COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
5. HILBERT SPACE OPERATORS, TIHANY (HUNGARY), 1970

Operators with a representation as multiplication by *x* on a Sobolev space

H. HELTON

I shall speak on a structure theorem for a certain class of non selfadjoint operators based on the representation of such operators as multiplication by x on a type of Sobolev space.

Let T be a bounded operator on a Hilbert space and set $V(s) = e^{isT}$. If we define the operator-valued entire function $R(s) = V(\bar{s})^* V(s) = e^{-isT^*} e^{isT} = \sum_{n=0}^{\infty} A_n s^n$, then T is self adjoint if and only if $R(s) = 1$ and hence it seems reasonable to classify operators by the behavior of $R(s)$. In particular, a class of operators which seems nearly as natural and amenable to study as the self-adjoint operators is the operators T for which $R(s)$ is a polynomial in s, that is

(POL) $$R(s) = V(\bar{s})^* V(s) = \sum_{n=0}^{N} A_n s^n .$$

Under certain additional hypotheses, it will be proved that such operators have a representation as multiplication by x on a type of Sobolev space.

The type of Sobolev space we need is this. Let $\mu = <\mu_{ij}>$ be a matrix of $(M+1)^2$ measures on the interval $[a,b]$ and define a bilinear form $(\,,\,)_\mu$ on $C^\infty[a,b]$ by

(REP) $$(f,g)_\mu = \sum_{i,j=0}^{M} \int_a^b f^{(j)} \bar{g}^{(j)} d\mu_{ij}$$

where $h^j(x) = \frac{d^j}{dx^j} h(x)$. If $(\,,\,)_\mu$ is positive definite we let $H(\mu)$ stand for the completion of $C^\infty[a,b]$ in $\|f\|_\mu = \sqrt{(f,f)_\mu}$. We will say that $H(\mu)$ has order M.

The object of this talk will be to sketch a proof of

Theorem. *If* T *satisfies* POL, *if* T *has a cyclic vector, and if* T *has spectrum* $[a,b]$, *then there is a space* $H(\mu)$ *and a unitary map* $U: H \to H(\mu)$ *such that* UTU^{-1} *is multiplication by* x *on* $H(\mu)$.

Before we begin the proof of this theorem, we make a few remarks. First of all, the theorem is much weaker than what can be proved, but for the sake of clear exposition we postpone a description of its generalizations to the end of this talk. Secondly, it is easy to prove that the degree N of the polynomial $R(s)$ is even. Undoubtedly the space $H(\mu)$ can be chosen to have order $N/2$. I have a heuristic proof for this. Lastly the POL condition can be expressed in more concerte terms. Namely, POL is equivalent to the condition $[T^*-T]^{[N+1]} = 0$, where $[A-B]^{[M]}$ is defined to be the operator $\sum_{K=0}^{M} \binom{M}{K} (-1)^{M-K} A^K B^{M-K}$. To check the equivalence of the two conditions note that for any bounded operator

$$\frac{d}{ds} e^{-isT^*} A e^{isT} = -i e^{-isT^*} [T^*A - AT] e^{isT}$$

and hence

$$\frac{d^{N+1}}{ds^{N+1}} e^{-isT^*} e^{isT} = (-i)^{N+1} e^{isT^*} C_T^{N+1}(I) e^{isT}$$

where C_T is the map of $\mathcal{L}(H)$, the set of all bounded operators on H, into $\mathcal{L}(H)$ defined by $C_T(A) = T^*A - AT$. The POL condition is equivalent to $\frac{d^{N+1}}{ds^{N+1}} R(s) = 0$ and consequently to $C_T^{N+1}(I) = 0$. A straightforward computation (cf. proof of

Theorem 2.2 [1]) shows that $[T^*-T]^{[M]} = (-1)^M C_T^M(I)$.

