Simultaneous controllability of several elastic strings

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Abstract

Controllability problems for N strings controlled from a common endpoint are studied. We give necessary and sufficient conditions for simultaneous spectral and approximate controllability and describe the space of simultaneously reachable states as a function of the position of the joint. For each type of controllability result the sharp controllability time is obtained. The proves are based on new results in nonharmonic Fourier series which describe Riesz bases of exponential divided differences.

1 Introduction

Boundary controllability of elastic systems has been intensively studied in recent years (see, for instance [1], [2], [3] and the references therein). In particular, the following challenging problem was discussed for some distributed parameter systems in [4] and [2, Chapter 5]: if we consider several exactly controllable systems, find the assumptions allowing control of all systems by using the same input function. This property is called *simultaneous controllability.* In the present paper, we consider this question for several strings controlled from a common endpoint. Necessary and sufficient conditions for simultaneous spectral and approximate controllability are obtained. Our main result is the description of the spaces of simultaneously reachable states as a function of the position of the joint. This generalizes the results concerning simultaneous controllability of two elastic strings obtained in [5] and [6].

To prove our results we apply the method of moments which has been widely used in control theory of distributed parameter systems since the classical papers of H.O. Fattorini and D.L. Russell in the end of 60-ies – beginning of 70-ies. The method is based on properties of exponential families (usually in space $L^2(0,T)$), the most important from which for control theory are minimality and the Riesz basis property and also $\mathcal{L}$-basis property which is Riesz basis in closure of a linear span of the family.

Problems of simultaneous control have raised a number of new difficult problems in the theory of exponential families. The principal questions which we consider in this paper are connected with the basis property of linear combinations from exponentials $e^{i\lambda_n t}$ in the case when the distance between some points λ_n tends to zero and therefore the family of exponentials $\left\{e^{i\lambda_n t}\right\}$ does not form a Riesz basis in $L^2(0,T)$. One of the known results in this direction, obtained by Ullrich [7], concerns the case when $\{\lambda_n\} = \bigcup \Lambda_k$ and the groups Λ_k consist of equal number of points all of them are close enough to $k, k \in \mathbf{Z}$. The results of a different kind were obtained by S. Jaffard, M. Tucsnak and E. Zuazua [8] and C. Baiocchi, V. Komornik and P. Loreti [9]. They extended the classical Ingham inequality to the case when groups may contain one or two points. We have generalized these results in two directions: for groups with arbitrary number of points and for the case of arbitrary complex sequence $\{\lambda_n\}$ in a strip parallel to the real axis. Moreover, we obtained a full description of Riesz bases of special kinds linear combinations of exponentials — generalized divided differences.

2 Main results

For $\xi_j > 0\,, j = 1, \dots, N$, we consider the problems

$$\begin{cases} \frac{\partial^2 u_j}{\partial t^2}(x,t) - \frac{\partial^2 u_j}{\partial x^2}(x,t) = 0 \quad \forall\, x \in (0,\xi_j), \quad \forall\, t \in (0,\infty) \\ u_j(0,t) = f(t),\; u_j(\xi_j,t) = 0 \quad \forall\, t \in (0,\infty) \\ u_j(x,0) = 0,\; \frac{\partial u_j}{\partial t}(x,0) = 0 \quad \forall\, x \in (0,\xi_j) \end{cases} \tag{2.1}$$

The systems above model the vibrations of several strings joined at a common endpoint at $x = 0$.

It is well known that, if $f \in L^2(0,T)$, each of the systems in (2.1) is well posed in $L^2(0,\xi_j) \times H^{-1}(0,\xi_j)$ (see for instance [2], [3]). This allows us to define the bounded linear operator

$$U^T :\; L^2(0,T) \mapsto \prod_{j=1}^{N} L^2(0,\xi_j) \times H^{-1}(0,\xi_j) =: \mathcal{W}^0\,,$$

$$U^T u = (u_1(\cdot,T), \dot{u}_1(\cdot,T), u_2(\cdot,T), \dot{u}_2(\cdot,T), \dots, u_N(\cdot,T), \dot{u}_N(\cdot,T))\,.$$

The space of states simultaneously reachable by the system (2.1) in the time interval $[0,T]$ is defined as the range $\mathcal{R}^T := U^T(L^2(0,T))$ of the operator U^T. According to the properties of this space of simultaneously reachable states we can define several types of simultaneous controllability.

