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POSITIVE POLYNOMIALS IN SCALAR AND MATRIX VARIABLES, THE SPECTRAL THEOREM AND OPTIMIZATION

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Tibi Constantinescu, in memoriam

ABSTRACT. This is expanded from the original on behalf of Bill's classes.

We follow a stream of the history of positive matrices and positive functionals, as applied to algebraic sums of squares decompositions, with emphasis on the interaction between classical moment problems, function theory of one or several complex variables and modern operator theory. The second part of the survey focuses on recently discovered connections between real algebraic geometry and optimization as well as polynomials in matrix variables and some control theory problems. These new applications have prompted a series of recent studies devoted to the structure of positivity and convexity in a free $*$-algebra, the appropriate setting for analyzing inequalities on polynomials having matrix variables. We sketch some of these developments, add to them and comment on the rapidly growing literature.

1. Introduction

This is an essay, addressed to non-experts, on the structure of positive polynomials on semi-algebraic sets, various facets of the spectral theorem for Hilbert space operators, inequalities and sharp constraints for elements of a free $*-$algebra, and some recent applications of all of these to polynomial optimization and engineering. The circle of ideas exposed below is becoming increasingly popular but not known in detail outside the traditional groups of workers in functional analysis or real algebra who have developed parts of it. For instance, it is not yet clear how to teach and facilitate the access

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of beginners to this beautiful emerging field. The exposition of topics below may provide elementary ingredients for such a course.

The unifying concept behind all the apparently diverging topics mentioned above is the fact that universal positive functions (in appropriate rings) are sums of squares. Indeed, when we prove inequalities we essentially complete squares, and on the other hand when we do spectral analysis we decompose a symmetric or a hermitian form into a weighted (possibly continuous) sum or difference of squares. There are of course technical difficulties on each side, but they do not obscure the common root of algebraic versus analytical positivity.

We will encounter quite a few positivity criteria, expressed in terms of: matrices, kernels, forms, values of functions, parameters of continued fractions, asymptotic expansions and algebraic certificates. Dual to sums of squares and the main positive objects we study are the power moments of positive measures, rapidly decaying at infinity. These moments will be regarded as discrete data given by fixed coordinate frames in the correspondence between an algebra (of polynomials or operators) and its spectrum, with restrictions on its location. Both concepts of real spectrum (in algebraic geometry) and joint spectrum (in operator theory) are naturally connected in this way to moment problems. From the practitioner's point of view, moments represent observable/computable numerical manifestations of more complicated entities.

It is not a coincidence that the genius of Hilbert presides over all aspects of positivity we will touch. We owe him the origins and basic concepts related to: the spectral theorem, real algebra, algebraic geometry and mathematical logic. As ubiquitous as it is, a Hilbert space will show up unexpectedly and necessarily in the proofs of certain purely algebraic statements. On the other hand our limited survey does not aim at offering a comprehensive picture of Hilbert's much wider legacy.

Not unexpected, or, better later than never, the real algebraist's positivity and the classical analyst's positive definiteness have recently merged into a powerful framework; this is needed and shaped by several applied fields of mathematics. We will bring into our discussion one principal customer: control theory. The dominant development in linear systems engineering in the 1990's was matrix inequalities and many tricks and ad hoc techniques for making complicated matrix expressions into tame ones, indeed into the Linear Matrix Inequalities, LMIs, loved by all who can obtain them. Since matrices do not commute a large portion of the subject could be viewed as manipulation of polynomials and rational functions of non-commuting (free) variables, and so a beginning toward helpful mathematical theory would be a semi-algebraic geometry for free $*$-algebras, especially its implications for convexity. Such ventures sprung to life within the last five years and this article attempts to introduce, survey and fill in some gaps in this rapidly expanding area of noncommutative semi-algebraic geometry.

The table of contents offers an idea of the topics we touch in the survey and what we left outside. We are well aware that in a limited space while viewing a wide angle, as captives of our background and preferences, we have omitted key aspects. We apologize in advance for all our omissions in this territory, and for inaccuracies when stepping on outer domains; they are all non-intentional and reflect our limitations. Fortunately, the reader will have the choice of expanding and complementing our article with several recent excellent surveys and monographs (mentioned throughout the text and some recapitulated in the last section).

The authors thank the American Institute of Mathematics, Palo Alto, CA, for the unique opportunity (during a 2005 workshop) to interact with several key contributors to the recent theory of positive polynomials. They also thank the organizers of the "Real Algebra Fest, 2005", University of the Saskatchewan, Canada, for their interest and enthusiasm. The second author thanks the Real Algebra Group at the University of Konstanz, Germany, for offering him the possibility to expose and discuss the first sections of the material presented below.

We dedicate these pages to Tibi Constantinescu, old time friend and colleague, master of all aspects of matrix positivity.[1]

Contents

[1]**Advice to the reader.** *Although the ordering of the material below follows a logic derived from a general theoretical and historical perspective, most of the sections, and sometimes even subsections, can be read independently. We have tried to keep to a minimum the number of cross references and we have repeated definitions.*

For instance, the reader oriented towards optimization and engineering applications of the decompositions of polynomials into sums of squares can start with sections on the Positivstellensatz §5.4, §3.4, then turn to sections on global optimization and engineering §6 and sections §9, §10 on systems whose structure does not depend on the size of the system (is dimension free). After this we suggest a tour for the bigger picture through neighboring sections.

The operator theorist should find most of the paper straightforwrd to read with the exception of Section 5 which relates this very functional analytic topic to logic. Also an experienced operator theorist could start reading in many places, for example, in §8 or in §3 or in §6 or even at the beginning. We reassure everyone that the important general Positivstellensatz in §5.4 whose proof requires logic, is stated in a self-contained way.

Intriguing for the algebraist and logician might be Sections 2, 3, and 4 which contain analytic material (mostly derived from the spectral theorem and its many facets) which has cousins and even closer relatives in algebra.

In any case, all readers should be aware of the modular structure of the text, and try non-sequential orderings to access it.

2. The spectral theorem

The modern proof of the spectral theorem for self-adjoint or unitary operators uses commutative Banach algebra techniques, cf. for instance [D03]. This perspective departs from the older, and more constructive approach imposed by the original study of special classes of integral operators. In this direction, we reproduce below an early idea of F. Riesz [R13] for defining the spectral scale of a self-adjoint operator from a minimal set of simple observations, one of them being the structure of positive polynomials on a real interval.

2.1. **Self-adjoint operators.** Let H be a separable, complex Hilbert space and let $A \in \mathcal{L}(H)$ be a linear, continuous operator acting on H. We call A *self-adjoint* if $A = A^*$, that is $\langle Ax, x\rangle \in \mathbb{R}$ for all vectors $x \in H$. The continuity assumption implies the existence of bounds

$$m\|x\|^2 \leq \langle Ax, x\rangle \leq M\|x\|^2, \quad x \in H. \tag{2.1}$$

The operator A is called *non-negative*, denoted in short $A \geq 0$, if

$$\langle Ax, x\rangle \geq 0, \quad x \in H.$$

The operator A is *positive* if it is non-negative and $(\langle Ax, x\rangle = 0) \Rightarrow (x = 0)$.

We need a couple of basic observations, see §104 of [RN90]. The real algebraists should enjoy comparing these facts with the axioms of an order in an arbitrary ring.

a). *A bounded monotonic sequence of self-adjoint operators converges (in the strong operator topology) to a self-adjoint operator.*

Indeed, assume $0 \leq A_1 \leq A_2 \leq ... \leq I$ and take $B = A_{n+k} - A_n$ for some fixed values of $n, k \in \mathbb{N}$. Observe that $0 \leq B \leq I$, so Cauchy-Schwarz' inequality holds for the bilinear form $\langle Bx, y\rangle$. Use this to get: $\langle Bx, Bx\rangle^2 \leq \langle Bx, x\rangle\langle B^2x, Bx\rangle \leq \langle Bx, x\rangle\langle Bx, Bx\rangle$, from which

$$\|Bx\|^2 = \langle Bx, Bx\rangle \leq \langle Bx, x\rangle$$

Thus, for every vector $x \in H$:

$$\|A_{n+k}x - A_nx\|^2 \;\leq\; \langle A_{n+k}x, x\rangle - \langle A_nx, x\rangle.$$

Since the sequence $\langle A_nx, x\rangle$ is bounded and monotonic, it has a limit. Hence $\lim_n A_nx$ exists for every $x \in H$, which proves the statement.

b). *Every non-negative operator A admits a unique non-negative square root $\sqrt{A}$: $(\sqrt{A})^2 = A$.*

For the proof one can normalize A, so that $0 \leq A \leq I$ and use a convergent series decomposition for $\sqrt{x} = \sqrt{1-(1-x)}$, in conjunction with the above remark. See for details §104 of [RN90].

Conversely, if $T \in L(H)$, then $T^*T \geq 0$.

c). *Let A, B be two commuting non-negative (linear bounded) operators. Then AB is also non-negative.*

Note that, if $AB = BA$, the above proof implies $\sqrt{B}A = A\sqrt{B}$. For the proof we compute directly

$$\langle ABx, x\rangle = \langle A\sqrt{B}\sqrt{B}x, x\rangle =$$

$$\langle \sqrt{B}A\sqrt{B}x, x\rangle = \langle A\sqrt{B}x, \sqrt{B}x\rangle \geq 0.$$

With the above observations we can enhance the polynomial functional calculus of a self-adjoint operator. Let $\mathbb{C}[t], \mathbb{R}[t]$ denote the algebra of polynomials with complex, respectively real, coefficients in one variable and let $A = A^*$ be a self-adjoint operator with bounds (2.1). The expression $p(A)$ makes sense for every $p \in \mathbb{C}[t]$, and the polynomial functional calculus for A which is the map ϕ

$$p \overset{\phi}{\mapsto} p(A)$$

is obviously linear, multiplicative and unital (1 maps to I). Less obvious is the key fact that that ϕ is positivity preserving:

Proposition 2.1. *If the polynomial $p \in \mathbb{R}[t]$ satisfies $p(t) \geq 0$ for all t in $[m, M]$ and the self-adjoint operator A satisfies $mI \leq A \leq MI$, then $p(A) \geq 0$.*

Proof. A decomposition of the real polynomial p into irreducible, real factors yields:

$$p(t) = c\prod_i (t-\alpha_i) \prod_j (\beta_j - t) \prod_k [(t-\gamma_k)^2 + \delta_k^2],$$

with $c > 0$, $\alpha_i \le m \le M \le \beta_j$ and $\gamma_k \in \mathbb{R}, \delta_k \in \mathbb{R}$. According to the observation c) above, we find $p(A) \ge 0$. ■

The proposition immediately implies

Corollary 2.2. *The homomorphism ϕ on $\mathbb{C}[t]$ extends to $C[m, M]$ and beyond. Moreover,*

$$\|p(A)\| \le \sup_{[m,M]} |p| =: \|p\|_\infty.$$

Proof. The inequality follows because $\sup_{[m,M]} |p| \pm p$ is a polynomial non-negative on $[m, M]$, so $\|p\|_\infty I \ge \pm p(A)$ which gives the required inequality. Thus ϕ is sup norm continuous and extends by continuity to the completion of the polynomials, which is of course the algebra $C[m, M]$ of the continuous functions.

The Spectral Theorem immediately follows.

Theorem 2.3. *If the self adjoint bounded operator A on H has a cyclic vector ξ, then there is a positive Borel measure μ on $[m, M]$ and a unitary operator $U : H \mapsto L^2(\mu)$ identifying H with $L^2(\mu)$ such that*

$$UAU^* = M_x.$$

Here for any g in L^∞ the multiplication operator M_g is defined by $M_g f = gf$ on all $f \in L^2(\mu)$.

The vector ξ *cyclic* means

$$span\ \{A^k\xi : \ k = 0, 1, 2. \cdots\} = \{p(A)\xi : \ p \ a \ polynomial\ \}$$

is dense in H.

Proof Define a linear functional $L : C([m, M]) \mapsto \mathbb{C}$ by

$$L(f) := \langle f(A)\xi, \xi\rangle \qquad for\ all\ f \in C([m, M]).$$

The Representation Theorem (see Proposition 3.2 for more detail) for such L says there is a Borel measure μ such that

$$L(f) = \int_{[m,M]} f d\mu;$$

moreover, μ is a positive measure because if $f \geq 0$ on $[m, M]$, then $L(f) \geq 0$. A critical feature is

$$\int p\overline{q}d\mu = \langle p(A)\xi, q(A)\xi\rangle \tag{2.2}$$

which holds, since $= L(p\overline{q}) = \langle p(A)\overline{q}(A)\xi, \xi\rangle$. We have built our representing space (using a formula which haunts the rest of this paper) and now we identify H with this space.

Define U by $Up(A)\xi = p$ which specifies it on a dense set (by the cyclic assumption) provided $Up_1(A)\xi = Up_2(A)\xi$ implies $e(A)\xi := p_1(A)\xi - p_2(A)\xi = 0$; in other words, $0 = \langle e(A)\xi, q(A)\xi\rangle$ for all polynomials q. Thus $0 = \int e\overline{q}d\mu$, so $e = 0$ a.e. wrt μ. Now to properties of U:

(1) U is isometric. (That is what (2.2) says.) Thus U extends to H and has closed range.
(2) The range of U is dense since it contains the polynomials.
(3) $UAp(A)\xi = xp(x) = xUp(A)\xi$ for all polynomials p. By the density imposed by cyclicity for any v in H we have

$$UAv = M_x Uv.$$

Note the constrction gives $U\xi = 1$. ■

2.2. **A bigger functional calculus and spectral measures.** Our next aim is to consider a bounded, increasing sequence p_n of real polynomial functions on the interval $[m, M]$ and define, according to observation a):

$$f(A)x = \lim p_n(A)x, \quad x \in H,$$

where f is a point-wise limit of p_n. A standard argument shows that, if q_n is another sequence of polynomials, monotonically converging on $[m, M]$ to f, then

$$\lim q_n(A)x = \lim_n p_n(A)x, \quad x \in H.$$

See for details §106 of [RN90]. The new calculus $f \mapsto f(A)$ remains linear and multiplicative.