Now we sketch a proof of the representation theorem based on distribution theory. Recall (cf. Chapter 1 and 2 of Hörmander [2]) that $\mathcal{S}(R^n)$ is the space of infinitely differentiable functions on R^n which along with their derivatives decrease faster than any polynomial at infinity. Frequently we shall use $\mathcal{S}$ to denote $\mathcal{S}(R^1)$. The POL condition guarantees that $\|e^{isT}\| = O(|s|^N)$ and thus T has a C^n functional calculus (cf. Colojoară and Foiaş [1] for a discussion of such operators).

Define $\mathcal{U}_f$ by

(1) $$\mathcal{U}_f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{i\zeta T} d\zeta$$

for any $f \in \mathcal{S}$, where $\hat{f}$ denotes the Fourier transform of f. This is a slight modification of the usual construction of a C^n functional calculus for T. The integral is norm convergent since the function $\hat{f}$ is in $\mathcal{S}$, and $\mathcal{U}_f$ has the usual properties

(2) $$\mathcal{U}_{f+g} = \mathcal{U}_f + \mathcal{U}_g \quad \text{and} \quad \mathcal{U}_{fg} = \mathcal{U}_f \mathcal{U}_g .$$

One can show that

(3) $$e^{isT}\mathcal{U}_{f(x)} = \mathcal{U}_{e^{isx} f(x)} \quad \text{and} \quad T\mathcal{U}_{f(x)} = \mathcal{U}_{xf(x)} .$$

Let ψ_0 be a cyclic vector for T. Define a bilinear form $[\,,\,]$ on $\mathcal{S}$ by

$$[f,g] = (\mathcal{U}_f\psi_0, \mathcal{U}_g\psi_0).$$

The next few paragraphs will be devoted to proving that $[\,,\,]$ has the form $(\,,\,)_\mu$ given in REP. After this is accomplished we will show that the map V: $\mathcal{S} \to \mathcal{H}$ defined by

$$Vf = \mathcal{U}_f\psi_0$$

induces a map V_r from $H(\mu) \cap \mathcal{S}$ into a dense subspace of H. The extension of V_f to $H(\mu)$ will be the map $\mathcal{U}^{-1}$ required by our theorem.

The Schwartz Nuclear Theorem (cf. Theorem 2.1 of Streater and Wightman [4]) implies that there is a continuous linear functional (distribution) on $\mathcal{S}(R^2)$ such that

(4) $$\ell(f(x)\bar{g}(y)) = [f,g] .$$

The POL condition says that

$$\frac{d^{N+1}}{ds^{N+1}}[e^{isx}f(x), e^{isx}g(x)] = 0$$

for any $f, g \in \mathcal{S}$ or equivalently

$$\frac{d^{N+1}}{ds^{N+1}} \ell(e^{isx}f(x)\bar{g}(y)) = 0 .$$

Thus

$$\ell([x-g]^{N+1} f(x)\bar{g}(y)) = 0$$

and since linear combinations of functions of the form $f(x)\bar{g}(y)$ are dense in $\mathcal{S}(R^2)$,

$$\ell([x-y]^{N+1} h(x,y)) = 0$$

for any function $h \in \mathcal{S}(R^2)$. This implies that if k is in $\mathcal{S}(R^2)$ and has a zero of order $N+1$ on $E = \{(x,y) \in R^2 : x = y\}$, then $\ell(k) = 0$.

Now we need a change of variables

$$x - y \longrightarrow \tau \qquad x + y \longrightarrow \beta .$$

Any function h in $\mathcal{S}(R^2)$ has a finite Taylor expansion

$$h(\tau,\beta) = \sum_{j=0}^{N} \frac{\partial^j}{\partial \tau^j} h(0,\beta)\tau^j + \psi(\tau,\beta) ,$$

where the remainder term $\psi(\tau,\beta)$ has a zero of order $N+1$ in τ at $\tau = 0$. Hence $\ell(\psi) = 0^*$ and thus we can obtain

