NONCOMMUTATIVE CONVEXITY

Question: Consider the noncommutative polynomial

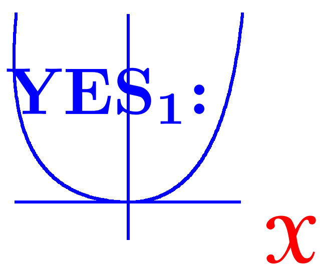
$$p(X) := X^4 + (X^4)^T.$$

Is it matrix convex?

Function p of noncommutative variables $\vec{X} := (X_1, X_2)$ is MATRIX CONVEX (geometric def.) $0 \le \alpha \le 1$

$$p(\alpha\vec{\mathfrak{X}} + (1-\alpha)\vec{\mathfrak{Y}}) \preceq \alpha p(\vec{\mathfrak{X}}) + (1-\alpha)p(\vec{\mathfrak{Y}})$$

Test convexity on $\mathfrak{X} \in R^{1\times 1}$. YES$_1$:



Test convexity on two matrices $\mathfrak{X}, \mathfrak{Y} \in R^{2\times 2}$:

ONE $R^{2\times 2}$ TEST OF CONVEXITY

$$p(X) := X^4 + (X^4)^T$$

$$\frac{1}{2}p(\mathcal{X}) + \frac{1}{2}p(\mathcal{Y}) - p(\frac{1}{2}\mathcal{X} + \frac{1}{2}\mathcal{Y}) \succeq 0?$$ **Try matrices**

$$\mathcal{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{Y} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \ \tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}p(\mathcal{X}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^4 + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\tfrac{1}{2}p(\mathcal{Y}) = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$ and $p(\tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y}) = \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix}$. **Then**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} - \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix} \succeq 0$$ **i.e., is PSD**

RIDDLE

Q1. Is
$$p(X) := X^4 + (X^4)^T?$$
matrix convex? How would you prove it?

Q2. Two vars $\vec{X} = (X_1, X_2)$. Is
$$p(\vec{X}) := X_1{}^4 + (X_1{}^4)^T + 5X_2{}^4 + 5(X_2{}^4)^T + X_1{}^T X_2 + X_2{}^T X_1$$
matrix convex? How would you prove it?

Q3. List all matrix convex polynomials that you can think of, e.g. degree ≤ 2 is easy.

Recall **MATRIX CONVEX** means
$$p(\alpha\vec{\mathfrak{X}} + (1-\alpha)\vec{\mathfrak{Y}}) \preceq \alpha p(\vec{\mathfrak{X}}) + (1-\alpha)p(\vec{\mathfrak{Y}})$$

Convexity vs. Linear Matrix Inequalities

Scott McCullough **University of Florida**

Victor Vinnikov **Ben Gurion University**

J. F. Camino, R. E. Skelton, J. Ye **UC San Diego**

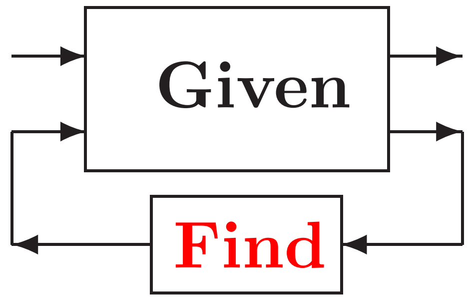
I am Bill Helton **UCSD**

Grad Student Programing: Jason Bell, Josh Griffin, Jeff Ovall,Tony Shaheen

NCAlgebra[a]

[a] Helton, Stankus (CalPoly SanLObispo), Miller (Archemedies Inc)

Systems problems $\longrightarrow$ Matrix Inequalities



Many such problems Eg. H^∞ control

Freq. domain	**Nev-Pick**		**1975-85**
State space	**tricks**	**Riccatis**	**1985-95**
Matrix ineqs	**tricks**	**LMIs**	**1990 -**

Solve numerical LMIs, up to 150×150 (unstructured) matrices, via semidef prog., bundle methods, etc.

$$\left.\begin{array}{rcl} F_1(\mathcal{X}) := A\mathcal{X} + \mathcal{X}A^T + BB^T & \succeq & 0 \\ F_2(\mathcal{X}, \mathcal{Y}) & \succeq & 0 \end{array}\right\} \qquad \textbf{PosSD}$$

MATRIX INEQUALITIES

Polynomial or Rational function of matrices are PosSDef.