In particular, we can apply the above definition to the step functions

$$\chi_s(t) = \begin{cases} 1, & t \le s, \\ 0, & t > s. \end{cases}$$

This yields a monotonic, operator valued function

$$F_A(s) = \chi_s(A),$$

with the additional properties $F_A(s) = F_A(s)^* = F_A(s)^2$ and

$$F_A(s) = \begin{cases} 0, & s < m, \\ I, & s \ge M. \end{cases}$$

With the aid of this *spectral scale* one can interpret the functional calculus as an operator valued Riemann-Stieltjes integral

$$f(A) = \int_m^M f(t) dF_A(t).$$

The *spectral measure* E_A of A is the operator valued measure associated to the monotonic function F_A, that is, after extending the integral to Borel sets σ,

$$E_A(\sigma) = \int_{\sigma \cap [m,M]} dF_A(t).$$

Thus $E_A(\sigma)$ is a family of mutually commuting orthogonal projections, subject to the multiplicativity constraint

$$E_A(\sigma \cap \tau) = E_A(\sigma) E_A(\tau).$$

As a matter of notation, we have then for every bounded, Borel measurable function f:

$$f(A) = \int_m^M f(t) E_A(dt). \tag{2.3}$$

This is a form of the Spectral Theorem which does not assume cyclicity.

A good exercise for the reader is to identify the above objects in the case of a finite dimensional Hilbert space H and a self-adjoint linear transformation A acting on it. A typical infinite dimensional example will be discussed later in connection with the moment problem.

2.3. **Unitary operators.** The spectral theorem for a unitary transformation $U \in L(H),\ U^*U = UU^* = I$, can be derived in a very similar manner.

The needed structure of positive polynomials is contained in the following classical result.

Lemma 2.4 (Riesz-Fejér). *A non-negative trigonometric polynomial is the modulus square of a trigonometric polynomial.*

Proof. Let $p(e^{i\theta}) = \sum_{-d}^{d} c_j e^{ij\theta}$ and assume that $p(e^{i\theta}) \geq 0, \quad \theta \in [0, 2\pi]$. Then necessarily $c_{-j} = \overline{c_j}$. By passing to complex coordinates, the rational function $p(z) = \sum_{-d}^{d} c_j z^j$ must be identical to $\overline{p(1/\overline{z})}$. That is its zeros and poles are symmetrical (in the sense of Schwarz) with respect to the unit circle.

Write $z^d p(z) = q(z)$, so that q is a polynomial of degree $2d$. One finds, in view of the mentioned symmetry:

$$q(z) = cz^{\nu} \prod_j (z - \lambda_j)^2 \prod_k (z - \mu_k)(z - 1/\overline{\mu_k}),$$

where $c \neq 0$ is a constant, $|\lambda_j| = 1$ and $0 < |\mu_k| < 1$.

For $z = e^{i\theta}$ we obtain

$$p(e^{i\theta}) = |p(e^{i\theta})| = |q(e^{i\theta}| =$$

$$|c| \prod_j |e^{i\theta} - \lambda_j|^2 \prod_k \frac{|e^{i\theta} - \mu_k|^2}{|\mu_k|^2}.$$

■

Returning to the unitary operator U we infer, for $p \in \mathbb{C}[z]$,

$$\Re p(e^{i\theta}) \geq 0 \quad \Rightarrow \Re p(U) \geq 0.$$

Indeed, according to the above Lemma, $\Re p(e^{i\theta}) = |q(e^{i\theta})|^2$, whence

$$\Re p(U) = q(U)^* q(U) \geq 0.$$

Then, exactly as in the preceding section one constructs the spectral scale and spectral measure of U.

For an operator T we denote its "real part" and "imaginary part" by $\Re T = (T + T^*)/2$ and $\Im T = (T - T^*)/2i$.

The reader will find other elementary facts (à la Riesz-Fejér's Lemma) about the decompositions of non-negative polynomials into sums of squares in the second volume of Polya and Szegö's problem book [PS25]. This particular collection of observations about positive polynomials reflects, from the mathematical analyst point of view, the importance of the subject in the first two decades of the XX-th century.

2.4. **Riesz-Herglotz formula.** The practitioners of spectral analysis know that the strength and beauty of the spectral theorem lies in the effective dictionary it establishes between matrices, measures and analytic functions. In the particular case of unitary operators, these correspondences also go back to F. Riesz. The classical Riesz-Herglotz formula is incorporated below in a more general statement. To keep the spirit of positivity of the last sections, we are interested below in the *additive* (rather than multiplicative) structure of polynomials (or more general functions) satisfying Riesz-Fejér's condition:

$$\Re p(z) \geq 0, \quad |z| < 1.$$

We denote by $\mathbb{D}$ the unit disk in the complex plane. Given a set X by a *positive semi-definite kernel* we mean a function $K : X \times X \longrightarrow \mathbb{C}$ satisfying

$$\sum_{i,j=1}^{N} K(x_i, x_j) c_i \overline{c_j} \geq 0,$$

for every finite selection of points $x_1, ..., x_N \in X$ and complex scalars $c_1, ..., c_N$.

Theorem 2.5. *Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be an analytic function. The following statements are equivalent:*

a). $\Re f(z) \geq 0, \quad z \in \mathbb{D}$,

b). (Riesz-Herglotz formula). There exists a positive Borel measure μ on $[-\pi, \pi]$ and a real constant C, such that:

$$f(z) = iC + \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad z \in \mathbb{D},$$

c). The kernel $K_f : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$,

$$K_f(z,w) = \frac{f(z) + \overline{f(w)}}{1 - z\overline{w}}, \quad z, w \in \mathbb{D},$$

is positive semi-definite,

d). There exists a unitary operator $U \in \mathcal{L}(H)$*, a vector* $\xi \in H$ *and a constant* $a \in \mathbb{C}$, $\Re a \geq 0$*, such that:*

$$f(z) = a + z\langle (U - z)^{-1}\xi, \xi\rangle, \quad z \in \mathbb{D}.$$

Proof. We merely sketch the main ideas in the proof. The reader can consult for details the monograph [AM02].

a) $\Rightarrow$ *b)*. Let $r < 1$. As a consequence of Cauchy's formula:

$$f(z) = i\Im f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{re^{it} + z}{re^{it} - z}\Re f(re^{it})dt, \quad |z| < r.$$

Since the positive measures $\frac{1}{2\pi}\Re f(re^{it})dt$ have constant mass on $[-\pi, \pi]$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Re f(re^{it})dt = \Re f(0), \quad r < 1,$$

they form a weak$-*$ relatively compact family (in the space of finite measure). Any weak$-*$ limit will satisfy the identity in b) (hence all limit points coincide).

b) $\Rightarrow$ *c)*. A direct computation yields:

$$K_f(z,w) = \int_{-\pi}^{\pi} \frac{2}{(e^{it} - z)(e^{-it} - \overline{w})}d\mu(t), \quad z, w \in \mathbb{D}. \tag{2.4}$$

Since for a fixed value of t, the integrand is positive semi-definite, and we average over a positive measure, the whole kernel will turn out to be positive semi-definite.

c) $\Rightarrow$ *a)*. Follows by evaluating K_f on the diagonal:

$$2\Re f(z) = (1 - |z|^2)K_f(z,z) \geq 0.$$

b) $\Rightarrow$ *d)*. Let $H = L^2(\mu)$ and $Uf(t) = e^{it}f(t)$. Then U is a unitary operator, and the constant function $\xi = \sqrt{2}$ yields the representation d).

d) ⇒ *b*). In view of the spectral theorem, we can evaluate the spectral measure E_U on the vector ξ and obtain a positive measure μ satisfying:

$$f(z) = a + z\langle (U-z)^{-1}\xi, \xi\rangle = a + z\int_{-\pi}^{\pi} \frac{d\mu(t)}{e^{it}-z} =$$

$$a + \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) - \frac{1}{2}\int_{-\pi}^{\pi} d\mu(t), \quad z \in \mathbb{D}.$$

By identifying the constants we obtain, up to the factor 2, conclusion b). ■

The theorem above has far reaching consequences in quite divergent directions: function theory, operator theory and control theory of linear systems, see for instance [AM02, FF90, M03, RR97]. We confine ourselves to describe only a generic consequence.

First, we recall that, exactly as in the case of finite matrices, a positive semi-definite kernel can be written as a sum of squares. Indeed, if $K : X \times X \longrightarrow \mathbb{C}$ is positive semi-definite, one can define a sesqui-linear form on the vector space $\oplus_{x\in X}\mathbb{C}$, with basis $e(x), x \in X$, by

$$\|\sum_i c_i e(x_i)\|^2 = \sum_{i,j=1}^{N} K(x_i, x_j) c_i \overline{c_j}.$$

This is a positive semi-definite inner product. The associated separated (i.e. Hausdorff) Hilbert space completion H carries the classes of the vectors $[e(x)] \in H$. They factor K into a sum of squares:

$$K(x,y) = \langle [e(x)], [e(y)]\rangle = \sum_k \langle [e(x)], f_k\rangle\langle f_k, [e(y)]\rangle,$$

where (f_k) is any orthonormal basis of H. For details, see for instance the Appendix to [RN90].

The following result represents the quintessential bounded analytic interpolation theorem.

Theorem 2.6 (Nevanlinna-Pick). *Let $\{a_i \in \mathbb{D};\ i \in I\}$ be a set of points in the unit disk, and let $\{c_i \in \mathbb{C};\ \Re c_i \geq 0,\ i \in I\}$ be a collection of points in the right half-plane, indexed over the same set.*

There exists an analytic function f in the unit disk, with $\Re f(z) \geq 0,\ |z| < 1$, and $f(a_i) = c_i,\ i \in I$, if and only if the kernel

$$\frac{c_i + \overline{c_j}}{1 - a_i\overline{a_j}}, \quad i, j \in I,$$

is positive semi-definite.

Proof. Point c) in the preceding Theorem shows that the condition is necessary.

A Moebius transform in the range ($f \mapsto g = (f-1)/(f+1)$) will change the statement into:

$$g : \mathbb{D} \longrightarrow \mathbb{D}, \quad g(a_i) = d_i,$$

if and only if the kernel

$$\frac{1 - d_i\overline{d_j}}{1 - a_i\overline{a_j}}, \quad i, j \in I,$$

is positive semi-definite.

To prove that the condition in the statement is also sufficient, assume that the latter kernel is positive semi-definite. As before, factor it (into a sum of squares):

$$\frac{1 - d_i\overline{d_j}}{1 - a_i\overline{a_j}} = \langle h(i), h(j)\rangle, \quad i, j \in I,$$

where $h : I \longrightarrow H$ is a function with values in an auxiliary Hilbert space H.

Then

$$1 + \langle a_i h(i), a_j h(j)\rangle = d_i\overline{d_j} + \langle h(i), h(j)\rangle, \quad i, j \in I.$$

The preceding identity can be interpreted as an equality between scalar products in $\mathbb{C} \oplus H$:

$$\langle \begin{pmatrix} 1 \\ a_i h(i) \end{pmatrix}, \begin{pmatrix} 1 \\ a_j h(ij) \end{pmatrix} \rangle = \langle \begin{pmatrix} d_i \\ h(i) \end{pmatrix}, \begin{pmatrix} d_j \\ h(j) \end{pmatrix} \rangle, \quad i, j \in I.$$

Let $H_1 \subset \mathbb{C} \oplus H$ be the linear span of the vectors $(1, a_i h(i))^T, \quad i \in I$. The map

$$V \begin{pmatrix} 1 \\ a_i h(i) \end{pmatrix} = \begin{pmatrix} d_i \\ h(i) \end{pmatrix}$$

extends then by linearity to an isometric transformation $V : H_1 \longrightarrow H$. Since the linear isometry V can be extended (for instance by zero on the orthogonal complement of H_1) to a contractive linear operator $T : \mathbb{C} \oplus H \longrightarrow \mathbb{C} \oplus H$, we obtain a block matrix decomposition of T satisfying:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} 1 \\ a_i h(i) \end{pmatrix} = \begin{pmatrix} d_i \\ h(i) \end{pmatrix}.$$

Since $\|D\| \leq 1$, the operator $I - zD$ is invertible for all $z \in \mathbb{D}$. From the above equations we find, after identifying A with a scalar:

$$h(i) = (I - a_i D)^{-1} C1, \quad d_i = A + a_i B h(i).$$

We define the analytic function

$$g(z) = A + zB(I - zD)^{-1} C1, \quad |z| < 1.$$

It satisfies, as requested: $g(a_i) = d_i, \ i \in I$.

By reversing the above reasoning we infer, with $h(z) = (I - zD)^{-1} C1 \in H$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} 1 \\ zh(z) \end{pmatrix} = \begin{pmatrix} g(z) \\ h(z) \end{pmatrix}.$$

Since T is a contraction,

$$\|g(z)\|^2 + \|h(z)\|^2 \leq 1 + \|zh(z)\|^2 \leq 1 + \|h(z)\|^2, \quad |z| < 1,$$

whence

$$|g(z)| \leq 1, \quad |z| < 1.$$

■

The above proof contains the germ of what experts in control theory call "realization theory". For the present survey it is illustrative as a constructive link between matrices and analytic functions with bounds; it will also be useful as a model to follow in more general, non-commutative settings.

A great deal of research was done in the last two decades on analogs of Riesz-Herglotz type formulas in several complex variables. As expected, when generalizing to $\mathbb{C}^n$, there are complications and surprises on the road. See for instance [AM02, BT98, CW99, EP02] and in several non-commuting variables [BGM05, K05]. We will return to some of these topics from the perspective of positive polynomials and moment sequences.

2.5. **von Neumann's inequality.** We have just seen that the heart of the spectral theorem for self-adjoint or unitary operators was the positivity of the polynomial functional calculus. A surprisingly general inequality, of the same type, applicable to an arbitrary bounded operator, was discovered by von Neumann [vN2].

Theorem 2.7. *Let $T \in \mathcal{L}(H), \|T\| \leq 1$, be a contractive operator. If a polynomial $p \in \mathbb{C}[z]$ satisfies $\Re p(z) \geq 0,\ z \in \mathbb{D}$, then $\Re p(T) \geq 0$.*

Proof. According to Riesz-Herglotz formula we can write

$$p(z) = iC + \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),\ |z| < 1,$$

where $C \in \mathbb{R}$ and μ is a positive measure.