(5) $$\ell(h(\tau,\beta)) = \sum_{j=0}^{N} \ell_j\left(\frac{\partial^j}{\partial \tau^j} h(0,\beta)\right) ,$$

where each ℓ_j is a distribution on $\mathcal{S}(R^1)$. Now any distribution s on $\mathcal{S}(R^1)$ has a representation (Schwartz Kernel Theorem, sec. 2.3 Shilov [3] or equation 2-11 [4]) of the form

*Technically ψ is not in $\mathcal{S}(R^2)$, however a standard argument involving cutoff functions makes this proof of equation (5) rigorous.

$$s(f) = \sum_{\ell=0}^{K} \int_{-\infty}^{\infty} b_\ell(x) f^{(\ell)}(x)\,dx$$

for some continuous functions b_ℓ wich grow at worst like a polynomial at infinity. If we put together the two representations above and note that

$$\frac{\partial}{\partial\tau} m(x)n(y)\Big|_{x=y} = \frac{1}{2}\Big[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big] m(x)n(y)\Big|_{x=y} = \frac{1}{2} m'(x)n(x) - \frac{1}{2} m(x)n'(x)$$

we get

$$\ell(f(x)\bar{g}(y)) = \sum_{i,j=0}^{M} \int_{-\infty}^{\infty} c_{ij}(x) f^{(i)}(x)\bar{g}^{(j)}(x)\,dx$$

for $f, g \in \mathcal{S}$. We must restrict the interval of integration to $[a,b]$ and in order to do this we need the following lemmas:

Lemma I. *If* $f \in \mathcal{S}$ *and* $f \equiv 0$ *in a neighborhood of* $\sigma(T)$, *then* $U_f = 0$. *Moreover, the norm closure* $\bar{\mathcal{P}}$ *of* $\mathcal{P} = \{p(T) : p$ *is a polynomial*$\}$ *is equal to the norm closure* $\bar{S}$ *of* $S = \{U_f : f \in \mathcal{S}\}$.

Proof. First we show that $\bar{\mathcal{P}}$ contains S. Since for each s the operator e^{isT} is in $\bar{\mathcal{P}}$ and since the integral in (1) is norm convergent, $\bar{\mathcal{P}}$ contains each operator U_f and thus contains S.

It is easy to see from properties (2) and (3) of U_f that the Gelfand map of $\bar{\mathcal{P}}$ into the continuous functions on its maximal ideal space maps U_f into f. Thus U_f is an invertible operator if and only if f is an invertible function on the maximal ideal space of $\bar{\mathcal{P}}$ that is, on $\sigma(T)$. Now suppose that f is a function in $\mathcal{S}$ which is zero on a neighborhood $\sigma(T)$. There is certainly some function h in $\mathcal{S}$ which is invertible on $\sigma(T)$ and which is 0 on the support of f. Thus $U_f U_h = U_{fh} = 0$ and $U_f = 0$. The first half of Lemma I has been proved.

Now we show that $\bar{\mathcal{P}} = \bar{S}$. This is very easy for if p is a polynomial and the function h in $\mathcal{S}$ is never zero in a neighborhood of $\sigma(T)$, then $p(T) = [U_h]^{-1} U_{hp}$ and so $p(T)$ is the product of two operators in $\bar{S}$. This concludes the proof of Lemma I.

Lemma II. *If* $\sigma(T) = [a,b]$ *where* $a \neq b$, *we have that if* $f \in \mathcal{S}$ *and* $f \equiv 0$ *on* $[a,b]$ *then* $U_f = 0$.