Definition 2.1 *1. The systems* (2.1) *are called simultaneously approximately controllable in the time interval $[0,T]$ if $\mathcal{R}^T$ is dense in $\mathcal{W}^0$.*

2. The systems (2.1) *are called simultaneously spectrally controllable in the time interval $[0,T]$ if, for all $n \geq 1$, the states*

$$\left(\sin\left(\frac{n\pi x}{\xi_1}\right), 0, 0, \ldots, 0\right), \left(0, \sin\left(\frac{n\pi x_1}{\xi}\right), 0, \ldots, 0\right), \ldots,$$

$$\left(0, 0, \ldots, 0, \sin\left(\frac{n\pi x}{\xi_N}\right), 0\right), \left(0, 0, \ldots, 0, \sin\left(\frac{n\pi x}{\xi_N}\right)\right),$$

are reachable at time T, i.e. if they belong to $\mathcal{R}^T$.

3. The systems (2.1) *are called simultaneously exactly controllable in the time interval $[0,T]$ with respect to a space $\mathcal{U} \subseteq \mathcal{W}^0$ if $\mathcal{R}^T \supseteq \mathcal{U}$.*

The main results in this paper are the following three theorems concerning, respectively, simultaneous approximate controllability, simultaneous spectral controllability and characterization of the simultaneously reachable space.

Let us set $T_* = 2\sum_{\mathrm{j}=1}^{N} \xi_j$ and $\theta_{jk} = \xi_j/\xi_k\,, 1 \leq j < k \leq N$.

Theorem 2.2 *For any ξ_j, the systems* (2.1) *are not approximately controllable in time $T < T_*$. If at least one of θ_{jk} is rational, the systems* (2.1) *are not approximately controllable for any $T > 0$.*

Theorem 2.3 *For any irrational θ_{jk}, the systems* (2.1) *are simultaneously spectrally controllable in the time interval $[0,T]$ for $T \geq 2$.*

Denote by $\mathcal{S}$ the set of all irrational numbers ρ such that if $[a_0, a_1, \ldots, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then the sequence of partial quotients (a_n) is bounded. This is the set of "badly approximable numbers". Note that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation (cf. [10, p.120]), its Lebesgue measure is equal to zero.

Let us consider the operators $A_j, j = 1, \ldots, N$, defined by

$$\mathcal{D}(A_j) = H^2(0,\xi_j) \cap H_0^1(0,\xi_j),$$

$$A_j : \mathcal{D}(A_j) \mapsto L^2(0,\xi_j),\ A_j h = -\frac{d^2 h}{dx^2}$$

and the spaces W_j^s and $\mathcal{W}^s$ defined as follows:

$$W_j^s = \mathcal{D}\left(A_j^{s/2}\right) \quad \text{for} \quad s > 0\,,$$

$$W_j^0 = L^2(0,\xi_j)\,,\ W_j^s = \left(W_j^{-s}\right)' \quad \text{for} \quad s < 0\,,$$

$$\mathcal{W}^s = \prod_{j=1}^{N} W_j^s \times W_j^{s-1}\,.$$

The following theorem gives the characterization of the simultaneously reachable space as function of ξ_j.

Theorem 2.4 *Suppose that $T \geq T_*$. Then the following statements are valid :*

(a) If all θ_{jk} belong to $\mathcal{S}$, the systems (2.1) *are exactly controllable with respect to $\mathcal{W}^{N-1}$.*

(b) For almost all ξ_j and any $s > N-1$, the systems (2.1) *are exactly controllable with respect to $\mathcal{W}^s$.*

(c) For all ξ_j and any $\varepsilon > 0$, the systems (2.1) *are not simultaneously exactly controllable with respect to the space $\prod_{j=1}^{N} H_0^{1-\varepsilon}(0,\xi_j) \times H^{-\varepsilon}(0,\xi_j)$.*

3 Bases of exponential divided differences

In 1979 B. Pavlov obtained necessary and sufficient conditions for an exponential family to form a Riesz basis in the space $L^2(0,T)$ [11].