Example: Riccati expressions like

$$\boldsymbol{A X + X A^T - X B B^T X + C^T C \geq 0} \tag{1}$$

OR linear matrix inequalities LMI's like

$$\begin{pmatrix} \boldsymbol{A X + X A^T + C^T C} & \boldsymbol{X B} \\ \boldsymbol{B^T X} & \boldsymbol{I} \end{pmatrix} \geq \boldsymbol{0} \tag{2}$$

which is equivalent to the Riccati inequality.

Here $\boldsymbol{A, B, C}$ are typically given matrices and $\boldsymbol{X = X^T}$

Convex is good!

How can we determine easily if a matrix expression is convex?

Prevailing methods

a. **Use tricks - (Schur Complements) to convert to Linear Matrix Inequalities.**

b. **Set problem up so it tends to produce LMIs.**

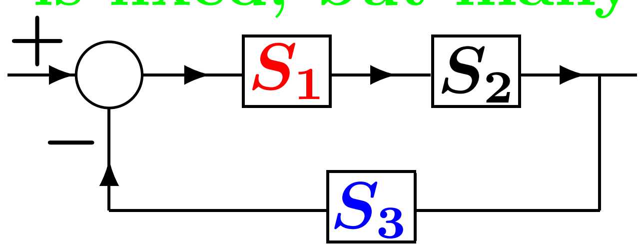
How can we determine if a matrix expression is equivalent to an LMI? Symbolically compute that LMI?

Can we treat many more problems with convex techniques than LMI techniques?

FORMULAS WHICH SCALE

1. **OUTER SCALABLE - NOT THIS TALK - Have "one" type of system and protocols for joining them together to form large networks without destroying stability. "One" system many topologies.**
   – **F. Kelly, G. Vinnicombe, T. Basar, Low, Paganini, Doyle**

2. **INNER SCALABILITY – THIS TALK**
   Topology is fixed; but many systems . E.g.



WANT FORMULAS: which hold regardless of the dimension of system S_1, S_2, S_3. E.g. almost all control text problems.

– **Get noncommutative formulas**

INEQUALITIES WHICH SCALE

Does Scale Gracefully

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{3}$$

Looks the same regardless of system size.

Does NOT Scale Gracefully

If dimensions of the matrices A, B, C, X are specified write (3) with matrices $L_0, \ldots, L_m$ as,

$$\sum_{j=0}^{m} L_j x_j \succeq 0 \tag{4}$$

unknown numbers x_j, (from $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$)

THE RANGE OF QUESTIONS

Characterize and analyze these situations:

	LMI	$CONVEX$
$SCALABLE(NC)$	?	?
$NONSCALE(COM)$	?	$usual\ \ convexity$

Can we treat many more problems with convex techniques than LMI techniques?

Ultimately we want Symbolic Algorithms

ORDINARY NON-SCALABLE LMI's

Commutative variables

Which Sets have LMI REPRESENTATIONS?

QUESTION (Vague) - **ARE CONVEX PROBLEMS ALL TREATABLE WITH LMI's?**

DEFINITION: A set $\mathcal{C} \subset \mathrm{R}^m$ has an **Linear Matrix Inequality (LMI) REPRESENTATION** provided that there are symmetric matrices $L_0, L_1, L_2, \cdots, L_m$ for which the **Linear Pencil** ,

$L(\vec{x}) := L_0 + L_1 x_1 + \cdots + L_m x_m$, has positivity set,

$$\mathcal{G} := \{(x_1, x_2, \cdots, x_m) : L_0 + L_1 x_1 + \cdots + L_m x_m \quad \text{is PosSD}\}$$

equals the set $\mathcal{C}$; that is, $\mathcal{C} = \mathcal{G}$.

Which convex sets have an LMI representation?

Parrilo and Sturmfels preprint 2000

EXAMPLE

$$\mathcal{C} := \{(x_1, x_2) : 1 + 2x_1 + 3x_2 - (3x_1 + 5x_2)(3x_1 + 2x_2) \geq 0\}$$

has the LMI Rep

$$\mathcal{C} = \{\vec{x} : L(\vec{x}) \succeq 0\} \qquad \text{here } \vec{x} := (x_1, x_2)$$