Fix $r < 1$, close to 1, and evaluate the above representation at $z = rT$:

$$p(rT) = iC + \int_{-\pi}^{\pi} (e^{it} + rT)(e^{it} - rT)^{-1} d\mu(t).$$

Therefore

$$p(rT) + p(rT)^* =$$

$$\int_{-\pi}^{\pi} (e^{it} - rT)^{-1}[(e^{it} + rT)(e^{-it} - rT^*) + (e^{it} - rT)(e^{-it} + rT^*)](e^{-it} - rT^*)^{-1} d\mu(t) =$$

$$2 \int_{-\pi}^{\pi} (e^{it} - rT)^{-1}[I - r^2 TT^*](e^{-it} - rT^*)^{-1} d\mu(t) \geq 0.$$

Letting $r \to 1$ we find $\Re p(T) \geq 0$. ■

A Moebius transform argument, as in the proof of Nevanlinna-Pick Theorem, yields the equivalent statement (for a contractive linear operator T):

$$(|p(z)| \leq 1,\quad |z| < 1)\ \Rightarrow\ \|p(T)\| \leq 1.$$

Von Neumann's original proof relied on the continued fraction structure of the analytic functions from the disk to the disk. The recursive construction of the continued fraction goes back to Schur [S18] and can be explained in a few lines.

Schur's algorithm. Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function. Then, in view of Schwarz Lemma, there exists an analytic function $f_1 : \mathbb{D} \longrightarrow \mathbb{D}$ with the property:

$$\frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)} = z f_1(z),$$

or equivalently, writing $s_0 = f(0)$:

$$f(z) = \frac{s_0 + z f_1(z)}{1 + \overline{s_0} z f_1(z)}.$$

In its turn,

$$f_1(z) = \frac{s_1 + z f_2(z)}{1 + \overline{s_1} z f_2(z)},$$

with an analytic $f_2 : \mathbb{D} \longrightarrow \mathbb{D}$, and so on.

This algorithm terminates after finitely many iterations for finite Blashcke products

$$f(z) = \prod_{k=1}^{N} \frac{z - \lambda_k}{1 - \overline{\lambda_k} z}, \quad |\lambda_k| < 1.$$

Its importance lies in the fact that the finite section of Schur parameters $(s_0, s_1, ..., s_n)$ depends via universal expressions on the first section (same number) of Taylor coefficients of f at $z = 0$. Thus, the conditions

$$|s_0(c_0)| \leq 1, \ |s_1(c_0, c_1)| \leq 1, \ldots$$

characterize which power series

$$c_0 + c_1 z + c_2 z^2 + ...,$$

are associated to analytic functions from the disk to the disk. For details and a variety of applications, see [Constantinescu96, FF90, RR97].

One notable application is to solve the classical Carathéodory-Fejér interpolation problem, a close relative of the Nevanlinna-Pick problem we presented earlier. Here one specifies complex numbers $c_0, \cdots, c_m$ and seeks $f : \mathbb{D} \to \mathbb{D}$ analytic for which

$$\frac{1}{j!} \frac{d^j f}{dz^j}(0) = c_j, \qquad j = 0, \cdots, m.$$

The Schur Algorithm constructs such a function and in the same time gives a simple criterion when the solution exists. Alternatively, a special type of matrix $(c_{n-m})_{n,m=0}^{m}$, with zero entries under the diagonal ($c_j = 0, \ j < 0$), called a Toeplitz matrix, based on $c_0, \cdots, c_m$ is a contraction if and only if a solution to the Carathéodory-Fejér problem exists. A version of this fact in the right half plane (rather than the disk) is proved in Theorem 3.3.

As another application, we can derive (also following Schur) an effective criterion for deciding whether a polynomial has all roots inside the unit disk. Let

$$p(z) = c_d z^d + c_{d-1} z^{d-1} + ... + c_0 \in \mathbb{C}[z],$$

and define

$$p^\flat(z) = z^d \overline{p(1/\overline{z})} = \overline{c_0} z^d + \overline{c_1} z^{d-1} + ... + \overline{c_d}.$$

It is clear that

$$|p(e^{it})| = |p^\flat(e^{it})|, \quad t \in [-\pi, \pi],$$

and that the roots of $p^\flat$ are symmetric with respect to the unit circle to the roots of p. Therefore, p has all roots contained in the open unit disk if and only if $\frac{p}{p^\flat}$ is an analytic function from the disk to the disk, that is, if and only if the kernel

$$\frac{p^\flat(z)\overline{p^\flat(w)} - p(z)\overline{p(w)}}{1 - z\overline{w}}, \quad z, w \in \mathbb{D},$$

is positive definite. As a matter of fact $\frac{p}{p^\flat}$ is a finite Blashcke product, and Schur's algorithm terminates in this case after finitely many iterations.

In general, regarded as a Hermitian form, evaluated to the variables $Z_i = z^i, 0 \leq i \leq d$, the signature of the above kernel (that is the number of zeros, negative and positive squares in its canonical decomposition) counts how many roots the polynomial p has inside the disk, and on its boundary. For many more details see the beautiful survey [KN81].

3. Moment problems

In this section we return to Hilbert space and the spectral theorem, by unifying the analysis and algebra concepts we have discussed in the previous sections. This is done in the context of power moment problems, one of the oldest and still lively sources of questions and inspiration in mathematical analysis.

As before, $x = (x_1, ..., x_g)$ stands for the coordinates in $\mathbb{R}^g$, and, at the same time, for a tuple of commuting indeterminates. We adopt the multi-index notation $x^\alpha = x_1^{\alpha_1}...x_g^{\alpha_g}$, $\alpha \in \mathbb{N}^g$. Let μ be a positive, rapidly decreasing measure on $\mathbb{R}^g$. The *moments* of μ are the real numbers:

$$a_\alpha = \int x^\alpha d\mu(x), \quad \alpha \in \mathbb{N}^g.$$

For its theoretical importance and wide range of applications, the correspondence

$$\{\mu; \text{ positive measure}\} \longrightarrow \{(a_\alpha); \text{ moment sequence}\}$$

can be put on an equal level with the Fourier-Laplace, Radon or wavelet transforms. It is the positivity of the original measure which makes the analysis of this category of moment problems interesting and non-trivial, and appropriate for our discussion. For general aspects and applications of moment problems (not treated below) the reader can consult the monographs [Akh65, BCR98, FF90, ST43] and the excellent survey [F83]. The old article of Marcel Riesz [MR23] remains unsurpassed for the classical aspects of the one variable theory.

Given a multi-sequence of real numbers $(a_\alpha)_{\alpha \in \mathbb{N}^g}$ a linear functional representing the potential integral of polynomials can be defined as:

$$L : \mathbb{R}[x] \longrightarrow \mathbb{R}, \quad L(x^\alpha) = a_\alpha, \ \alpha \in \mathbb{N}^g,$$

and vice-versa. When necessary we will complexify L to a complex linear functional on $\mathbb{C}[x]$.

If $(a_\alpha)_{\alpha \in \mathbb{N}^g}$ are the moments of a positive measure, then for a polynomial $p \in \mathbb{R}[x]$ we have

$$L(p^2) = \int_{\mathbb{R}^g} p^2 d\mu \geq 0.$$

Moreover, in the above positivity there is more structure: we can define on $\mathbb{C}[x]$ a pre-Hilbert space bracket by:

$$\langle p, q \rangle = L(p\overline{q}), \quad p, q \in \mathbb{C}[x].$$

The inner product is positive semi-definite, hence the Cauchy-Schwarz inequality holds:

$$|\langle p, q \rangle|^2 \leq \|p\|^2 \|q\|^2.$$

Thus, the set of null-vectors $N = \{p \in \mathbb{C}[x];\ \|p\| = 0\}$ is a linear subspace, invariant under the multiplication by any polynomial. Let H be the Hilbert space completion of $\mathbb{C}[x]/N$ with respect to the induced Hermitian form. Let $\mathcal{D} = \mathbb{C}[x]/N$ be the image of the polynomial algebra in H. It is a dense linear subspace, carrying the multiplication operators:

$$M_{x_i} : \mathcal{D} \longrightarrow \mathcal{D}, \quad M_{x_i} p = x_i p.$$

Note that these are well defined, symmetric linear operators:

$$\langle M_{x_i} p, q\rangle = L(x_i p \overline{q}) = \langle p, M_{x_i} q\rangle, \quad p, q \in \mathcal{D},$$

and they commute

$$M_{x_i} M_{x_j} = M_{x_j} M_{x_i}.$$

Finally the (constant function) vector $\xi = 1$ is cyclic, in the sense that $\mathcal{D}$ is the linear span of repeated actions of $M_{x_1}, ..., M_{x_g}$ on ξ:

$$\mathcal{D} = \bigvee_{\alpha \in \mathbb{N}^g} M_{x_1}^{\alpha_1} ... M_{x_g}^{\alpha_g} \xi.$$

We collect these observations into a single statement.

Proposition 3.1. *There is a bijective correspondence between all linear functionals*

$$L \in \mathbb{R}[x]', \quad L|_{\Sigma^2 \mathbb{R}[x]} \geq 0,$$

and the pairs (M, ξ) of g-tuples $M = (M_1, ..., M_g)$ of commuting, symmetric linear operators with a cyclic vector ξ (acting on a separable Hilbert space). The correspondence is given by the relation

$$L(p) = \langle p(M)\xi, \xi\rangle, \quad p \in \mathbb{R}[x].$$

Above the word commuting has to be taken with caution: implicitly it is understood that we define the span $\mathcal{D}$ as before, and remark that every M_i leaves $\mathcal{D}$ invariant. Then M_i commutes with M_j as endomorphisms of $\mathcal{D}$.

Having a positive measure μ represent the functional L adds in general new constraints in this dictionary.

Let $\mathcal{P}_+(K)$ be the set of all polynomials which are non-negative on the set $K \subset \mathbb{R}^g$ and note that this is a convex cone.

Proposition 3.2. *A linear functional $L \in \mathbb{R}[x]'$ is representable by a positive measure μ:*

$$L(p) = \int p d\mu, \quad p \in \mathbb{R}[x]$$

if and only if $L|_{\mathcal{P}_+(\mathbb{R}^g)} \geq 0$.

Although this observation (in several variables) is attributed to Haviland, see [Akh65], it is implicitly contained in Marcel Riesz article [MR23]. Again we see exactly *the gap*

$$\Sigma^2\mathbb{R}[x] \subset \mathcal{P}_+(\mathbb{R}^g),$$

which we must understand in order to characterize the moments of positive measures (as already outlined in Minkowski's and Hilbert's early works).

Proof. If the functional L is represented by a positive measure, then it is obviously non-negative on all non-negative polynomials.

To prove the converse, assume that $L|_{\mathcal{P}_+(\mathbb{R}^g)} \geq 0$. Let $C_{pBd}(\mathbb{R}^g)$ be the space of continuous functions f having a polynomial bound at infinity:

$$|f(x)| \leq C(1 + |x|)^N,$$

with the constants $C, N > 0$ depending on f. We will extend L, following M. Riesz [MR23], to a non-negative functional on $C_{pBd}(\mathbb{R}^g)$.

This extension process, parallel and arguably prior to the Hahn-Banach Theorem, works as follows. Assume that

$$\hat{L} : V \longrightarrow \mathbb{R}$$

is a positive extension of L to a vector subspace $V \subset C_{pBd}(\mathbb{R}^g)$. That is:

$$(h \in V,\ h \geq 0) \ \Rightarrow (\hat{L}(h) \geq 0).$$

Remark that L is defined on all polynomial functions. Assume V is not the whole space and choose a non-zero function $f \in C_{pBd}(\mathbb{R}^g) \setminus V$. Since f has polynomial growth, there are elements $h_1, h_2 \in V$ satisfying

$$h_1 \leq f \leq h_2.$$

By the positivity of $\hat{L}$, we see $\hat{L}h_1 \leq \hat{L}f \leq \hat{L}h_2$, that is

$$\sup_{h_1 \leq f} \hat{L}(h_1) \leq \inf_{f \leq h_2} \hat{L}(h_2).$$

Choose any real number c between these limits and define

$$L'(h + \lambda f) = \hat{L}(h) + \lambda c, \quad h \in V, \ \lambda \in \mathbb{R}.$$

This will be a positive extension of L to the larger space $V \oplus \mathbb{R}f$.

By a standard application of Zorn's Lemma, we find a positive extension of L to the whole space. Finally, F. Riesz Representation Theorem provides a positive measure μ on $\mathbb{R}^g$, such that $L(p) = \int p d\mu, \quad p \in \mathbb{R}[x]$. ■

Next we focus on a few particular contexts (either low dimensions, or special supporting sets for the measure) where the structure of the positive functionals and tuples of operators appearing in our dictionary can be further understood.

3.1. **The trigonometric moment problem.** We specialize to dimension $n = 2$ and to measures supported on the unit circle (torus) $\mathbb{T} = \{z \in \mathbb{C}; \quad |z| = 1\}$. The group structure of $\mathbb{T}$ identifies our moment problem to the Fourier transform. It is convenient in this case to work with complex coordinates $z = x + iy \in \mathbb{C} = \mathbb{R}^2$, and complex valued polynomials. In general, we denote by $\Sigma_h^2 \mathbb{C}[x]$ the sums of moduli squares (i.e. $|q|^2$) of complex coefficient polynomials.

The ring of regular functions on the torus is

$$A = \mathbb{C}[z, \overline{z}]/(1 - z\overline{z}) = \mathbb{C}[z] \oplus \overline{z}\mathbb{C}[\overline{z}],$$

where $(1 - z\overline{z})$ denotes the ideal generated by $1 - z\overline{z}$. A non-negative linear functional L on $\Sigma_h^2 A$ necessarily satisfies

$$L(\overline{f}) = \overline{L(f)}, \quad f \in A.$$

Hence L is determined by the complex moments $L(z^n), \ n \geq 0$. The following result gives a satisfactory solution to the trigonometric moment problem on the one dimensional torus.