Proof. Suppose that $f \equiv 0$ on $[a,b]$. Define $f^{\pm}$ by

$$f^+ = \begin{cases} f(x) & x \geq b \\ 0 & x \leq b \end{cases}$$

$$f^- = \begin{cases} f(x) & x \leq a \\ 0 & x \geq a . \end{cases}$$

Since $a \neq b$ both f^+ and f^- are in $\mathcal{S}$. The functions $f_n^\pm$ defined by $f_n^\pm(x) = f_n^\pm(x \mp 1/n)$ converge to $f^\pm$ in the $\mathcal{S}$ norm because

$$\lim_{n\to\infty} \left| x^\ell \frac{d^k}{dx^k}[f^\pm(x) - f^\pm(x \mp 1/n)] \right| =$$

$$= \lim_{n\to\infty} \frac{1}{n} \left| x^\ell \frac{d^k}{dx^k}[f^\pm(x) - f^\pm(x \mp 1/n)]/1/n \right| = 0 .$$

Since $f_n^\pm \equiv 0$ on a neighborhood of $[a,b]$, the operators $U_{f_n^\pm}$ are all 0. However, U_f is a continuous map of $\mathcal{S}$ into $\mathcal{L}(H)$ and hence $U_{f^\pm} = 0$. Therefore $U_f = U_{f^+} + U_{f^-} = 0$.

At this point let us remark that if $a = b$, then Lemma II fails to be true. For example, if T is an operator whose square is zero, then the spectrum of T is $\{0\}$ and U_f defined by (1) for T is equal to zero if and only if $f(0) = 0$ and $f'(0) = 0$.

We had shown, several paragraphs ago, that the distribution ℓ on $\mathcal{S}(R^2)$ with property (4) has support on E. Lemma I above implies further that ℓ has support on the set $E_{\sigma(T)} = \{(x,x) \in E : x \in \sigma(T)\}$. Now under the simplifying assumption that $\sigma(T)$ is an interval $[a,b]$ it is easy to show (with the help of the fact that any distribution with support at a point is a linear combination of derivatives of the dirac δ-function) that the representation REP holds for the form $[\ ,\ ]$.

Now we conclude the proof of the representation theorem. Define $V : \mathcal{S} \to H$ by

$$Vf = U_f \psi_0 .$$

Since ψ_0 is cyclic for T, Lemma I implies that the orbit of ψ_0 under S is dense in H , and hence V maps onto a dense subspace of H . By Lemma II we may define a map V_r of $C^\infty[a,b]$ into H by

$$V_r g = V\tilde{g}$$

where g is a function in $C^\infty[a,b]$ and $\tilde{g}$ is any extension of g which belongs to $\mathcal{S}$. From the statement that $[\ ,\ ]$ has a representation $(\ ,\ )_\mu$ which was prove above we can conclude that

$$(f,g)_\mu = (V_r f, V_r g) .$$

The bilinear form $(\ ,\ )_\mu$ must be positive since $(\ ,\ )$ is positive. Moreover, it is positive definite since $(f,f)_\mu = 0$ implies that $V_r f = 0$, which implies that $U_f \psi_0 = Vf = 0$ which implies that $U_f U_g \psi_0 = 0$ for any g . This in turn implies that $U_f \equiv 0$, which (by the proof of Lemma I) implies that $f \equiv 0$ on $\sigma(T)$. Thus $C^\infty[a,b]$ can be completed in $(\ ,\ )_\mu$ to produce a space $H(\ )$. Naturally V_r extends to an isometry $\bar{V}$ from $H(\mu)$ onto H . If $f \in C^\infty[a,b]$, then $\bar{V}T\bar{V}^{-1}f(x) = \bar{V}TV_r f(x) = \bar{V}TU_{\tilde{f}} = \bar{V}U_{x\tilde{f}} = xf(x)$. The equation $\bar{V}T\bar{V}^{-1}f(x) = xf(x)$ extends by continuity to any $f \in H(\mu)$. This completes the proof of the representation theorem.