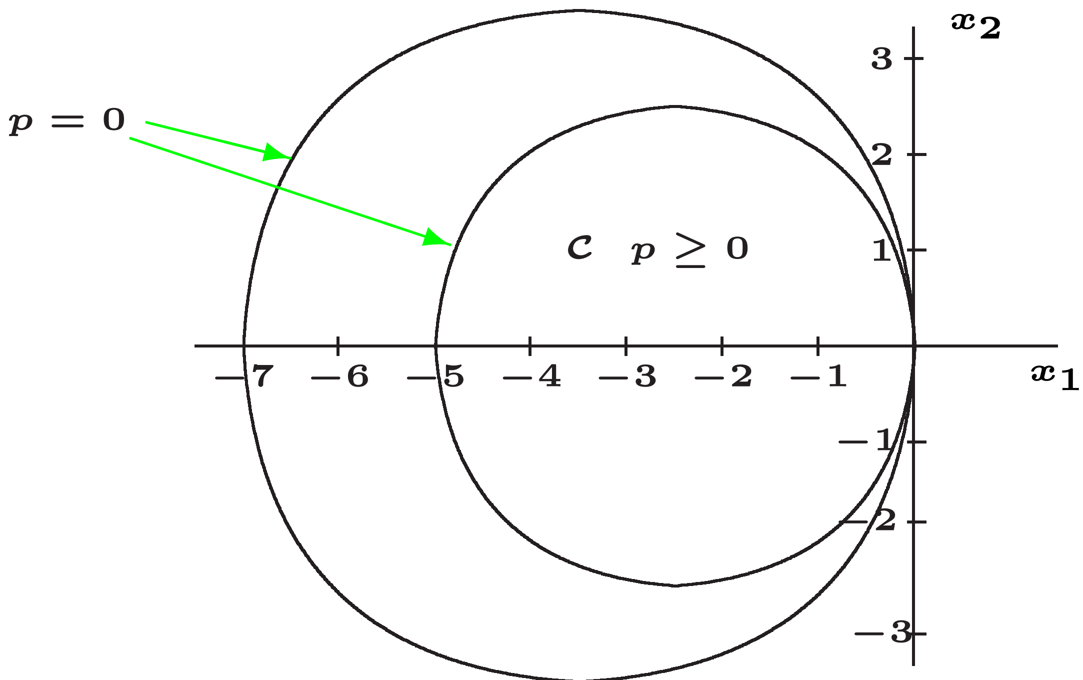
with

$$L(\vec{x}) = \begin{pmatrix} 1 + 2x_1 + 3x_2 & 3x_1 + 5x_2 \\ 3x_1 + 2x_2 & 1 \end{pmatrix}$$

Pf: The determinant of $L(\vec{x})$ is pos iff $L(\vec{x})$ is PosSD.

QUESTION 1

Does this set $\mathcal{C}$ which is the middle component of



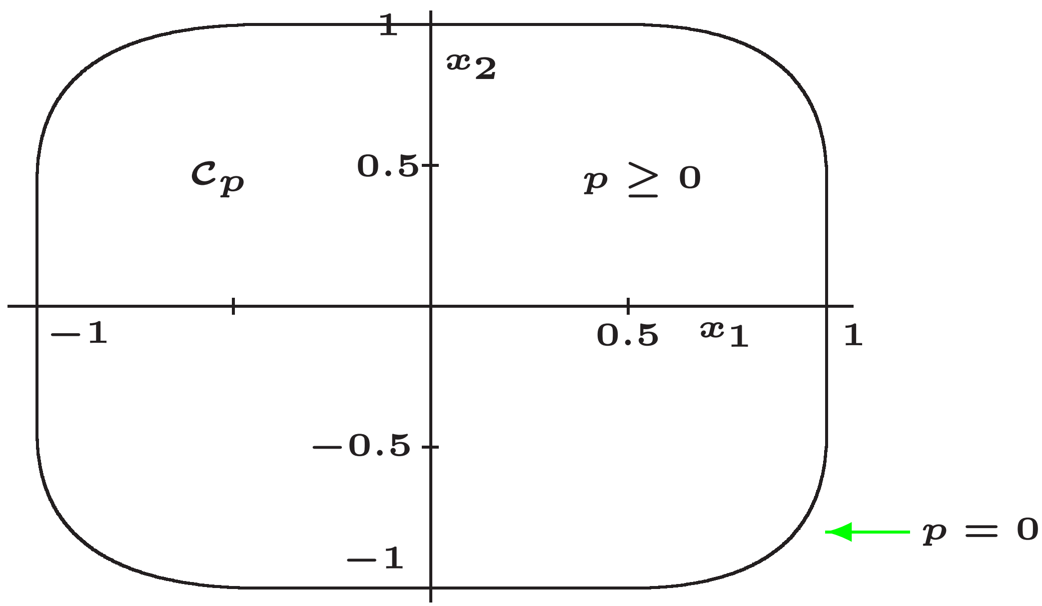
have an LMI representation?