Theorem 3.3. *Let $(c_n)_{n=-\infty}^{\infty}$ be a sequence of complex numbers subject to the conditions $c_0 \geq 0, \quad c_{-n} = \overline{c_n}, \ n \geq 0$. The following assertions are equivalent:*

a). There exists a unique positive measure μ on $\mathbb{T}$, such that:

$$c_n = \int_{\mathbb{T}} z^n d\mu(z), \quad n \geq 0;$$

b). The Toeplitz matrix $(c_{n-m})_{n,m=0}^{\infty}$ is positive semi-definite;

c). There exists an analytic function $F : \mathbb{D} \longrightarrow \mathbb{C}, \ \Re F \geq 0$, such that

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k, \quad |z| < 1;$$

d). There exists a unitary operator $U \in L(H)$ and a vector $\xi \in H$ cyclic for the pair (U, U^*), such that

$$\langle U^n \xi, \xi \rangle = c_n, \quad n \geq 0.$$

Proof. Let $L : \mathbb{C}[z, \overline{z}]/(1 - z\overline{z}) \longrightarrow \mathbb{C}$ be the linear functional defined by

$$L(z^n) = c_n, \ n \geq 0.$$

Condition b) is equivalent to

$$L(|p|^2) \geq 0, \quad p \in \mathbb{C}[z, \overline{z}]/(1 - z\overline{z}).$$

Indeed, assume that $p(z) = \sum_{j=0}^{g} \alpha_j z^j$. Then, since $\overline{z}z = 1$,

$$|p(z)|^2 = \sum_{j,k=0}^{g} \alpha_j \overline{\alpha_k} z^{j-k},$$

whence

$$L(|p|^2) = \sum_{j,k=0}^{g} \alpha_j \overline{\alpha_k} c_{j-k}.$$

Thus $a) \Rightarrow b)$ trivially. In view of the Riesz-Fejér Lemma, the functional L is non-negative on all non-negative polynomial functions on the torus. Hence, in view of Proposition 3.2 it is represented by a positive measure. The uniqueness is assured by the compactness of $\mathbb{T}$ and Stone-Weierstrass Theorem (trigonometric polynomials are uniformly dense in the space of continuous functions on $\mathbb{T}$). The rest follows from Theorem 2.5. ■

Notable in the above Theorem is the fact that the main objects are in bijective, and constructive, correspondence established essentially by Riesz-Herglotz formula. Fine properties of the measure μ can be transferred in this way into restrictions imposed on the generating function F or the unitary operator U.

For applications and variations of the above result (for instance a matrix valued analog of it) the reader can consult [AM02, Akh65, FF90, RR97].

3.2. **Hamburger's moment problem.** The passage from the torus to the real line reveals some unexpected turns, due to the non-compactness of the line. One may argue that the correct analog on the line would be the continuous Fourier transform. Indeed, we only recall that Bochner's Theorem provides an elegant characterization of the Fourier transforms of positive measures.

Instead, we remain consistent and study polynomial functions and positive measures acting on them. Specifically, consider an $\mathbb{R}$-linear functional

$$L : \mathbb{R}[x] \longrightarrow \mathbb{R}, \; L|_{\Sigma^2\mathbb{R}[x]} \geq 0.$$

By denoting

$$c_k = L(x^k), \quad k \geq 0,$$

the condition $L|_{\Sigma^2\mathbb{R}[x]}$ is equivalent to the positive semi-definiteness of the Hankel matrix

$$(c_{k+l})_{k,l=0}^{\infty} \geq 0,$$

since

$$0 \leq \sum_{k,l} f_k c_{k+l} f_l = \sum_{k,l} L(f_k x^k x^l f_l) = L(\sum_k f_k x^k \sum_l x^l f_l) = L(f(x)^2).$$

Next use that every non-negative polynomial on the line is a sum of squares of polynomials, to invoke Proposition 3.2 for the proof of the following classical fact.

Theorem 3.4 (Hamburger)**.** *Let* $(c_k)_{k=0}^{\infty}$ *be a sequence of real numbers. There exists a rapidly decaying, positive measure* μ *on the real line, such that*

$$c_k = \int_{-\infty}^{\infty} x^k d\mu(x), \ \ k \geq 0,$$

if and only if the matrix $(c_{k+l})_{k,l=0}^{\infty}$ *is positive semi-definite.*

Now we sketch *a second proof of Hamburger Theorem*, based on the Hilbert space construction we have outlined in the previous section. Namely, start with the positive semi-definite matrix $(c_{k+l})_{k,l=0}^{\infty}$ and construct a Hilbert space (Hausdorff) completion H of $\mathbb{C}[x]$, satisfying

$$\langle x^k, x^l \rangle = c_{k+l}, \quad k, l \geq 0.$$

Let $\mathcal{D}$ denote as before the image of the algebra of polynomials in H; the image is dense. The (single) multiplication operator

$$(Mp)(x) = xp(x), \quad p \in \mathcal{D},$$

is symmetric and maps $\mathcal{D}$ into itself. Moreover, M commutes with the complex conjugation symmetry of H:

$$\overline{Mp} = M\overline{p}.$$

By a classical result of von-Neumann [vN1] there exists a self-adjoint (possibly unbounded) operator A which extends M to a larger domain. Since A possesses a spectral measure E_A (exactly as in the bounded case), we obtain:

$$c_k = \langle x^k, 1 \rangle = \langle M^k 1, 1 \rangle =$$
$$\langle A^k 1, 1 \rangle = \int_{-\infty}^{\infty} x^k \langle E_A(dx)1, 1 \rangle.$$

The measure $\langle E_A(dx)1, 1 \rangle$ is positive and has prescribed moments (c_k). ∎

This second proof offers more insight into the uniqueness part of Hamburger's problem. Every self-adjoint extension A of the symmetric operator

M produces a solution $\mu(dx) = \langle E_A(dx)1, 1\rangle$. The set K of all positive measures with prescribed moments (c_k) is convex and compact in the weak-$*$ topology. The subset of Nevanlinna extremal elements of K are identified with the measures $\langle E_A(dx)1, 1\rangle$ associated to the self-adjoint extensions A of M. In particular one proves in this way the following useful uniqueness criterion.

Proposition 3.5. *Let (c_k) be the moment sequence of a positive measure μ on the line. Then a positive measure with the same moments coincides with μ if and only if the subspace*

$$(iI + M)\mathcal{D} \ \text{ is dense in } H,$$

or equivalently, there exists a sequence of polynomials $p_n \in \mathbb{C}[x]$ satisfying

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |(i+x)p_n(x) - 1|^2 \, d\mu(x) = 0.$$

Note that both conditions are intrinsic in terms of the initial data (c_k). For the original function theoretic proof see [MR23]. For the operator theoretic proof see for instance [Akh65].

There exists a classical analytic function counterpart of the above objects, exactly as in the previous case (see §2.4, §2.5) of the unit circle. Namely, assuming that

$$c_k = \langle A^k 1, 1\rangle = \int_{-\infty}^{\infty} x^k d\mu(x), \quad k \geq 0,$$

as before, the analytic function

$$F(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x - z} = \langle (A - z)^{-1} 1, 1\rangle$$

is well defined in the upper half-plane $\Im z > 0$ and has the asymptotic expansion at infinity (in the sense of Poincaré, uniformly convergent in wedges $0 < \delta < \arg z < \pi - \delta$):

$$F(z) \approx -\frac{c_0}{z} - \frac{c_1}{z^2} - \cdots, \quad \Im(z) > 0.$$

One step further, we have a purely algebraic recursion which determines the continued fraction development

$$-\frac{c_0}{z} - \frac{c_1}{z^2} - \cdots = -\cfrac{c_0}{z - \alpha_0 - \cfrac{\beta_0}{z - \alpha_1 - \cfrac{\beta_1}{z - \alpha_2 - \cfrac{\beta_2}{\ddots}}}}, \quad \alpha_k \in \mathbb{R}, \ \beta_k \geq 0.$$

It was Stieltjes, and then Hamburger, who originally remarked that (c_k) is the moment sequence of a positive measure if and only if the elements β_k in the continued fraction development of the generating (formal) series are non-negative. Moreover, in this case they proved that there exists a unique representing measure if and only if the continued fraction converges in the upper half-plane. For details and a great collection of classical examples see Perron's monograph [Per50]. A well known uniqueness criterion was obtained via this formalism by Carleman [C26]. It states that uniqueness holds if

$$\sum_1^\infty \frac{1}{c_{2k}^{1/(2k)}} = \infty.$$

The condition is however not necessary for uniqueness.

The alert reader has seen the great kinship between the continued fraction recursion just elucidated and the recursion called the Schur Algorithm in §2.5. These are essentially the same thing, but one is in the disk setting while the other is in the half plane.

3.2.1. *Moments on the semiaxis* $[0, \infty]$. The above picture applies with minor modifications to Stieltjes problem, that is the power moment problem on the semi-axis $[0, \infty)$.

Example 3.6. We reproduce below an example found by Stieltjes, and refined by Hamburger. See for details [Per50]. Let ρ and δ be positive constants, and denote

$$\alpha = \frac{1}{2+\delta}, \quad \gamma = \rho^{-\alpha}.$$

Then

$$a_n = (2+\delta)\rho^{n+1}\Gamma[(2+\delta)(n+1)] = \int_0^\infty x^n e^{-\gamma x^\alpha}\, dx, \quad n \geq 0,$$

is a moment sequence on the positive semi-axis. A residue integral argument implies

$$\int_0^\infty x^n \sin\bigl(\gamma x^\alpha \tan(\pi\alpha)\bigr)\, e^{-\gamma x^\alpha}\, dx = 0, \quad n \geq 0.$$

Hence

$$a_n = \int_0^\infty x^n \bigl(1 + t\sin(\gamma x^\alpha \tan(\pi\alpha))\bigr) e^{-\gamma x^\alpha}\, dx,$$

for all $n \geq 0$ and $t \in (-1,1)$. This shows that the moment sequence (a_n) does not uniquely determine μ even knowing its support is $[0,\infty)$. □

Summing up the above ideas, we have bijective correspondences between the following sets ($\mathbb{C}_+$ stands for the open upper half plane):

A). *Rapidly decaying positive measures μ on the real line;*

B). Analytic functions $F : \mathbb{C}_+ \longrightarrow \overline{\mathbb{C}_+}$, satisfying $\sup_{t>1} |tF(it)| < \infty$*;*

C). Self-adjoint operators A with a cyclic vector ξ.

More precisely:

$$F(z) = \langle (A-z)^{-1}\xi, \xi\rangle = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x-z}, \quad z \in \mathbb{C}_+.$$

The moment sequence $c_k = \int_{-\infty}^{\infty} x^k d\mu(x), k \geq 0$, appears in the asymptotic expansion of F, at infinity, but it does not determine F, (A,ξ) or μ. For further details about Hamburger and Stieltjes moment problems see Akhiezer's monograph [Akh65].

3.3. **Several variables.** The moment problem on $\mathbb{R}^g$, $g > 1$, is considerably more difficult and less understood. Although we have the general correspondence remarked in Proposition 3.1, the gap between a commuting tuple of unbounded symmetric operators and a strongly commuting one (i.e.

by definition one possessing a joint spectral measure) is quite wide. A variety of strong commutativity criteria came to rescue; a distinguished one, due to Nelson [N59], is worth mentioning in more detail.

Assume that $L : \mathbb{R}[x_1, ..., x_g] \longrightarrow \mathbb{R}$ is a functional satisfying (the non-negative Hankel form condition) $L|_{\Sigma^2\mathbb{R}[x]} \geq 0$. We complexify L and associate, as usual by now, the Hilbert space H with inner product:

$$\langle p, q\rangle = L(p\overline{q}), \quad \mathbb{C}[x].$$

The symmetric multipliers M_{x_k} commute on the common dense domain $\mathcal{D} = \mathbb{C}[x] \subset H$. Exactly as in the one variable case, there exists a positive measure μ on $\mathbb{R}^g$ representing L if and only if there are (possibly unbounded) self-adjoint extensions $M_{x_k} \subset A_k,\ 1 \leq k \leq n$, commuting at the level of their resolvents:

$$[(A_k - z)^{-1}, (A_j - z)^{-1}] := (A_k - z)^{-1}(A_j - z)^{-1} - (A_j - z)^{-1}(A_k - z)^{-1} = 0,$$

$$\text{for } \Im z > 0,\ 1 \leq j, k \leq n.$$

See for details [F83]. Although individually every M_{x_k} admits at least one self-adjoint extension, it is the joint strong commutativity (in the resolvent sense) of the extensions needed to solve the moment problem.

Nelson's theorem gives a sufficient condition in this sense: if $(1 + x_1^2 + ... + x_g^2)\mathcal{D}$ is dense in H, then the tuple of multipliers $(M_{x_1}, ..., M_{x_g})$ admits an extension to a strongly commuting tuple of self-adjoint operators. Moreover, this insures the uniqueness of the representing measure μ. For complete proofs and more details see [Berg87, F83].

A tantalizing open question in this area can be phrased as follows:

Open problem. *Let $(c_{\alpha+\beta})_{\alpha,\beta\in\mathbb{N}^g}$ be a positive semi-definite Hankel form. Find effective conditions insuring that (c_α) are the moments of a positive measure.*

Or equivalently, in predual form, find effective criteria (in terms of the coefficients) for a polynomial to be non-negative on $\mathbb{R}^g$.

We know from Tarski's principle that the positivity of a polynomial is decidable. The term "effective" above means to find exact rational expressions in the coefficients which guarantee the non-negativity of the polynomial.

We do not touch in this discussion a variety of other aspects of the multivariate moment problem such as uniqueness criteria, orthogonal polynomials, cubature formulas and the truncated problem. See for instance [Berg87, Berg91, CF05, GV61, KM70].

3.4. **Positivstellensätze on compact, semi-algebraic sets.** Now we look at a very popular classes of Positivstellensätze. The hypotheses are more restrictive (by requiring bounded sets) than the general one, but the conclusion gives a simpler certificate of positivity. The techniques of proof are those used in the multivariate moment problem but measures with compact semi-algebraic support allow much more detail.

To state the theorems in this section requires the notions of preorder, $PO(F)$ and of quadratic module which we now give, but the treatment of them in Section 5.4 on the general Positivstellensatz gives more properties and a different context than done here. Let $F = \{f_1, ..., f_p\}$ denote a set of real polynomials. The *preordering* generated by F is

$$PO(F) = \{ \sum_{\sigma \in \{0,1\}^r} s_\sigma f_1^{\sigma_1} ... f_r^{\sigma_r}; \quad s_\sigma \in \Sigma^2 R[x]\}.$$

The *quadratic module generated by* F is defined to be:

$$QM(F) = \sum_{f \in F \cup \{1\}} f \Sigma^2 \mathbb{R}[x].$$

We start with a fundamental result of Schmüdgen, proved in 1991 ([S91]), which makes use in an innovative way of Stengle's general Positivstellensatz.