Theorem 3.1 *(Pavlov) Let $\Lambda := \{\lambda_k | k \in \mathbb{Z}\}$ be a countable set of the complex plane. The family $\mathcal{E} := \left\{e^{i\lambda_k t}\right\}$ forms a Riesz basis in $L^2(0,T)$ if and only if the following three conditions are satisfied:*

(i) the sequence $\{\lambda_k\}$ lies in a strip parallel to the real axis,

$$\sup_{k\in\mathbb{Z}} |\Im\lambda_k| < \infty\,;$$

(ii) the set Λ is separated, i.e.

$$\inf_{k\neq n} |\lambda_k - \lambda_n| > 0\,; \tag{3.2}$$

(iii) there exists an entire function F of exponential type with indicator diagram of width T and zero set Λ such that, for some real h, the function $|F(x+ih)|^2$ satisfies the Muckenhoupt condition (A_2):

$$\sup_{I\in\mathcal{J}}\left\{\frac{1}{|I|}\int_I |F(x+ih)|^2\ dx\ \frac{1}{|I|}\int_I |F(x+ih)|^{-2}\ dx\right\} < \infty, \tag{3.3}$$

where $\mathcal{J}$ is the set of all intervals of the real axis.

It is well known that the condition (3.3) is equivalent to the Helson–Szegö condition: functions $u, v \in L^\infty(\mathbb{R}), \|v\|_{L^\infty(\mathbb{R})} < \pi/2$ may be found such that

$$|F(x+ih)|^2 = \exp u(x) + \tilde{v}(x)\,.$$

Here the map $v \mapsto \tilde{v}$ denotes the Hilbert transform for bounded functions:

$$\tilde{v}(x) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{\infty} v(t) \left\{\frac{1}{x-t} + \frac{t}{t^2+1}\right\} dt.$$

Definition 3.2 *An entire function of exponential type with indicator diagram of width T and zero set Λ is called a generating function of the family $\mathcal{E}$ in $L^2(0,T)$.*

Our goal is to generalize Pavlov's theorem for the case when Λ does not satisfy the separability condition (3.2). We obtain a description of Riesz bases of *divided differences* (see the Definition 3.5 below) in terms of the generating function and the Muckenhoupt condition (A_2) (or equivalent to it the Helson–Szegö condition). We give also a description of $\mathcal{L}$-bases of divided differences which is important in applications to control theory.

Definition 3.3 *A family in a Hilbert space which forms a Riesz basis in the closure of its linear span is called an $\mathcal{L}$-basis.*

In what follows $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ is a countable set ordered in such a way that $\Re\lambda_n$ form a nondecreasing sequence. We assume also that $\sup|\Im\lambda_n| < \infty$.

Let us denote by $\#\mathcal{A}$ the number of elements in the set $\mathcal{A}$, write

$$n^+(r) := \sup_{x\in\mathbb{R}} \#\{\Re\Lambda \cap [x, x+r]\},$$

$$n^-(r) := \inf_{x\in\mathbb{R}} \#\{\Re\Lambda \cap [x, x+r]\}$$

and define the upper $\mathcal{D}^+$ and lower $\mathcal{D}^-$ uniform densities of Λ to be respectively

$$\mathcal{D}^\pm := \lim_{r\to\infty} \sup \frac{n^\pm(r)}{r}.$$

The set Λ is called *uniformly discrete* or *separated* if

$$\delta(\Lambda) := \inf_{m\neq n} |\lambda_m - \lambda_n| > 0$$

and *relatively uniformly discrete* if it can be represented as a finite union of uniformly discrete sets. To each Λ, we associate an exponential family $\mathcal{E}(\Lambda) := \left\{e^{i\lambda_n t}\right\}_{\lambda_n\in\Lambda}$.

Let us introduce notations needed to formulate further results. For any $\lambda \in \mathbb{C}$ denote by $D_\lambda(r)$ a disk with center λ and radius r. Let $G^{(p)}(r)$, $p = 1, 2, \ldots$, be connected components of the union $\cup_{\lambda\in\Lambda} D_\lambda(r)$ and write $\Lambda^{(p)}(r) = \{\lambda_{j,p}\}$ for the set of points from Λ lying in $G^{(p)}$: $\Lambda^{(p)}(r) := \Lambda \cap G^{(p)}(r)$.

The following statement is quite obvious.

Lemma 3.4 *Let a relatively uniformly discrete set Λ be a union of N uniformly discrete sets Λ_j,*

$$\delta_j := \inf_{\lambda\neq\mu;\ \lambda,\mu\in\Lambda_j} |\lambda - \mu|,\ \delta := \min_j \delta_j.$$

Then for $r < r_0 := \delta/(2N)$ we have $\mathcal{N}^{(p)}(r) := \#\Lambda^{(p)} \leq N$.