$$p(x_1, x_2) = (x_1{}^2 + x_2{}^2)(x_1{}^2 + x_2{}^2 + 12x_1 - 1) + 36x_1{}^2 \geq 0$$

$$\mathcal{C} := \{\vec{x} \in R^2 : p(\vec{x}) \geq 0\} \text{ component of (-1, 0)}$$

QUESTION 2

Does this set have an LMI representation?



$$p(x_1, x_2) = 1 - {x_1}^4 - {x_2}^4 \geq 0$$

$\mathcal{C}_p := \{\vec{x} \in R^2 : p(\vec{x}) \geq 0\}$ **has degree 4.**

Rigid Convexity

DEFINE: A convex set $\mathcal{C}$ in R^m with minimal degree (denote degree by d) defining polynomial p to be **rigidly convex** provided

> for every point x^0 in $\mathcal{C}$ and every line ℓ through x^0 (except for maybe a finite number of lines), the line ℓ intersects the the zero set $\{\vec{x} \in R^m : p(\vec{x}) = 0\}$ of p in exactly d points [a].

[a] In this counting one ignores lines which go thru $\boldsymbol{x^0}$ and hit the boundary of $\mathcal{C}$ at ∞.

p SATISFIES THE REAL ZEROES CONDITION - skip

DEF: A REAL ZERO polynomial in m variables satisfies for each εR^m,

$$(RZ) \qquad p(\mu x) = 0 \quad \text{implies} \quad \mu \quad \text{is real}$$

LEMMA:

If $\mathcal{C}$ contains 0 in its interior, and p is the minimal degree defining polynomial for $\mathcal{C}$, then p satisfies the Real Zeroes Condition if and only if $\mathcal{C}$ is rigidly convex.

If $\mathcal{C}$ does not contain 0 shift it.

IN R^2 RIGID CONVEXITY RULES

THM (Vinnikov + H).

If $\mathcal{C}$ is a closed convex set in R^m with an LMI representation, then $\mathcal{C}$ is rigidly convex.

When $m = 2$, the converse is true, namely, a rigidly convex degree d set has a LMI representation with symmetric matrices $L_j \in R^{d\times d}$.

The Proof of necessity is trivial. The Proof of sufficiency ($m = 2$) is not at all elementary. Uses algebraic geometry methods of Vinnikov. These have been refined and extended by Vinnikov and Joe Ball.

LMI REPS for

SCALABLE SETS

Noncommutative LMI's

Example: Take $X_j^T = X_j$. One noncommutative LMI is

$$L(\vec{X}) = \begin{pmatrix} 1 + 2X_1 + 3X_2 & 3X_1 + 5X_2 \\ 3X_1 + 5X_2 & 1 \end{pmatrix} \succeq 0.$$

Clearly, if we were to replace $\vec{X}$ by 2-tuples $\vec{\mathcal{X}} = (\mathcal{X}_1, \mathcal{X}_2)$ of 3×3 matrices $L(\vec{\mathcal{X}})$ would be a 6×6 matrix.

Consider, for all ℓ

$$\mathcal{G}_\ell := \{\vec{\mathcal{X}} : L(\vec{\mathcal{X}}) \succeq 0, \quad all \quad \vec{\mathcal{X}} \in R^{\ell \times \ell}\}$$

call it a **Symbolic Inequality Domain, SID,** denoted

$$\mathcal{G} := \{\vec{X} : L(\vec{X}) \geq 0\}$$

CONVEX SIDS

QUESTION: Suppose p is a polynomial with SID

$$\mathcal{G} := \{\vec{X} : p(\vec{X}) \geq 0\}$$

which is convex and bounded, then does $\mathcal{G}$ have an LMI representation? That is, there exist $L_0, L_1, \cdots, L_m$,

$$\mathcal{G} = \{\vec{X} : L_0 + L_1 X_1 + \cdots + L_m X_m \geq 0\}?$$

THMS (McCullough +H):

Yes, with $L_0, L_1, \ldots, L_m$ which are "infinite matrices"

Yes, Suppose $\mathcal{G}$ is "bounded" (in C^N) and p has form,

$p_1(\vec{X}) = \sum^N a_w \vec{X}^{w^T} \vec{X}^w$. Then p has LMI rep. (FinDim)

NC CONVEX POLYNOMIALS

(Before we did convex sets)

RIDDLE

Q1. Is

$$p(X) := X^4 + (X^4)^T?$$

matrix convex. How would you prove it?