Theorem 3.7 (Schmüdgen). *Let* $F = \{f_1, ..., f_p\}$ *be a set of real polynomials in* g *variables, such that the non-negativity set* $\mathcal{D}_F$ *is compact in* $\mathbb{R}^g$. *Then*

a). A functional $L \in \mathbb{R}[x]'$ *is representable by a positive measure supported on* K *if and only if*

$$L|_{PO(F)} \geq 0.$$

b). Every positive polynomial on $\mathcal{D}_F$ belongs to the preorder $PO(F)$.

Due to the compactness of the support, and Stone-Weierstrass Theorem, the representing measure is unique. We will discuss later the proof of b) in a similar context.

We call the quadratic module $QM(F)$ *archimedean* if there exists $C > 0$ such that

$$C - x_1^2 - ... - x_g^2 \in QM(F).$$

This implies in particular that the semi-algebraic set $\mathcal{D}_F$ is contained in the ball centered at zero, of radius $\sqrt{C}$. Also, from the convexity theory point of view, this means that the convex cone $QM(F) \subset \mathbb{R}[x]$ contains the constant function 1 in its algebraic interior (see [K69] for the precise definition). If the set $\mathcal{D}_F$ is compact, then one can make the associated quadratic module archimedean by adding to the defining set one more term, of the form $C - x_1^2 - ... - x_g^2$.

The key to Schmüdgen's Theorem and to a few forthcoming results in this survey is the following specialization of Proposition 3.1.

Lemma 3.8. [P93] *Let F be a finite set of polynomials in $\mathbb{R}[x]$ with associated quadratic module $QM(F)$ having the archimedean property. There exists a bijective correspondence between:*

a). Commutative g-tuples A of bounded self-adjoint operators with cyclic vector ξ and joint spectrum contained in $\mathcal{D}_F$;

b). Positive measures μ supported on $\mathcal{D}_F$;

c). Linear functionals $L \in \mathbb{R}[x]'$ satisfying $L|_{QM(F)} \geq 0$.

The correspondence is constructive, given by the relations:

$$L(p) = \langle p(A)\xi, \xi \rangle = \int_{\mathcal{D}_F} p d\mu, \qquad p \in \mathbb{R}[x].$$

Proof. Only the implication $c) \Rightarrow a)$ needs an argument. Assume c) holds and construct the Hilbert space H associated to the functional L. Let $M = (M_{x_1}, ..., M_{x_g})$ denote the tuple of multiplication operators acting on H. Due to the archimedean property,

$$\langle (C - x_1^2 - ... - x_g^2)p, p\rangle \geq 0, \quad p \in \mathbb{C}[x],$$

whence every M_{x_k} is a bounded self-adjoint operator. Moreover, the condition

$$\langle f_j p, p\rangle \geq 0, \quad p \in \mathbb{C}[x],$$

assures that $f_j(M) \geq 0$, that is, by the spectral mapping theorem, the joint spectrum of M lies on $\mathcal{D}_F$. Let E_M be the joint spectral measure of M. Then

$$L(p) = \int_{\mathcal{D}_F} p(x)\langle E_M(dx)1, 1\rangle,$$

and the proof is complete. ∎

For terminology and general facts about spectral theory in a commutative Banach algebra see [D03].

With this dictionary between positive linear functionals and tuples of commuting operators with prescribed joint spectrum we can improve Schmüdgen's result.

Theorem 3.9 ([P93])**.** *Let F be a finite set of real polynomials in g variables, such that the associated quadratic module $QM(F)$ is archimedean.*

Then a polynomial strictly positive on $\mathcal{D}_F$ belongs to $QM(F)$.

Proof. Assume by contradiction that p is a positive polynomial on $\mathcal{D}_F$ which does not belong to $QM(F)$. By a refinement of Minkowski separation theorem due to Eidelheit and Kakutani (see [K69]), there exists a linear functional $L \in \mathbb{R}[x]'$ such that $L(1) > 0$ and:

$$L(p) \leq 0 \leq L(q), \quad q \in QM(F).$$

(Essential here is the fact that the constant function 1 is in the algebraic interior of the convex cone $QM(F)$). Then Lemma 3.8 provides a positive measure μ supported on $\mathcal{D}_F$, with the property:

$$L(p) = \int_{\mathcal{D}_F} p d\mu \leq 0.$$

The measure is non-trivial because

$$L(1) = \mu(\mathcal{D}_F) > 0,$$

and on the other hand $p > 0$ on $\mathcal{D}_F$, a contradiction. ■

An algebraic proof of the latter theorem is due to Jacobi and Prestel, see [PD01].

4. Complex variables

The operator theoretic counterpart to positive functionals described in the previous section becomes more transparent in the case of complex variables. We present below, closely following [P06], a series of generalizations of Lemma 3.8 and Theorem 3.9. It is at this point when von Neumann's inequality becomes relevant.

Throughout this section $z = (z_1, ..., z_g)$ denote the complex coordinates in $\mathbb{C}^g$. Then real coordinates of the underlying space $\mathbb{R}^{2d}$ are denoted by $x = (x_1, ..., x_{2n})$, where $z_k = x_k + ix_{n+k}$. We will work as before in the polynomial algebra $\mathbb{C}[x] = \mathbb{C}[z, \overline{z}]$, and consider there the convex hulls of non-negative polynomials:

$$\Sigma^2 = \text{co}\{p^2;\ p \in \mathbb{R}[x]\},$$

and

$$\Sigma_h^2 = \text{co}\{|q|^2;\ q \in \mathbb{C}[z]\}.$$

It is easy to see that the cone of hermitian positive squares Σ_h^2 is a proper subset of Σ^2. Indeed, remark that

$$\frac{\partial}{\partial z_k}\frac{\partial}{\partial \overline{z}_k}|q|^2 \geq 0, \quad q \in \mathbb{C}[z],$$

while the same Laplace operator has negative values on $(y_k - x_k^2)^2$.

Let $F = \{p_1, ..., p_m\}$ be a finite subset of $\mathbb{R}[x]$ so that the basic semi-algebraic set

$$\mathcal{D}_F = \{x \in \mathbb{R}^{2d};\ p_1(x) \geq 0, ..., p_m(x) \geq 0\}$$

is compact. Let p be a real polynomial which is positive on $\mathcal{D}_F$. We will indicate conditions which insure the decompositions:

$$p \in \Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2, \tag{4.1}$$

or

$$p \in \Sigma^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2, \tag{4.2}$$

or

$$p \in QM(F) = \Sigma^2 + p_1\Sigma^2 + ... + p_m\Sigma^2. \tag{4.3}$$

The last one is covered by Theorem 3.9. The other two require stronger assumptions on p, as we shall see below.

We start by recalling an early, unrelated observation about strictly positive hermitian polynomials [Q68].

Theorem 4.1 (Quillen). *If a bi-homogeneous polynomial $p \in \mathbb{C}[z, \overline{z}]$ satisfies $p(z, \overline{z}) > 0$ for $z \neq 0$, then there exists $N \in \mathbb{N}$ such that:*

$$|z|^{2N} p(z, \overline{z}) \in \Sigma_h^2.$$

This result was rediscovered, and obtained by a different method, by Catlin and d'Angelo [CD96]. Their approach proved to be geometric in its nature and very flexible, leading to a variety of refinements of Quillen's theorem, see for instance [CD99, d'A05, d'AV03]. This line of research, not developed in the present survey, culminates in completely removing the strict positivity assumption. Specifically, the following characterization of bi-homogeneous quotients of polynomials of Σ_h^2 was recently discovered.

Theorem 4.2 (Varolin, [Var06]). *Let the bi-homogeneous polynomial $p \in \mathbb{C}[z, \overline{z}]$ satisfy $p(z, \overline{z}) \geq 0$ for $z \in \mathbb{C}^g$. Write $p(z, \overline{z}) = \sum_{\alpha,\beta} p_{\alpha,\beta} z^\alpha \overline{z}^\beta$ and, using the positive and negative spectral subspaces of the Hermitian matrix $(p_{\alpha,\beta})$, decompose $p = p_+ - p_-$, with $p_\pm \in \Sigma_h^2$.*

Then there are non-zero polynomials $s, t \in \Sigma_h^2$ with the property

$$sp = t$$

if and only if there exists a positive constant C, such that

$$p_+ + p_- \leq C(p_+ - p_-)$$

on $\mathbb{C}^g$.

Next we return to the compact semi-algebraic set $\mathcal{D}_F \subset \mathbb{C}^g$ and the three levels of positivity certificates described by the preceding convex cones.

We recall that a commutative g-tuple of linear bounded operators T acting on a Hilbert space H is called *subnormal* if there exists a larger Hilbert space $H \subset K$ and a commutative g-tuple of normal operators N acting on K, so that every N_j leaves H invariant and $N_j|_H = T_j$, $1 \leq j \leq g$. A commutative g tuple N of normal operators $N_j = X_j + iX_{g+j}$ consists of $2g$ mutually commuting self-adjoint operators $X_1, ..., X_{2g}$. Hence N possesses a joint spectral measure E_N, supported on the joint spectrum $\sigma(N) \subset \mathbb{C}^g$.

Assume from now on that the cone $\Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2$ is archimedean, that is, after a normalization:

$$1 - |z_1|^2 - ... - |z_g|^2 \in \Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2.$$

Let $L \in \mathbb{C}[x]'$ be a linear functional satisfying

$$L|_{\Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2} \geq 0.$$

Instead of constructing the completion of the whole ring of real polynomials, we consider the same L-inner product, defined only on complex polynomials $\mathbb{C}[z]$, in the variables z. Let H be the associated Hilbert space. The multiplication operators M_{z_j} act and commute on H. They are bounded due to the above mentioned archimedean property:

$$\|M_{z_j}q(z)\|^2 = L(|z_jq(z)|^2) \leq L(|q(z)|^2) = \|q\|^2.$$

The only difference to the framework of the multivariate Hamburger moment problem is that now M_{z_j} are not necessarily self-adjoint operators and the commutators $[M_{z_j}, M_{z_k}^*]$ may not vanish. The constant function vector 1 remains cyclic, in the sense that the span of the vectors $M_{z_1}^{\alpha_1}...M_{z_g}^{\alpha_g}1$ is the whole Hilbert space H.

Let $M = (M_{z_1}, ..., M_{z_g})$ be the commutative n-tuple of operators we have just constructed. For a polynomial $p(z, \overline{z})$ we define after Colojoara and Foias, cf. ([AM02]), the *hereditary functional calculus* $\tilde{p}(M, M^*)$ by putting all adjoints $M_{z_k}^*$ in the monomials of p to the left of the powers of M_{z_j}'s. For example,

$$(\widetilde{|z_1|^2 z_2^2 \overline{z_3}})(M) = M_{z_1}^* M_{z_3}^* M_{z_1} M_{z_2}^2.$$

We have thus established the first part of the following dictionary.

Proposition 4.3. *Let $F = \{p_1, ..., p_m\}$ be a finite set of real polynomials in $2g$ variables, such that*

$$1 - |z_1|^2 - ... - |z_g|^2 \in \Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2.$$

a). There exists a bijective correspondence between functionals $L \in \mathbb{C}[x]'$ which are non-negative on $\Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2$ and commutative g-tuples of linear Hilbert space operators T, with a cyclic vector ξ, subject to the conditions

$$\tilde{p_j}(T, T^*) \geq 0, \; 1 \leq j \leq m.$$

b). If, in addition L is non-negative on the larger cone $\Sigma^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2$, then, and only then, T is also subnormal.

In both cases, the correspondence is given by

$$L(p(z, \overline{z})) = \langle \tilde{p}(T, T^*)\xi, \xi \rangle.$$

Proof. For the proof of part b) remark that the Hilbert space completion H of the ring of complex polynomials with respect to the inner product $\langle p, q \rangle = L(p\overline{q}), \; p, q \in \mathbb{C}[z]$ carries the bounded, commutative g-tuple M of multiplication operators with the variables $z_1, ..., z_g$. The positivity of the functional L on Σ^2 is equivalent to the multivariate analog of the Halmos-Bram subnormality condition applied to M. See for details [Dem02, P93].

Conversely, if T is a cyclic subnormal tuple of operators with normal extension N, and $p(z, \overline{z})$ is a polynomial, then

$$\langle |\tilde{p}|^2(T, T^*)\xi, \xi \rangle = \langle |p|^2(N, N^*)\xi, \xi \rangle = \|p(N, N^*)\xi\|^2 \geq 0.$$

This uses the very definition of the hereditary calculus, for example,

$$\langle T_2^{*^4} T_1^{*^6} T_1^6 T_2^4 \xi,\ \xi \rangle = \langle T_1^6 T_2^4 \xi,\ T_1^6 T_2^4 \xi \rangle = \langle T_1^6 N_2^4 \xi,\ T_1^6 N_2^4 \xi \rangle$$

$$\langle N_1^6 N_2^4 \xi,\ N_1^6 N_2^4 \xi \rangle = \| N_1^6 N_2^4 \xi \|.$$

■

The following translation of the proposition shows that the class of all commutative tuples of operators serves as a better "spectrum" for the polynomial algebra in the variables $(z, \overline{z})$.

Corollary 4.4. *Let F be as in the Proposition and let $p(z,\overline{z})$ be a polynomial. If $\tilde{p}(T,T^*) > 0$ for every commutative g-tuple of linear Hilbert space operators T, satisfying $\tilde{p_j}(T,T^*) \geq 0,\ 1 \leq j \leq m$, then p belongs to $\Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2$.*

The cyclic vector condition is not relevant for this statement.