The above theorem can be generalized considerably. In fact most of its hypotheses can be dropped an still a strong conclusion holds. Firstly, we can replace the assumption that T has a cyclic vector ψ_0 with the assumption that T has a finite cyclic set $\{\psi_1, \dots, \psi_n\}$. In this case we study the bilinear form $[f, g] = (U_{f_1}\psi_1 + \dots + U_{f_n}\psi_n, U_{g_1}\psi_1 + \dots + U_{g_n}\psi_n)$ where F and G are C^∞ n-vector valued functions with components $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$. Under the POL assumption this bilinear form will have a representation which is the natural generalization of REP to vector valued functions. Note that the form $[\ ,\ ]$ in this vector valued case must be positive but not necessarily positive definite. Secondly, suppose that $T_1, \dots, T_k$ is a commuting family each operator of which satisfies POL. We then are forced to consider functions f not of one variable but of k variables and to build a functional calculus

$$U_f = \frac{1}{\sqrt{2\pi}^k} \int_{R^k} \hat{f}(\zeta_1, \dots, \zeta_k) e^{iT_1\zeta_1 + \dots + iT_k\zeta_k} d\zeta_1 \dots d\zeta_k .$$

The bilinear form $[\ ,\ ]$ on $\mathcal{S}(R^k)$ defined by

$$[f, g] = (U_f \psi_0, U_g \psi_0)$$

will have a representation which is the natural generalization of REP to R^k. Thirdly, the operator T need not be bounded. All that we really need to consider is a one parameter group of operators $V(s)$ which satisfies the POL condition. If we modify the cyclic vector assumption property, then REP holds with $b = \infty$, $a = -\infty$ and $V(s)$ maps into multiplication by e^{isx}. Fourthly, the three generalizations above can be put together to give the following theorem:

Theorem. *Suppose that*

a) $V_1(s), \dots, V_k(s)$ *are commuting one parameter groups of operators each of which satisfies the* POL *condition*

and

b) *there are* n*-vectors* $\psi_1, \dots, \psi_n$ *such that the orbit of the subspace spanned by them under the algebra generated by* $V_1(s), \dots, V_k(s)$ *is dense in* H.

Construct a map $\mathcal{S}(R^k) \to \mathcal{L}(H)$ *by*

$$U_f = \frac{1}{\sqrt{2\pi}^k} \int_{R^k} \hat{f}(\zeta_1, \dots, \zeta_k)\, e^{i[\zeta_1 T_1 + \dots + \zeta_k T_k]}\, d\zeta_1 \dots d\zeta_k .$$

Define $[\,,\,]$ *by*

$$[F,G] = (U_{f_1}\psi_1 + \dots + U_{f_n}\psi_n,\ U_{g_1}\psi_1 + \dots + U_{g_n}\psi_n) \qquad \textit{for} \quad (f_1, \dots, f_n)$$

and $G = (g_1, \dots, g_n)$ n*-vector valued functions with components* f_i, g_i *in* $\mathcal{S}(R^k)$. *Then* $[\,,\,]$ *has a representation*

$$[F,G] = \sum_{|\beta|,|\alpha| < M} \ \sum_{i,j=0}^{n} \int_{R^k}^{\infty} \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_i(x) \frac{\partial^{|\beta|}}{\partial x^\beta} \bar{g}_j(x)\, d\mu_{\alpha\beta ij}$$

where $\mu_{\alpha\beta ij}$ *are measures on* R^k *and where* $\frac{\partial^\alpha}{\partial x^\alpha}$ *stands for* $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}}$ *for each* k *tuple* $\alpha = (\alpha_1, \dots, \alpha_k)$ *of positive integers. Here* $|\alpha| = \alpha_1 + \dots + \alpha_k$.

Furthermore, $V_j(s)$ *acting on* H *corresponds to multiplication by* e^{isx_j} *on the space of* n*-vector valued functions with components in* $\mathcal{S}(R^k)$.

BIBLIOGRAPHY

[1] I. Colojoară and C. Foiaş, *Theory of generalized spectral operators* (New York, 1968).

[2] L. Hörmander, *Linear partial differential operators* (Berlin, 1963).

[3] G.E. Shilov, *Generalized functions and partial differential equations* (New York, 1968).

[4] R.F. Streater and A.S. Wightman, *PCT Spin and Statistics and all that* (New York, 1964).