Q2. Is

$$p(\vec{X}) := {X_1}^4 + ({X_1}^4)^T + 5{X_2}^4 + 5({X_2}^4)^T$$

matrix convex? How would you prove it?

Q3. List all matrix convex polynomials that you can think of, e.g. degree ≤ 2 is easy.

Recall **MATRIX CONVEX** means

$$p(\alpha \vec{\mathfrak{X}} + (1-\alpha)\vec{\mathfrak{Y}}) \preceq \alpha p(\vec{\mathfrak{X}}) + (1-\alpha)p(\vec{\mathfrak{Y}})$$

CONVEX POLYNOMIALS ARE WIMPS

The riddle revealed:

$$p(X) := X^4 + (X^4)^T \text{ is not matrix convex.}$$

THM: (McC + H)

Every symmetric noncommutative polynomial which is matrix convex has degree 2 or less.

COR A Convex NC Polys is the Schur complement of some linear pencil. **Proof:**

1. **NC Positive Polynomials**

2. **NC Second Derivatives**

3. **Put the two together**

1. "POSITIVE" NC POLYS

$\mathbf{p} \geq \mathbf{0}$ Means What ?

p is a symmetric polynomial in non-commutative variables $\vec{X} = \{X_1, X_2\}$ with real coefficients, eg.

$$p(\vec{X}) = X_1{}^2 + (X_1{}^2)^T + X_2{}^T X_2$$

Define MATRIX POSITIVE polynomial

Plug in $n \times n$ matrices $\mathcal{X}_j$ for X_j in p

always get

$p(\mathcal{X}_1, \mathcal{X}_2)$ is a PosSD $n \times n$ matrix.

Example $\quad \vec{X} = \{X_1, X_2\}$

$$
\begin{array}{lcccccc}
p(\vec{X}) & = (3X_1^2 + X_2)^T & (3X_1^2 + X_2) & +(X_1{}^5)^T & (X_1{}^5) \\
 & & \Downarrow & & \Downarrow \\
p(\vec{\mathcal{X}}) & = & \text{PosSD} & + & \text{PosSD}
\end{array}
$$

This is a sum of squares (SoS). More generally

$$p(\vec{X}) = \sum_{j=1}^{c} \mathcal{L}_j(\vec{X})^T \mathcal{L}_j(\vec{X})$$

is clearly always Matrix Positive.

TESTING POLYS FOR POSITIVITY

Write poly p as Sum of Squares (SoS)

$$p = \text{Sum of Squares (SoS)} \Rightarrow p \text{ is pos}$$

QUESTION: $\text{SoS} \overset{?}{\Leftarrow} p$ is pos?

COMMUTATIVE Poly on R^n: Hilbert 17th problem

$n = 1$	and pos	$\Rightarrow$ SoS	YES
$n \geq 2$	and pos	$\Rightarrow$ SoS	NOT ALWAYS

Algorithms (commutative) for SoS: - effective

- Reznick, then Powers - Wormann 1990's
- Parrilo's 2001 Thesis +;
- MTNS Session today

NON-COM POLYS ARE BETTER BEHAVED THAN COMMUTATIVE POLYS

THM:

$\boxed{\text{Matrix positive non-commutative polynomials are sums of squares}}$

Can compute SoS, thus CAN TEST Matrix Positivity.

Proof: Operator theory techniques

- **See 2001 H preprint.**
- **S. McCullough 2000 preprint - when all variables X_j are "unitary".**

2. Second NC Derivatives

Non-commutative rational function

$$\Gamma(X,Y) := X^T A X + D Y D^T + X Y X^T, \quad Y = Y^T, A = A^T$$

Directional derivative wrt X, $D_X \Gamma(X,Y)[H] :=$

$$= \lim_{t \to 0} \frac{1}{t} \left(\Gamma(X + tH, Y) - \Gamma(X,Y) \right) = \frac{d}{dt} \Gamma(X + tH, Y) \Big|_{t=0}$$

Examples:

- ✓ **First derivative of $\Gamma(X,Y)$ in X**

$$D_X \Gamma(X,Y)[H] = H^T A X + X^T A H + H Y X^T + X Y H^T$$

- ✓ **Second derivative of $\Gamma(X,Y)$ in X**

$$\mathcal{H}_X \Gamma(X,Y)[H] = 2 H Y H^T + 2 H^T A H$$

Convexity- Two Definitions

Geometrically matrix convex $\quad 0 \leq \alpha \leq 1$

$$\Gamma(\alpha \vec{\mathcal{X}}^1 + (1-\alpha)\vec{\mathcal{X}}^2) \preceq \alpha\Gamma(\vec{\mathcal{X}}^1) + (1-\alpha)\Gamma(\vec{\mathcal{X}}^2)$$

Matrix Convexity: Plugging any matrices into Hessian $\mathcal{H}\Gamma(\vec{X})[\vec{H}]$ of Γ wrt $\vec{X}$ gives a PosSD matrix.