Proof. The proof follows from now a known pattern. Assume by contradiction that $p \notin \Sigma_h^2 + p_1\Sigma_h^2 + ... + p_m\Sigma_h^2$. By Minkowski-Eidelheit-Kakutani separation theorem, there exists a linear functional $L \in \mathbb{C}[x]'$ satisfying the conditions of Proposition 4.3 and $L(p) \leq 0 < L(1)$. Then the commutative g-tuple M associated to the inner-product space defined by L satisfies

$$\langle p(M, M^*)1, 1 \rangle = L(p) \leq 0,$$

a contradiction. ■

Even the simple case of the unit ball or unit polydisk in $\mathbb{C}^g$ is interesting from this perspective. Assume first that $n = 1$. According to von-Neumann's inequality,

$$\|p(T)\| \leq 1,$$

whenever T is a contraction and $\sup_{z\in\mathbb{D}} |p(z)| \leq 1$. Thus, in view of the above proposition, for every polynomial $p(z)$ and constant $M > \sup_{z\in\mathbb{D}} |p(z)|$, we have

$$M^2 - |p(z)|^2 \in \Sigma_h^2 + (1 - |z|^2)\Sigma_h^2.$$

Needless to say that this statement is equivalent to von-Neumann's inequality.

In complex dimension two, a celebrated theorem of Ando (see [AM02, CW99]) asserts that, for every pair of commuting contractions (T_1, T_2) and every polynomial $p(z_1, z_2)$ one has a von-Neumann type inequality:

$$\|p(T_1, T_2)\| \leq \|p\|_{\infty, \mathbb{D}^2}.$$

And a not less celebrated example of Varopoulos (see again [AM02]) shows that it is no more the case in dimension $n = 3$ and higher. Specifically, according to our corollary, for every polynomial $p(z)$ and $\epsilon > 0$, we have

$$(\|p\|_{\infty, \mathbb{D}^2} + \epsilon)^2 - |p(z)|^2 \in \Sigma_h^2 + (1 - |z_1|^2)\Sigma_h^2 + (1 - |z_2|^2)\Sigma_h^2,$$

but the statement is not true (for the unit polydisk) in higher dimensions.

Open problem. *It would be interesting to find an algebraic explanation of Ando's Theorem via the above equivalent sums of squares decomposition.*

On the other hand, by exploiting part b) of Proposition 4.3 one can prove a sharper weighted sums of squares decomposition on complex analytic polyhedra.

Theorem 4.5. *Let $S = \{z \in \mathbb{C}^g; |p_j(z)| \leq 1, 1 \leq j \leq m\}$ be a compact semi-algebraic set, where p_j are complex polynomials. Assume that the convex cone*

$$C = \Sigma^2 + (1 - |p_j|^2)\Sigma_h^2 + ... + (1 - |p_m|^2)\Sigma_h^2$$

is archimedean. Then every real polynomial p which is strictly positive on S belongs to C.

For a proof see [P93, P06]. The article [BGM05] contains a similar approach to bounded analytic multipliers on the Hardy space of the bi-disk. The recent note [P06] contains a few other sums of squares translations of some recently proved inequalities in operator theory.

Finally, we reproduce from [HMP06] the following general Nichtnegativstellensatz over the complex affine space. Note that when evaluating on tuples of commutative matrices, we do not have to impose the strict positivity of the polynomial to be decomposed.

Theorem 4.6. *Let $p(z, \overline{z})$ be a real valued polynomial, where $z \in \mathbb{C}^g$. Then there are polynomials $q_i \in \mathbb{C}[z]$, $1 \leq i \leq k$, with the property*

$$p(z, \overline{z}) = \sum_{i=1}^{k} |q_i(z)|^2,$$

if and only if, for all tuples of commuting matrices $X = (X_1, ..., X_g) \in M_d(\mathbb{C})$, $d \geq 1$, we have

$$p(X, X^*) \geq 0.$$

The proof follows the general scheme outlined in this section and we omit it. See [HMP06] for full details.

5. Real algebra and mathematical logic

Keeping in mind the main theme of our essay (sums of squares decompositions), we briefly recall below, without aiming at completeness, some classical facts of real algebra and mathematical logic. We follow an approximate chronological order. For a more comprehensive, recent and very authoritative survey of real algebra and real algebraic geometry aspects of sums of squares we refer to Scheiderer [S03].

5.1. **Minkowski and Hilbert.** In the same time to, and even before, the analysis aspects of sums of squares decompositions we have discussed have been discovered, similar questions have appeared in number theory and algebra. Lagrange's famous theorem (that every positive integer can be written as a sum of squares of four integers) was the origin of many beautiful studies, see Chapter XX of Hardy and Wright's monograph [HW93].

According to Hilbert [Hilb1910], after the teenager Minkowski won the 1882 "Grand Prix" of the French Academy of Sciences on a theme related to Lagrange's four squares theorem, he has started working in Könisberg on his thesis devoted to quadratic forms of a higher number of variables. It was in Minkowski's inaugural dissertation, with Hilbert as opponent, that he remarked that *"it is not probable that every positive form can be represented as a sum of squares"* [M86].

The opponent (Hilbert) produced the first (non-explicit) example, see [Hilb1888]. His idea is the following. Consider nine points $a_1, ..., a_9$ in $\mathbb{R}^2$, as the base of a pencil of cubics (that is, a family of curves obtained via a linear combination of their third degree defining equations), so that every cubic polynomial vanishing at the first eight points $a_1, ..., a_8$ will automatically vanish at a_9. By a rather involved geometric argument, one can prove the existence of a polynomial $p(x, y)$, of degree six, which is non-negative on $\mathbb{R}^2$, vanishes at $a_1, ..., a_8$ and satisfies $p(a_9) > 0$. Then clearly p cannot be written as a sum of squares of polynomials:

$$p = q_1^2 + ... + q_N^2,$$

because every q_i would have degree at most three and therefore would be null at a_9, too. Hilbert's argument is reproduced at the end of Chapter II of Gelfand and Vilenkin's monograph [GV61].

The first explicit example based on Hilbert idea was constructed by Robinson in 1969, see [R92] for details. Robinson's homogenized polynomial is :

$$P(x, y, z) = x^6 + y^6 + z^6 -$$

$$-(x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2.$$

About the same time (some six dozen years after Hilbert's article) Motzkin has produced a very simple polynomial (again shown in homogenized form):

$$Q(x, y, z) = z^6 + x^2y^2(x^2 + y^2 - z^2).$$

The reader will find easily why Q is non-negative, but not a sum of squares. More examples of non-negative polynomials which are not sums of squares were discovered by Choi, Lam and Reznick, and separately Schmüdgen. We refer to Reznick's monograph [R92] for more details and for an elegant geometric duality method (based on the so called Fisher inner product) adapted to the analysis of the convex cones of such polynomials.

One of Hilbert's celebrated 1900 problems in mathematics was about the structure of positive polynomials and the logical implications of the existence of a constructive way of testing positivity. Here are his words:

Problem 17. Expression of definite forms by squares:

"...the question arises whether every definite form may not be expressed as a quotient of sums of squares of forms...

... it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented." [Hilb1910].

Hilbert's intuition proved to be correct on both conjectures raised by his question. His query was the origin of a series of remarkable results in algebra and logic, see [BCR98, PD01, S03].

5.2. **Real fields. Many may want to skip this section which defines a real closed field, a generalization of the real numbers. Then in the rest of this section, when we say real close field, just think real numbers.** Hilbert's 17-th problem was solved in the affirmative by E. Artin [Art26] in 1927, as an application of the theory of real fields he has developed with Schreier [AS26]. For history and self-contained introductions to real algebra, and complete proofs of Artin's Theorem, we refer to either one of the following monographs [BCR98, J89, PD01]. We merely sketch below the main ideas of Artin-Schreier theory (as exposed in Jacobson's algebra book [J89]), to serve as a comparison basis for the computations we will develop in later sections in the framework of star algebras.

An *ordered field* is a characteristic zero field F with an *ordering*, that is a prescribed subset P of positive elements, satisfying:

$$P + P \subset P, \quad P \cdot P \subset P, \quad F = P \cup \{0\} \cup \{-P\}.$$

Since $a^2 \in P$ for all $a \neq 0$, if $\sum_i a_i^2 = 0$, then every $a_i = 0$. Or equivalently, -1 cannot be written as a sum of squares in F. By a theorem of Artin and Schreier, every field with the latter property can be ordered. An ordered field R is *real closed* if

(1) Every positive element has a square root. In this case, exactly as in the case of real numbers, the extension $R(\sqrt{-1})$ is algebraically closed.
(2) Any polynomial of odd degree with coefficients in R has at least one root in R.

A central result in Artin-Schreier theory is the existence and uniqueness of the real closure R of an ordered field F: that is the extension $F \subset R$ is algebraic and $x \in F$ is positive in R if and only if is positive in F. Interestingly enough, the proof of this fact uses Sturm's algorithm for the determination of the number of roots of a polynomial with real coefficients (or more generally with coefficients in a real closed field).

Sturm's algorithm. Let R be a real closed field and let

$$p(x) = a_0 + a_1 x + ... + x^d \in R[x]$$

be a polynomial. Let

$$C = 1 + |a_{d-1}| + ... + |a_1| + |a_0|.$$

Define the sequence of polynomials:

$$p_0 = p, \quad p_1 = p' \ \text{(the derivative)},$$

$$p_{j+1} = p_j q_j - p_{j-1}, \quad \deg p_{j+1} < \deg p_j.$$

Then for a large n, $p_n = 0$. Sturm's Theorem asserts that the number of roots of p in R is $N(-C) - N(C)$, where $N(a)$ is the number of sign changes in the sequence $p_0(a), p_1(a), ..., p_n(a)$.

As an application of the existence of the real closure of an ordered field, one can prove that an element $x \in F$ is a sum of squares if and only if it is positive in every order on F. Or, equivalently, if $y \in F$ is not a sum of squares, then there exists an order on F with respect to which $y < 0$.

5.3. **Another Hilbert Question goes Down.**

Theorem 5.1 (Artin). *Let F be a subfield of $\mathbb{R}$ which has a unique ordering and let f be a rational function with coefficients in F. If $f(a_1, ..., a_g) \geq 0$ for all $(a_1, ..., a_g) \in F^g$ for which f is defined, then f is a sum of squares of rational functions with coefficients in F.*

The idea of the proof is to admit by contradiction that f is not a sum of squares, hence it does not belong to an ordering of the field of rational functions $K = F(x_1, ..., x_g)$ in g variables. By completing K to a real closed field R, one finds "ideal" points $b_1, ..., b_g \in R$, so that $f(b_1, ..., b_g) < 0$. By Sturm's counting theorem one shows then that there are points $a_1, ..., a_g \in F$ with the property $f(a_1, ..., a_g) < 0$. For details see for instance §11.4 in [J89].

Examples of fields with a unique ordering are $\mathbb{Q}$ and $\mathbb{R}$. Artin's Theorem prompts a series of natural questions, as for instance : how many squares are necessary, is there a universal denominator in the decomposition of f as a sum of squares, are there degree bounds? All these problems were thoroughly studied during the last decades, [BCR98, PD01].

It was Tarski who in the late 1920-ies put Sturm's Algorithm into a very general and surprising statement. His work had however an unusually long gestation and has remained unknown to the working mathematician until mid XX-th Century. His articles are available now from a variety of parallel sources, see for instance [T86] and the historical notes in [PD01]. His main thesis is contained in the following principle, cited from his original 1948 RAND publication [T86]:

Tarski's elimination theory for real closed fields. "*To any formula $\phi(x_1, ..., x_g)$ in the vocabulary $\{0, 1, +, \cdot, <\}$ and with variables in a real closed field, one can effectively associate two objects:*

(i) a quantifier free formula $\overline{\phi}(x_1, ..., x_g)$ in the same vocabulary, and

(ii) a proof of the equivalence $\phi \equiv \overline{\phi}$ that uses only the axioms of real closed fields."

He aimed this theorem at the completeness of the logical system of elementary algebra and geometry, very much in the line of Hilbert's programme in the foundations of mathematics. As a consequence one obtains the transfer principle alluded above, in the proof of Artin's Theorem: let $R_1 \subset R_2$ be real closed fields. A system of polynomial inequalities and equalities with coefficients in R_1 has a solution in R_2 if and only if it has a solution in R_1.

Let R be a real closed field. We recall that a *semi-algebraic set* in R^g is a finite union of finite intersections of sets of the form

$$\{x \in R^g;\ p(x) = 0\}, \quad \{x \in R^g;\ p(x) > 0\}, \quad \text{or} \quad \{x \in R^g;\ p(x) \geq 0\}.$$

A self-contained account of Tarski's theorem can be found in [S54]. See also [BCR98, J89, PD01]. In practice, the most useful form of Tarski's result is the following theorem.

Theorem 5.2 (Tarski-Seidenberg)**.** *If R is a real closed field and S is a semi-algebraic set in $R^g \times R^m$, then the projection of S onto R^g is also semi-algebraic.*

For a self contained proof of the above theorem see the Appendix in [Horm83].

Applications of Tarski's principle came late, but were spectacular. We only mention for illustration one of them:

Hörmander's inequality. (1955) : *For every polynomial $f(x_1, ..., x_g) \in \mathbb{R}[x_1, ..., x_g]$ there are positive constants c, r such that*

$$|f(x)| \geq c \operatorname{dist}(x, V(f))^r, \quad x \in \mathbb{R}^g, |x| \leq 1.$$

Above $V(f)$ stands for the real zero set of f. The inequality was generalized to real analytic functions by Lojasiewicz in 1964, and served as the origin of fundamental discoveries in modern analytic geometry and the theory of partial differential operators, see for instance [BCR98, Horm83].

5.4. **The general Positivstellensatz.** A great jewel of real algebraic geometry, which is now causing excitement in applications, is the Positivstellen-sätze and the real Nullstellensatz (which it contains). This section states these theorems and can be read independently of earlier parts of this paper.

The Positivstellen-sätze lives in a polynomial ring with coefficients in a real closed field (as opposed to the complex numbers) and were discovered only in the 1960-ies (see [Dub69, K64]) before being rediscovered and refined by Stengle in 1974 ([S74]). The statement of the Nullstellensatz departs from Hilbert's Nullstellensatz over an algebraically closed field, by imposing an additional sum of squares term in the characterization of a radical ideal, as we shall see below. It is interesting to remark that Stengle's article makes specific references, as origins or motivations of his investigation, to works in mathematical logic and mathematical programming.

In order to state Stengle's Positivstellen-sätze we need first a few definitions and conventions. Let R be a real closed field (many readers will be happy to think of R as the real numbers) and denote $x = (x_1, ..., x_g) \in R^g$ and also regard x as a g-tuple of commuting indeterminates. Let $\Sigma^2 A$ denote the set of all sums of squares in the algebra A .