Let μ_k, $k = 1, \ldots, m$, be arbitrary complex numbers, not necessarily different.

Definition 3.5 *Generalizated divided difference (GDD) of the first order of a point μ is $[\mu](t) := e^{i\mu t}$. GDD of the order n, $n \leq m$, of points $\mu_1, \ldots, \mu_n$ is*

$$[\mu_1, \ldots, \mu_n] := \begin{cases} \frac{[\mu_1,\ldots,\mu_{n-1}]-[\mu_2,\ldots,\mu_n]}{\mu_1-\mu_n}, & \mu_1 \neq \mu_n, \\ \frac{\partial}{\partial\mu}[\mu, \mu_2, \ldots, \mu_{n-1}]|_{\mu=\mu_1}, & \mu_1 = \mu_n. \end{cases}$$

One can easily derive the explicit formulae for GDD. For instance, if all μ_k are different

$$[\mu_1, \ldots, \mu_n] = \sum_{k=1}^{n} \frac{e^{i\mu_k t}}{\prod_{j\neq k}(\mu_k - \mu_j)}.$$

The following theorem describes Riesz bases of GDD. It is proved in [12] using the methods developed in [3, Sections II.2, II.3].

Theorem 3.6 *Let Λ be a relatively uniformly discrete set, $r < r_0$ and $\mathcal{E}^{(p)}(r)$ be a family of GDD of the order from 1 to $\mathcal{N}^{(p)}(r) \leq N$ corresponding to the points $\Lambda^{(p)}(r)$. Then*

(i) the family $\{\mathcal{E}^{(p)}\}$ forms a Riesz basis in $L^2(0,T)$ if and only if the generating function of the family $\left\{e^{i\lambda_n t}\right\}$ has the indicator diagram of width T and satisfies the Helson-Szegö condition;

(ii) for any finite sequence $\{a_{p,j}\}$ the inequality

$$\|\sum_{p,j} a_{p,j} e^{i\lambda_{j,p} t}\|^2_{L^2(0,T)} \geq C \sum_{p,j} |a_{p,j}|^2 \delta_p^{2(\mathcal{N}^{(p)}-1)}$$

is valid, where a constant C does not depend on $\{a_{p,j}\}$ and δ_p is the diameter of $\Lambda^{(p)}$,

$$\delta_p := \min\{|\lambda_{j,p} - \lambda_{k,p}| \,\Big|\, k, j \leq \mathcal{N}^{(p)}, k \neq j\}.$$

The proof of the following theorem is presented in [13]. It is based on the '$1/4$ in the mean' theorem (see [14], [3, Section II.4]).

Theorem 3.7 *Under conditions of the previous theorem the following statements are valid.*

(i) For any $T > 2\pi\mathcal{D}^+$ the family $\{\mathcal{E}^{(p)}\}$ forms an $\mathcal{L}$-basis in $L^2(0,T)$.

(ii) For any $T < 2\pi\mathcal{D}_-$ there exists a subfamily of $\{\mathcal{E}^{(p)}\}$ which forms a Riesz basis in $L^2(0,T)$.

4 Proof of the main results

The proofs of the Theorems 2.2 and 2.3 can be carried out by the scheme of the paper [6] using the Theorems 3.6 and 3.7.

The proof of the Theorem 2.4 is based on the following generalization of the classical inequality of Ingham which extends previous results [8] and [9] in this direction for sets contained an arbitrary number of close points. We formulate it here in the form adapted to prove Theorem 2.4 (see [13] for the general case).

Let all θ_{jk} be irrational and

$$\Lambda := \bigcup_{j=1}^{N} \left\{\frac{k\pi}{\xi_j}\right\}_{k\in\mathbb{Z}}.$$

It is easy to prove that the set Λ can be presented in the form

$$\Lambda = \bigcup_{n=1}^{N} \Lambda^{(n)} \;, \;\; \Lambda^{(n)} = \bigcup_{p=1}^{\infty} \Lambda_p^{(n)} \;,$$

where the number of points in each set $\Lambda_p^{(n)}$ is equal n and

$$dist\left(\Lambda_p^{(n)}\,,\,\Lambda\setminus\Lambda_p^{(n)}\right) \geq \delta > 0 \qquad \forall\, n,p\,.$$