Geometric and matrix convexity are the same, provided $\Gamma(\vec{X})$ is smooth and there are no constraints on $\vec{X}$. (cf. Merino+H, CDC1997)

Notation: $\quad \mathcal{H}\Gamma(X_1, \ldots, X_m)[H_1, \ldots, H_m]$

3. $p = X^4 + X^{T^4}$ IS NOT CONVEX

Idea of Proof: $p = XXXX + (XXXX)^T$

$D_X p(X)[H] =$

$$HXXX + XHXX + XXHX + XXXH + (same)^T$$

$\mathcal{H}_X p(X)[H] =$ $\mathbf{2}HHXX + \mathbf{2}HXHX + etc$

Convex p says its Hessian is SoS:

$$\mathcal{H}_X p(X)[H] = \Sigma_j \; \mathcal{L}_j(X)[H]^T \mathcal{L}_j(X)[H].$$

The highest order terms of $\mathcal{L}_j(X)[H]$ are linear in H. So HXX is a term of $\mathcal{L}_j(X)[H]$. Thus the degree 6 term $X^T X^T H^T HXX$ is a term of the degree 4 polynomial $\mathcal{H}_X p(X)[H]$. Contradiction. $\square$

REGIONS OF CONVEXITY

CONVEX ON A SI DOMAIN

$$\mathcal{G} := \{\vec{X} : \delta(\vec{X}) \geq 0 \geq 0\}$$

"Open" Symbolic Inequality Domain: E.g.

$$\mathcal{G}_\varepsilon := \{\vec{X} : \varepsilon^2 I - ({X_1}^T X_1 \cdots + X_m^T X_m) \geq 0\}$$

is "open", since matrices $\vec{\mathcal{X}}$ of high dimension satisfying

$$\varepsilon^2 I - ({\mathcal{X}_1}^T \mathcal{X}_1 \cdots + \mathcal{X}_m^T \mathcal{X}_m) \succeq 0.$$

THM (McC +H):

If the noncommutative symmetric p is convex on an "open" set, then p has degree ≤ 2.

p is Positive on a (SID) -Definition

$$\mathcal{G} := \{\overrightarrow{Z} : \delta_1(\overrightarrow{Z}) \geq 0, \ldots, \delta_c(\overrightarrow{Z}) \geq 0\}$$

$p(\overrightarrow{Z})$ is **MATRIX POSITIVE on SID $\mathcal{G}$** means: For any matrices $\overrightarrow{\mathfrak{Z}}$, satisfying

$$\delta_1(\overrightarrow{\mathfrak{Z}}) \geq 0, \ldots, \delta_c(\overrightarrow{\mathfrak{Z}}) \geq 0,$$

we have $p(\overrightarrow{\mathfrak{Z}})$ is a Positive Semidefinite matrix.

Example $\quad p$ is a weighted sum of squares:

$$p(\overrightarrow{Z}) = \sum_{j=1}^{c} \mathcal{L}_j(\overrightarrow{Z})^T \delta_j(\overrightarrow{Z}) \mathcal{L}_j(\overrightarrow{Z})$$

clearly is MATRIX POS on the SID $\mathcal{G}$.

NC POSITIVSTELLENSATZ

THM (McC +H):

If p is "operator positive" on the bounded SID

$$\mathcal{G} := \{\vec{Z} : \delta_1(\vec{Z}) \succeq 0, \ldots, \delta_c(\vec{Z}) \succeq 0\},$$

then

$$p(\vec{Z}) := \sum_{k=1} \rho_k(\vec{Z})^T \rho_k(\vec{Z}) +$$

$$\sum_{j=1}^{c} \mathcal{L}_j(\vec{Z})^T \delta_j(\vec{Z}) \mathcal{L}_j(\vec{Z}).$$

If $\mathcal{G}$ is convex "matrix positive" will do.

PROOF. "Operator case" follows outline of M. Putinar + F. Vasilescu proof for the commutative case, then add small algebra trickery to get convex case.