Let $S \subset R[x]$ be a subset, and write

$$\mathcal{D}_S = \{x \in R^g;\ p(x) \geq 0,\ \forall p \in S\},$$

for the positivity set of of the functions S. If $S = \{p_1, ..., p_r\}$ is finite then $\mathcal{D}_S$ is a *basic closed semi-algebraic set* . The *preordering* generated by S is

$$PO(S) = \{ \sum_{\sigma \in \{0,1\}^r} s_\sigma p_1^{\sigma_1} ... p_r^{\sigma_r};\quad s_\sigma \in \Sigma^2 R[x]\}.$$

The *quadratic module* generated by S is

$$QM(S) = \Sigma^2 R[x] + p_1 \Sigma^2 R[x] + \ldots + p_r \Sigma^2 R[x].$$

Note that a preordering satisfies conditions similar to an ordering in a field:

$$PO(S) + PO(S) \subset PO(S),\quad PO(S) \cdot PO(S) \subset PO(S),\quad \Sigma^2 R[x] \subset PO(S),$$

while the quadratic module fails to be closed under multiplication, but still satisfies:

$$QM(S)+QM(S) \subset QM(S), \ \ \Sigma^2 R[x] \subset QM(S), \ \ \Sigma^2 R[x]\cdot QM(S) \subset QM(S),$$

and clearly,

$$QM(S) \subset PO(S).$$

Theorem 5.3 (Stengle)**.** *Let R be a real closed field and let $p_1, ..., p_r \in R[x_1, ..., x_g]$. Let $\mathcal{S} = \mathcal{D}(p_1, ..., p_r)$ and let $T = PO(p_1, ..., p_r)$ be the preorder generated by p_i. Let $f \in R[x_1, ..., x_g]$. Then*

(a). $f > 0$ on $\mathcal{S}$ if and only if there are $s, t \in T$ satisfying $sf = 1 + t$;

(b). $f \geq 0$ on $\mathcal{S}$ if and only if there are $s, t \in T$ and an integer $N \geq 0$, with the property $sf = f^{2N} + t$;

(c). $f = 0$ on $\mathcal{S}$ if and only if there exists an integer $N \geq 0$ with the property $-f^{2N} \in T$.

We derive a few particular consequences. For instance, the following real Nullstellensatz is contained in the preceding result: assume that $p, q \in R[x]$ and

$$(p(x) = 0) \ \Rightarrow (q(x) = 0).$$

Then point (c) applies to $f = q$ and $p_1 = -p^2$. We infer: there exists $N \geq 0$ such that

$$-q^{2N} = s_1 - p^2 s_2, \quad s_1, s_2 \in \Sigma^2 R[x],$$

therefore:

$$q^{2N} + s \in (p), \quad s \in \Sigma^2 R[x],$$

where (p) denotes the ideal generated by p. Obviously, if the latter condition holds, then q vanishes on the zero set of p. Thus, we have proved:

$$[(p(x) = 0) \ \Rightarrow (q(x) = 0)] \Leftrightarrow [\exists (N \geq 0, \ s \in \Sigma^2 R[x]) : \ q^{2N} + s \in (p)].$$

Variants of this are pleasurable, and we suggest as an exercise the reader repeat the above but take $f = q$ and $p_1 = p, \ p_2 = -p$.

As another example, assume that $p(x) \geq 0$ for all $x \in R^g$. Then the theorem applies to $f = p$ and $p_1 = 1$ and we obtain: there exists an integer $N \geq 0$ and elements $s_1, s_2 \in \Sigma^2 R[x]$, such that

$$s_1 p = p^{2N} + s_2.$$

In particular,

$$s_1^2 p \in \Sigma^2 R[x],$$

which is exactly the conclusion of Artin's Theorem.

The concepts of (pre)ordering and quadratic module can be defined for an arbitrary commutative ring with unit; these, together with the important construct of the real spectrum provide the natural framework for developing modern real algebra and real algebraic geometry. For the general versions of the Sätze outlined in this section the reader can consult as a guide [S03], and for complete details and ramifications, the monographs [BCR98, PD01].

6. Applications of semi-algebraic geometry

The prospect of applying semi-algebraic geometry to a variety of areas is the cause of excitement in many communities; and we list a few of them here.

6.1. **Global optimization of polynomials.** An exciting turn in the unfolding of real algebraic geometry are applications to optimization. To be consistent with the non-commutative setting of the subsequent sections we denote below by $x \in \mathbb{R}^g$ a generic point in Euclidean space, and in the same time the g-tuple of indeterminates in the polynomial algebra.

6.1.1. *Minimizing a Polynomial on* $\mathbb{R}^g$. A classical question is: given a polynomial $q \in \mathbb{R}[x]$, find

$$\min_{x \in \mathbb{R}^g} q(x)$$

and the minimizer x^{opt}. The goal is to obtain a numerical solution to this problem and it is daunting even in a modest dimension such as $g = 15$. Finding a local optimum is numerically "easy" using the many available variations of gradient descent and Newton's method. However, polynomials are notorious for having many many local minima.

A naive approach is to grid $\mathbb{R}^g$, lets say with 64 grid points per dimension (a fairly course grid), and compare values of q on this grid. This requires $64^{15} \sim 10^9 10^7$ function evaluations or something like 10,000 hours to compute. Such prohibitive requirements occur in many high dimensional spaces and go under the heading of the "curse of dimensionality".

The success of sums of squares and Positivstellensätze methods rides on the heels of semi-definite programming, a subject which effectively goes back a decade and a half ago, and which effectively allows numerical computation of a sum of squares decomposition of a given polynomial q. The cost of the computation is determined by the number of terms of the polynomial q and is less effected by the number g of variables and the degree of q. To be more specific, this approach to optimization consists of starting with a number q^{**} and numerically solve

$$q - q^{**} = s,$$

for $s \in \Sigma^2$. If this is possible, lower q^{**} according to some algorithm and try again. If not, raise q^{**} and try again. Hopefully, one obtains q^{*o} at the transition (between being possible to write $q - q^{**}$ as a sums of squares and not) and obtains

$$q - q^{*o} \in \Sigma^2$$

and conclude that this is an optimum. This method was proposed first by Shor [S87] and subsequently refined by Lasserre [L01] and by Parrilo [ParThesis].

Parrilo and Sturmfels [PS03] reported experiments with a special class of 10,000 polynomials for which the true global minimum could be computed explicitly. They found in all cases that q^{*o} determined by sums of squares optimization equals the true minimum.

Theoretical evidence supporting this direction is the following observation, see [BCR98] §9.

Theorem 6.1. *Given a polynomial $q \in \mathbb{R}[x]$, the following are equivalent:*

(1) $q \geq 0$ on the cube $[-1,1]^g$.

(2) For all $\varepsilon > 0$, there is $s \in \Sigma^2$ such that

$$\|q - s\|_{L^1([-1,1]^g)} < \varepsilon.$$

A refinement of this result was recently obtained by Lasserre and Netzer [LN06]. Namely, the two authors prove that an additive, small perturbations with a fixed polynomial, produces a sum of squares which is close to the original polynomial in the L^1 norm of the coefficients. We reproduce, without proofs, their main result.

Theorem 6.2. [LN06] *Let $p \in \mathbb{R}[x_1, ..., x_g]$ be a polynomial of degree d, and let*

$$\Theta_r = 1 + x_1^{2r} + ... + x_g^{2r},$$

where $r \geq d/2$ is fixed. Define

$$\epsilon_r^* = \min_L \{L(p);\ L \in \mathbb{R}_{2r}[x_1, ..., x_g]',\ L(\Theta_r) \leq 1,\ L|_\Sigma^2 \geq 0\}.$$

Then $\epsilon_r^* \leq 0$ and the minimum is attained. The polynomial

$$p_{\epsilon,r} = p + \epsilon\Theta_r$$

is a sum of squares if and only if $\epsilon \geq -\epsilon_r^*$.

Moreover, if the polynomial p is non-negative on the unit cube $[-1,1]^g$, then $\lim_{r\to\infty} \epsilon_r^* = 0$.

Variations of the above theorem, with supports on semi-algebraic sets, relevant examples and an analysis of the degree bounds are contained in the same article [LN06].

For quite a few years by now, Lasserre has emphasized the tantamount importance of such perturbation results for optimization using sums of squares (henceforth abbreviated SOS) methods, see [L01], in that it suggests that determining if a given p is nonnegative on a bounded region by computing a sums of squares has a good probability of being effective.

We shall not prove the stated perturbation results, but remark that a free algebra version of them holds, [KS05].

In the opposite pessimistic direction there are the precise computations of Choi-Lam-Reznick (see [R92]) and a recent result due to Bleckermann [Blec04].

As a backup to the above optimization scheme, if a $q - q^{*o} \in \Sigma^2$ fails to be a sum of squares, then one can pick a positive integer m and attempt to solve

$$(1 + |x|^2)^m (q - q^{*o}) \in \Sigma^2.$$

Reznick's Theorem [R95] tells us that for some m this solves the optimization problem exactly, provided the highest degree homogeneous part of q is strictly positive. Engineers call using the term with some non zero m "relaxing the problem", but these days they call most modifications of almost anything a "relaxation" .

6.1.2. *Constrained optimization.* Now we give Jean Lasserre's interpretation of Theorem 3.9. Let $\mathcal{P}$ denote a collection of polynomials. The standard constrained optimization problem for polynomials is:

$$\textit{minimize } q(x) \textit{ subject to } x \in \mathcal{D}_{\mathcal{P}} := \{x \in \mathbb{R}^g; p(x) \geq 0,\ p \in \mathcal{P}\}.$$

Denote the minimum value of q by q^{opt}. We describe the idea when $\mathcal{P}$ contains but one polynomial p. Assume $\nabla p(x)$ does not vanish for $x \in \partial \mathcal{D}_p$.

The standard first order necessary conditions for $x^{opt} \in \partial \mathcal{D}_{\mathcal{P}}$ to be a local solution to this problem is

$$\nabla q(x^{opt}) = \lambda \nabla p(x^{opt})$$

with $\lambda > 0$. We emphasize, this is a local condition and λ is called the Lagrange multiplier.

Now we turn to analyzing the global optimum. Suppose that q can be expressed in the form:

$$q - q^{**} = s_1 + s_2 p, \quad s_{1,2} \in \Sigma^2,$$

which implies $q(x) \geq q^{**}$ for all $x \in \mathcal{D}_p$. So q^{**} is a lower bound. This is a stronger form of the Positivstellensatz than is always true. Then this optimistic statement can be interpreted as a global optimality condition when $q^{**} = q^{opt}$. Also it implies the classical Lagrange multiplier linearized condition, as we now see. At the *global* minimum x^{opt} we have

$$0 = q(x^{opt}) - q^{opt} = s_1(x^{opt}) + s_2(x^{opt})p(x^{opt})$$

which implies $0 = s_1(x^{opt})$ and, since s_1 is a sum of squares, we get $\nabla s_1(x^{opt}) = 0$. Also $s_2(x^{opt} = 0$, $\nabla s_2(x^{opt}) = 0$ whenever $p(x^{opt}) \neq 0$. Calculate

$$\nabla q \;=\; \nabla s_1 + p \nabla s_2 + s_2 \nabla p.$$

If $p(x^{opt}) = 0$, we get

$$\nabla q(x^{opt}) \;=\; s_2(x^{opt}) \nabla p(x^{opt})$$

and if $p(x^{opt}) \neq 0$ we get $\nabla q(x^{opt}) = 0$, the classic condition for an optimum in the interior. Set $\lambda = s_2(x^{opt})$ to get $\lambda \nabla p(x^{opt}) = \nabla q(x^{opt})$ the classic Lagrange multiplier condition as a (weak) consequence of the Positivstellensatz.

The reference for this and more general (finitely many p_j in terms of the classical Kuhn-Tucker optimality conditions) is [L01] Proposition 5.1.

Also regarding constrained optimization we mention that, at the technical level, the method of moments has re-entered into polynomial optimization. Quite specifically, Lasserre and followers are relaxing the original problem

$$\min_{x \in \mathcal{D}} q(x)$$

as

$$\min_{\mu} \int_{\mathcal{D}} q d\mu,$$

where the minimum is taken over all probability measures supported on $\mathcal{D}$. They prove that it is a great advantage to work in the space of moments (as free coordinates), see [HL05, L01, L04].

6.2. **Primal-dual optimality conditions.** In this section we explain in more detail Lasserre's point of view [L00, L01] of linearizing polynomial optimization via sums of squares decompositions and via moment data seen as independent variables.

Specifically, we start with a polynomial $f \in \mathbb{R}[x]$ and seek values of the scalar λ for which $f - \lambda \in \Sigma^2$. To this aim we consider a variable linear functional $L \in \mathbb{R}[x]'$ and denote the corresponding moments

$$y_\alpha = L(x^\alpha), \quad |\alpha| \leq 2d.$$

The integer d is fixed throughout the whole section and will not explicitly appear in all coming formulas. We denote $y = (y_\alpha)_{|\alpha| \leq 2d}$ and consider the associated Hankel matrix

$$M_y = (y_{\alpha+\beta})_{|\alpha|,|\beta| \leq d}.$$

In all these considerations it is important to fix an ordering (such as the graded lexicographic one) on the multi-indices α. Let

$$V(x) = (1, x_1, \cdots, x_g, x_1^2, \cdots)$$

be the "tautological" vector consisting of all monomials of degree less than or equal to d. Let $\alpha_0 = (0, 0, \cdots, 0)$, so $x^{\alpha_0} = 1$.

The matrix valued polynomial

$$\sum_\alpha B_\alpha x^\alpha = V(x) \cdot V(x)^T,$$

produces a sequence of matrix coefficients B_α which carry a Hankel type structure and whose entries are either 0 or 1. Write

$$f = \sum_\alpha f_\alpha x^\alpha.$$

The next lemma equates "minimization" of a polynomial via a sum of squares to a matrix problem.

