (NC Plush) mean?

for comparison: NC Plush implies NC harmonic.

PLUSH POLYS

A symmetric NC polynomial p in free variables is p is called **NC plurisubharmonic polynomial** if its "**Complex Hessian**" is matrix positive.

For p containing x and x^T and we replace x^T by y and define the following to be the NC "**Complex Hessian**" by

$$\mathcal{L}(p)[h] := \frac{\partial^2 p}{\partial t \partial s}(x + th, y + sk)_{|t,s=0}|_{y=x^T, k=h^T} \tag{3}$$

Commuting Case: $\mathcal{L}(p)[h] := \sum_{ij} \bar{h}_i \frac{\partial^2 p(x)}{\partial \bar{x}_i \partial x_j} h_j$

PLUSH POLYNOMIALS ARE:

$$\mathcal{L}(p)[h] := \frac{\partial^2 p}{\partial t\, \partial s}(x + th, y + sk)_{|t,s=0}|_{y=x^T, k=h^T} \tag{4}$$

Commuting Case: $\mathcal{L}((p))[h] = \sum_{ij} \bar{h}_i \frac{\partial^2 p(x)}{\partial \bar{x}_i \partial x_j} h_j$

Theorem 1. Jeremy Greene -Victor V + H *A symmetric NC polynomial* p *in free variables is NC Plurisubharmonic iff*

$$p = \sum f_j^T f_j + \sum g_j g_j^T + F + F^T \tag{5}$$

where each f_j, g_j, F *is NC analytic.*

Q?: In the commutative setting:
Is every Plush poly c also "SOS Plush", that is, the Complex Hessian of c is a SOS?
Background: Not every convex poly is SOS convex (Amir Ali Ahmadi)

ALGORITHMS and IMPLEMENTATIONS

1. **Convexity Checker - Camino, Skelton, H de Oliveria Public**

2. **Realization Builder: Convex Rational to LMI - Slinglend, Shopple in progress**

3. **Numerical matrix unknowns - de Oliveira, Camino, H, Skelton in house**

4. **LMI Producer (uses existing methods on special problems) de Oliveira, H in house**

Try NCAlgebra

END