A Weighted SoS Decomposition

Suppose $Q(\vec{Z})[\vec{H}]$ is rational in $\vec{Z}$ and quadratic in $\vec{H}$.

We seek a weighted SoS decomposition

$$Q(\vec{Z})[\vec{H}] = \Sigma_j \ \mathcal{L}_j(\vec{Z})[\vec{H}]^T \delta_j(\vec{Z}) \mathcal{L}_j(\vec{Z})[\vec{H}]$$

with $\mathcal{L}_j(\vec{Z})[\vec{H}]$ linear in $\vec{H}$.

THM (CHSY) Under modest hypotheses:

Any Q has a (computable) weighted SoS decomposition with $\mathcal{L}_j(\vec{Z})[\vec{H}]$ linear in $\vec{H}$ giving SID

$$\mathcal{G} := \{\vec{Z} : \delta_1(\vec{Z}) \geq 0, \ldots, \delta_c(\vec{Z}) \geq 0\}$$

which is the “biggest” SID on which Q is Mat Pos.

∞ EXTRAPOLATE ∞

ENGINEERING SPECULATION: Convex systems problems which are inner scalable can all be converted to LMI.

MATH QUESTION: Which NC rational functions Γ are matrix convex on some "open" set?

MATH SPECULATION: Such Γ is a "Schur complement" of some NC linear pencil

$$L(\vec{X}) := L_0 + L_1 X_1 + \cdots + L_m X_m.$$

Is there an algorithm which finds it?

WHERE WE STAND

Convexity Algorithm

The theory in the talk gives an algorithm

❶ **What is the domain of convexity for:**

$$F(X,Y) = (X - Y^{-1})^{-1}, \qquad X = X^T, Y = Y^T$$

☞ **APPLY:** NCConvexityRegion[F, {X, Y}]

$$\Updownarrow$$

$$\{2(X - Y^{-1})^{-1}, \;\; 2Y^{-1}\}$$

☞ **CONCLUDE:** $F(X,Y)$ **is matrix convex on the domain**

$$\mathcal{G} := \{(X,Y) : Y > 0 \quad \text{and} \quad X - Y^{-1} > 0\}$$

Convexity Algorithm -In Vivo

In[1] := `<< NCAlgebra.m;`

In[2] := `<< NCConvexity.m;`

In[3] := `SetNonCommutative[X, Y];`

In[4] := `F = inv[X - inv[Y]]`

Out[5] := **inv[X − inv[Y]]**

In[6] := `NCConvexityRegion[F,{X, Y}]`

Out[7] := `{{2inv[X - inv[Y]], 2inv[Y]}}`

Download NCAlgebra : www.math.ucsd ∼ ncalg

SUMMARY COMMUTATIVE

1. **COMMUTATIVE LMI's (Not Scalable).**

   THM 1 THERE IS A "LINE" TEST, ANY CONVEX SET $\mathcal{C}$ IN R^m WITH AN LMI REPRESENTATION MUST SATISFY. IN R^2 THIS IS SUFFICIENT. Valuable TECHNIQUE MULTI DIMENSIONAL SYS- WKSHOP and SESS

2. **COMMUTATIVE SUM OF SQUARES. HAVE MANY RECENT APPLICATIONS. SESSION THIS AFTERNOON**

SUMMARY - NONCOMMUTATIVE

1. **Inner SCALABLE SYS PROBS ARE NONCOM.**
2. THM **2** **EVERY "POSITIVE" NC POLYNOMIAL IS A SUM OF SQUARES.** $\leftarrow$ **COMPUTABLE.**
3. THM **3** **EVERY "CONVEX" NC POLYNOMIAL HAS DEGREE $\leq$ 2.** **Equiv. to an LMI.**
4. THM **4** **THMs 2 AND 3 LOCALIZE, eg NC POSITIVSTELLENSATZ**
5. ALGORTHIM **CHECKS CONVEXITY OF NC RATIONAL FUNCTION.** **TRY NCAlgebra**
6. **MAYBE** **ALL CONVEX (Inner) SCALABLE SYSTEMS PROBLEMS ARE TREATABLE BY LMIs.**

THE END

PLASTIC SLIDES

ONE $R^{2\times 2}$ TEST OF CONVEXITY

$$p(X) := X^4 + (X^4)^T$$

$$\frac{1}{2}p(\mathcal{X}) + \frac{1}{2}p(\mathcal{Y}) - p(\frac{1}{2}\mathcal{X} + \frac{1}{2}\mathcal{Y}) \succeq 0? \qquad \text{Try matrices}$$

$$\mathcal{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{Y} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}p(\mathcal{X}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^4 + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\tfrac{1}{2}p(\mathcal{Y}) = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$ and $p(\tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y}) = \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} - \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix} \succeq 0 \qquad \text{i.e., is PSD}$$

RIDDLE

Q1. Is

$$p(X) := X^4 + (X^4)^T?$$

matrix convex. How would you prove it?