For any finite sequence $(a_\lambda)_{\lambda\in\Lambda}$, we set

$$G_1 = \sum_{\lambda\in\Lambda^{(1)}} |a_\lambda|^2 \;,$$

$$G_2 = \sum_p \sum_{\{\lambda_1,\lambda_2\}\in\Lambda_p^{(2)}} \left(|a_{\lambda_1}+a_{\lambda_2}|^2 + |a_{\lambda_2}(\lambda_2-\lambda_1)|^2\right) \;,$$

$$\begin{aligned} G_3 \;\; = \;\; \sum_p \sum_{\{\lambda_1,\lambda_2,\lambda_3\}\in\Lambda_p^{(3)}} & \left(|a_{\lambda_1}+a_{\lambda_2}+a_{\lambda_3}|^2\right. \\ & +|a_{\lambda_2}(\lambda_2-\lambda_1)+a_{\lambda_3}(\lambda_3-\lambda_1)|^2 \\ & \left.+|a_{\lambda_3}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)|^2\right) \end{aligned}$$

and so on,

$$\begin{aligned} G_N \;\; = \;\; & \sum_p \sum_{\{\lambda_1,\lambda_2,\ldots,\lambda_N\}\in\Lambda_p^{(N)}} \left(\left|\sum_{k=1}^{N} a_{\lambda_k}\right|^2\right. \\ & \left. + \sum_{s=2}^{N}\left|\sum_{k=s}^{N} a_{\lambda_k}(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)\ldots(\lambda_k-\lambda_{s-1})\right|^2\right) \;. \end{aligned}$$

Theorems 3.6 and 3.7 imply the following result.

Theorem 4.1 *Let*

$$g(t) = \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda t} \;\; \textit{and} \;\; T \geq T_* = 2\sum_{j=1}^{N} \xi_j \;.$$

There exist positive constants C_1 and C_2 such that

$$C_1 \sum_{n=1}^{N} G_n \leq \|g\|^2_{L^2(0,T)} \leq C_2 \sum_{n=1}^{N} G_n$$

for any finite sequence $(a_\lambda)_{\lambda\in\Lambda}$.

This theorem together with results on the simultaneous diophantine approximation [10] implies the Theorem 2.4.

Using Theorems 3.6, 3.7 and 4.1 we can also obtain simultaneous controllability results for N connected beams (see [13] for details) generalizing the results of [9] proved for the case $N = 2$.

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References

[1] Lagnese J.E. and Lions J.-L. *Modelling, Analysis and Control of Thin Plates*. Masson, Paris, 1988.

[2] Lions J-L. *Controlabilité Exacte Perturbations et Stabilisation de Systèmes Distribués*, Volume 1, Masson, Paris, 1988.

[3] Avdonin S.A. and Ivanov S.A. *Families of exponentials - The method of moments in controllability problems for distributed parameter systems.* Cambridge University Press, New York, 1995.

[4] Russell D.L. *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region*, SIAM J. Control & Optim., Vol. 24, pp.199–229, 1986.

[5] Tucsnak M. and Weiss G. *Simultaneous exact controllability and some applications*, submitted.

[6] Avdonin S. and Tucsnak M. *Simultaneous controllability in short time of two elastic strings*, submitted.

[7] Ullrich D. *Divided differences and system of nonharmonic Fourier series.* Proceedings of the American Mathematical Society, Vol.80(1), pp.47–57, 1980.

[8] Jaffard S., Tucsnak M. and Zuazua E. *On a theorem of Ingham.* J. Fourier Anal. Appl., Vol. 3 , pp.577–582, 1997.

[9] Baiochi C., Komornik V. and Loreti P. *Ingham type theorems and applications to control theory*, Boll. U.M.I, Vol.2B, pp.33–63, 1999.

[10] Cassels J.W.S. *An Introduction to Diophantine Approximation*. Cambridge University Press, Cambridge, 1965.

[11] Pavlov B.S. *Basicity of an exponential systems and Muckenhoupt's condition*, Dokl. Akad. Nauk SSSR, Vol.247, No.1, pp.37–40, 1979 (Russian); English transl. in Soviet Math. Dokl., Vol.20, pp.655–659, 1979.

[12] Avdonin S.A. and Ivanov S.A. *Exponential Riesz bases of subspaces and divided differences*, submitted.

[13] Avdonin S.A. *Ingham type inequalities and Riesz bases of divided differences*, submitted.

[14] Avdonin S.A. *On Riesz bases of exponentials in L^2.* Vestnik Leningr. Univ. Math., Vol.7, pp.203–211, 1979.