Q2. Is

$$p(\vec{X}) := X_1{}^4 + (X_1{}^4)^T + 5X_2{}^4 + 5(X_2{}^4)^T$$

matrix convex? How would you prove it?

Q3. List all matrix convex polynomials that you can think of, e.g. degree ≤ 2 is easy.

Recall **MATRIX CONVEX** means

$$p(\alpha\vec{\mathfrak{X}} + (1-\alpha)\vec{\mathfrak{Y}}) \preceq \alpha p(\vec{\mathfrak{X}}) + (1-\alpha)p(\vec{\mathfrak{Y}})$$

$\mathrm{p} \geq 0$ Means What ?

p is a symmetric polynomial in non-commutative variables $\overrightarrow{X} = \{X_1, X_2\}$ with real coefficients, eg.

$$p(\overrightarrow{X}) = {X_1}^2 + (X_1^2)^T + {X_2}^T X_2$$

Define MATRIX POSITIVE polynomial

Plug in $n \times n$ matrices $\mathcal{X}_j$ for X_j in p

always get

$p(\mathcal{X}_1, \mathcal{X}_2)$ is a PosSD $n \times n$ matrix.

Riddle: 1. Is it easy to check Mat Pos computationally?

2. Is p above Matrix Positive?

Which Sets have LMI REPRESENTATIONS?

QUESTION (Vague) - ARE CONVEX PROBLEMS ALL TREATABLE WITH LMI's?

DEFINITION: A set $\mathcal{C} \subset \mathrm{R}^m$ has an **Linear Matrix Inequality (LMI) REPRESENTATION** provided that there are symmetric matrices $L_0, L_1, L_2, \cdots, L_m$ for which the **Linear Pencil** ,

$L(\vec{x}) := L_0 + L_1 x_1 + \cdots + L_m x_m$, has positivity set,

$$\mathcal{G} := \{(x_1, x_2, \cdots x_m) : L_0 + L_1 x_1 + \cdots + L_m x_m \quad \text{is PosSD}\}$$

equals the set $\mathcal{C}$; that is, $\mathcal{C} = \mathcal{G}$.

Which convex sets have an LMI representation?

Parrilo and Sturmfels - preprint 2000

Rigid Convexity

DEFINE: A convex set $\mathcal{C}$ in R^m with minimal degree (denote degree by d) defining polynomial p to be **rigidly convex** provided

> for every point x^0 in $\mathcal{C}$ and every line ℓ through x^0 (except for maybe a finite number of lines), the line ℓ intersects the the zero set $\{\vec{x} \in R^m : p(\vec{x}) = 0\}$ of p in exactly d points [a].

[a] In this counting one ignores lines which go thru x^0 and hit the boundary of $\mathcal{C}$ at ∞.

Polynomial or Rational function of matrices are PosSDef.

Example: Riccati expressions like

$$AX + XA^T - XBB^TX + C^TC \succeq 0 \tag{5}$$

OR linear matrix inequalities LMI's like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{6}$$

which is equivalent to the Riccati inequality.

Here A, B, C are typically given matrices and $X = X^T$

Does Scale Gracefully

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{7}$$

Looks the same regardless of system size.

Does NOT Scale Gracefully

If dimensions of the matrices A, B, C, X are specified write (7) with matrices $L_1, \ldots, L_m$ as,

$$\sum_{j=1}^{m} L_j x_j \succeq 0 \tag{8}$$

unknown numbers x_j, (from $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$)

UGLY SCALING

Example: If $A \in R^{2\times 2}, B \in R^{2\times 1}, C \in R^{1\times 2}$, then $X \in R^{2\times 2}$ and we would take

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$

and define the 4×4 symmetric matrices L_0, L_1, L_2, L_3

$$L_0 := \begin{pmatrix} C^T C & 0 \\ 0 & I \end{pmatrix}$$

$$L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$$

UGLY SCALING

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11} + a_{22} & b_{21} & b_{22} \\ a_{22} + a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix}$$

$$L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}$$

Plug $L's$ into

$$\sum_{j=1}^{m} L_j x_j \succeq 0$$

UGH.