Lemma 6.3. *The degree d polynomial $f - \lambda$ is a sum of squares with r squares of polynomials each having degree $\leq n$ if and only if there exists a positive semi-definite matrix $\Omega \in \mathbb{R}^{n\times n}$ of rank r such that*

$$\text{tr}(B_\alpha \Omega) = \ f_\alpha - \lambda \delta_{\alpha_0} \quad for\ all\ |\alpha| \leq d. \tag{6.1}$$

Proof. Write the symmetric positive semi-definite $n\times n$ matrix $\Omega = \sum_j^r q_j q_j^T$ were r is the rank of Ω and $q_j \in \mathbb{R}^n$. Then

$$\text{tr}(\sum_\alpha B_\alpha x^\alpha \Omega) = \ \sum_\alpha f_\alpha x^\alpha - \lambda \tag{6.2}$$

which gives

$$\text{tr}(\ V(x)V(x)^T \sum_j^r q_j q_j^T) = f(x) - \lambda$$

and

$$\sum_j q_j^T V(x) V(x)^T q_j = f(x) - \lambda$$

so we obtain

$$\sum_j Q_j^T(x) Q_j(x) = f(x) - \lambda$$

where $Q_j(x) := V(x)^T q_j$. The argument reverses, so f a sum of squares implies $\Omega = \sum_j^r q_j q_j^T$ which makes Ω positive semi-definite.■

Clearly, there are many matrices $\Omega \geq 0$ satisfying (6.1). A canonical choice, appearing in the next Lemma, was proposed by Nesterov.

Lemma 6.4. *Suppose there is a positive definite solution to (6.1), then one of them $\breve{\Omega}$ has inverse, $\breve{\Omega}^{-1}$ which is a Hankel matrix.*

Proof. We show that $\breve{\Omega}$ is the "maximum entropy solution" to (6.1), namely, $\breve{\Omega}$ is the (unique) solution to

$$\max_\Omega \ \ln\ \det\ \Omega \qquad \text{subject to (6.1)}.$$

It is standard that if a positive definite Ω exists maximizing entropy, then it "keeps its eigenvalues positive", so $\breve{\Omega}$ is positive definite. We use the standard formula

$$\frac{d\ \ln\ \det(\breve{\Omega} + t\Delta)}{dt}|_{t=0} = -\text{tr}[\breve{\Omega}^{-1}\Delta]$$

which is 0 for all Δ satisfying $\mathrm{tr}[B_\alpha \Delta] = 0$ for all α. Now $\mathrm{tr}[\breve{\Omega}^{-1}\Delta] = 0$ says that $\breve{\Omega}^{-1}$ is in the orthogonal complement of the orthogonal complement of *span* $\{B_\alpha\}$, thus $\breve{\Omega}^{-1} \in$ *span* $\{B_\alpha\}$, in other words, it is a Hankel matrix. ■

Our next step is the minimization problem:

$$\hat{L}_f := \min_L L(f) \quad \text{subject to } L(\Sigma^2) \geq 0 \text{ and } L(1) = 1.$$

Clearly,

$$\hat{\lambda}_f \leq \min_\mu \int f d\mu = \min_x f(x)$$

as μ ranges over all probability measure. If the minimum is attained, Dirac's measure at the optimal x yields $\hat{\lambda}_f = \min_x f(x)$.

Since $L = L_y$ corresponding to the moment sequence y satisfies $L_y(f) = \sum_\alpha f_\alpha y_\alpha$ our basic "primal" problem is:

$$(PRIMAL) \qquad \min_y \sum_\alpha f_\alpha y_\alpha \quad \text{subject to} \quad M_y \geq 0 \;\; y_{\alpha_0} = 1.$$

The "dual" problem is

$$(DUAL) \qquad \max \lambda \quad \text{subject to} \quad \Omega \geq 0 \quad \text{and} \quad \mathrm{tr}(B_\alpha \Omega) = f_\alpha - \lambda \delta_{\alpha_0}$$

for all α which we saw in Lemma 6.3 as solving the sum of squares problem is the same as

$$\hat{\lambda}_\Sigma := \max \lambda \quad \text{subject to} \quad f - \lambda \text{ is in } \Sigma.$$

To derive that these problems are indeed dual to each other define a "Lagrangian" by

$$\mathcal{L}(y, \Omega, \lambda) := \sum_\alpha f_\alpha y_\alpha - \mathrm{tr}(\sum_\alpha y_\alpha B_\alpha \Omega) - (y_{\alpha_0} - 1)\lambda$$

on the set $M_y \geq 0, \Omega \geq 0, \lambda \in \mathbb{R}$. For convenience write

$$\mathcal{L}(y, \Omega, \lambda) := \sum_\alpha [(f_\alpha - \mathrm{tr}(\sum_\alpha B_\alpha \Omega) - \delta_{\alpha_0}\lambda] y_\alpha - y_{\alpha_0}\lambda. \tag{6.3}$$

Then

$$\min_{M_y \geq 0} \max_{\Omega \geq 0, \lambda} \mathcal{L}(y, \Omega, \lambda) = \min_{M_y \geq 0} \sum_\alpha f_\alpha y_\alpha \;\text{ if } \;\; y_{\alpha_0} = 1 \quad (\text{is} + \infty \text{ if } y_{\alpha_0} \neq 1).$$

which is the primal problem. Next

$$\max_{\Omega\geq 0,\lambda}\ \min_{M_y\geq 0}\ \mathcal{L}(y,\Omega,\lambda) = \max\lambda \ \text{ if }\ f_\alpha - \mathrm{tr}(B_\alpha\Omega) - \lambda\delta_{\alpha_0} = 0 \ \ (\text{is } -\infty \text{ otherwise}).$$

which is the dual problem. We summarize with

Lemma 6.5. *For Ω, M_y in $\mathbb{R}^{n\times n}$, define sets*

$$\mathcal{N} := \{\Omega \geq 0, \lambda \in \mathbb{R} :\ f_\alpha - \mathrm{tr}(B_\alpha\Omega) - \lambda\delta_{\alpha_0} = 0 \ \ \textit{all}\ |\alpha| \leq d\}$$

$$\mathcal{M} := \{y :\ M_y \geq 0,\ y_{\alpha_0} = 1\}.$$

Then $\mathcal{N}$ is not empty for some dimension n if and only if $f - \lambda$ is a sum of squares and we have

$$\hat{\lambda}_\Sigma := \max_{\Omega,\lambda\in\mathcal{N}} \lambda = \max_{\Omega,\lambda\in\mathcal{N}}\min_{y\in\mathcal{M}} \mathcal{L}(y,\Omega,\lambda)$$

$$\leq \min_{y\in\mathcal{M}}\ \max_{\Omega,\lambda\in\mathcal{N}} \mathcal{L}(y,\Omega,\lambda) = \min_{y\in\mathcal{M}}\sum_\alpha f_\alpha y_\alpha =: \hat{\lambda}_f$$

A saddle point $\hat{\Omega}, \hat{\lambda}, \hat{y}$ is defined as one which satisfies

$$\hat{\lambda}_\Sigma = \mathcal{L}(\hat{y},\hat{\Omega},\hat{\lambda}) = \hat{\lambda}_f.$$

The lemma verifies our claim that the problems PRIMAL and DUAL are dual with respect to each other. Also if they have a saddle point they have the same optimal value $\hat{\lambda} := \hat{\lambda}_\Sigma = \hat{\lambda}_f$. Existence of a saddle point, because of the bilinearity of $\mathcal{L}$, is in the perview of the von Neumann Minmax Theorem, but we do not discuss this here. We refer the reader to [L00, L01, HL06] for further details. Although their approach is a bit different.

Now we make a few remarks. Firstly, the saddle point condition

$$\hat{\lambda} = \mathcal{L}(\hat{y},\hat{\Omega},\hat{\lambda}) = \sum_\alpha f_\alpha\hat{y}_\alpha - \mathrm{tr}(\sum_\alpha \hat{y}_\alpha B_\alpha\hat{\Omega}) - (\hat{y}_{\alpha_0} - 1)\hat{\lambda}$$

reduces to $0 = \mathrm{tr}(\sum_\alpha \hat{y}_\alpha B_\alpha\hat{\Omega}) = tr(M_{\hat{y}}\hat{\Omega})$. Since $M_{\hat{y}}, \hat{\Omega}$ are both positive semi-definite, this forces the "complementarity" of an optimal moment matrix and an optimal sum of squares representor $\hat{\Omega}$.

$$M_{\hat{y}}\hat{\Omega} = 0. \tag{6.4}$$

Secondly, an equation "balanced" between primal and dual is

$$tr(M_y\Omega) = L_y(f) - \lambda.$$

We will continue this line of thought in a separate article.

6.3. **Engineering.** For nonlinear systems, sum of squares techniques can be used to find Lyapunov functions by direct computation. Here the problem is to check if a differential equation

$$\frac{dx}{dt} = a(x)$$

on $\mathbb{R}^g$ is stable. The most common technique is to seek a function $V > 0$ except $V(0) = 0$ satisfying the differential inequality

$$\nabla V(x) \cdot a(x) \leq 0 \quad \text{for all } x \; in \; \mathbb{R}^g;$$

such V are called *Lyapunov functions* . If a is a vector field with polynomial entries, it is natural to seek V which is a sum of squares of polynomials, and this reduces to a semi-definite program. Solution can be attempted numerically and if successful produces V a Lyapunov function; if not one can modify the sum of squares from polynomials to some rational sum of squares and try again, see [ParThesis]. Such techniques lay out numerical solutions to midsized (dimension 8 or so) nonlinear control problems.

More generally for control problems one seeks to find a feedback law $u = k(x)$ which stabilizes

$$\frac{dx}{dt} = a(x) + b(x)u.$$

Here there is a function V, beautifully tamed by E. Sontag, called a "control Lyapunov function" generalizing the classical Lyapunov function. Unfortunately, no Positivstellensätz technique is known for for finding V. However, A. Rantzer cleverly introduced a "dual control Lyapunov function" and in [PPR04] showed that it is quite amenable to sum of squares techniques.

Another direction gives a generalization of the classical S-procedure which finds performance bounds on broad classes of problems, see [ParThesis]. There are also applications to combinatorial problems described there.

Recently, [HL06] have given a technique for converting system engineering problems to polynomial minimization. The wide scope of the technique is very appealing.

7. Linear matrix inequalities and computation of sums of squares

Numerical computation of a sum of squares and a Positivstellensatz is based on a revolution which started about 20 years ago in optimization; the rise of interior point methods. We avoid delving into yet another topic but mention the special aspects concerning us. Thanks to the work of Nesterov and Nemirovskii in the early 1990s one can solve Linear Matrix Inequalities (LMIs in short) numerically using interior point optimization methods, called *semi-definite programming* . An LMI is an inequality of the form

$$A_0 + A_1 x_1 + \cdots A_g x_g \geq 0 \tag{7.1}$$

where the A_j are symmetric matrices and the numerical goal is to compute $x \in \mathbb{R}^g$ satisfying this. The sizes of matrix unknowns treatable by year 2006 solvers exceed 100×100; with special structure dimensions can go much higher. This is remarkable because our LMI above has about $5000g$ unknowns.

7.1. SOS and LMIs.

Sum of squares and Positivstellensätze problems convert readily to LMIs and these provide an effective solution for polynomials having modest number of terms. These applications make efficiencies in numerics a high priority. This involves shrewd use of semi-algebraic theory and computational ideas to produce a semi-definite programming package, for a recent paper see [1]; also there is recent work of L. Vandenberghe. Semi-algebraic geometry packages are: SOS tools [PPSP04] and GloptiPoly [HL03].

A lament is that all current computational semi-algebraic geometry projects use a packaged semi-definite solver, none write their own. This limits efficiencies for sum of squares computation.

Special structure leads to great computational improvement as well as elegant mathematics. For example, polynomials which are invariant under a group action, the delight of classical invariant theory, succumb to rapid computation, see [GP04] [CKSprept].

7.2. **LMIs and the world.** LMIs have a life extending far beyond computational sum of squares and are being found in many areas of science. Later in this paper §10 we shall glimpse at their use in systems engineering, a use preceding sum of squares applications by 10 years. The list of other areas includes statistics, chemistry, quantum computation together with more; all to vast for us to attempt description.

A paradigm mathematical question here is:

Which convex sets $\mathcal{C}$ in $\mathbb{R}^g$ with algebraic boundary can be represented with some monic LMI?

That is,

$$\mathcal{C} = \{x \in \mathbb{R}^g : \ I + A_1 x_1 + \cdots A_g x_g \geq 0\},$$

where A_j are symmetric matrices. Here we have assumed the normalization $0 \in \mathcal{C}$. This question was raised by Parrilo and Sturmfels [PS03]. The paper [HVprept] gives an obvious necessary condition [2] on $\mathcal{C}$ for an LMI representation to exist and proves sufficiency when $g = 2$.

The main issue is that of determinantal representations of a polynomial $p(x)$ on $\mathbb{R}^g$, namely, given p express it in the form

$$p(x) = det(A_0 + A_1 x_1 + \cdots A_g x_g). \tag{7.2}$$

That this is possible for some matrices is due to the computer scientist Leslie Valiant [Val79]. That the matrices can be taken real and symmetric is in [HMVprept] as is the fact the a representation of det $p(X)$ always holds for polynomials in non-commuting (free) variables, as later appear in §8. A

[2]This is in contrast to the free algebra case where all evidence (like that in this paper) indicates that convexity is the only condition required.

symbolic computer algorithm due to N. Slinglend and implemented by J. Shopple runs under the Mathematica package NCAlgebra.

The open question is which polynomials can we represent monicaly; that is with $A_0 = I$. Obviously, necessary is the *real zero condition* , namely,

the polynomial $f(t) := p(tx)$ *in one complex variable* t
has only real zeroes,

but what about the converse? When $g = 2$ the real zero condition on p insures that it has a monic representation; this is the core of [HVprept].

What about higher dimensions? Lewis, Parrilo and Ramana [LPR05] showed that this $g = 2$ result (together with a counterexample they concocted) settles a 1958 conjecture of Peter Lax, which leads to the surmise that sorting out the $g > 2$ situation may not happen soon. Leonid Gurvitz pointed out the Valient connection to functional analysts and evangelizes that monic representations have strong implications for lowering the complexity of certain polynomial computations.

8. Non-commutative algebras

A direction in semi-algebraic geometry, recently blossoming still with many avenues to explore, concerns variables which do not commute. As of today versions of the strict Positivstellensätze we saw in §5.4 are proved for a free $*$- algebra and for the enveloping algebra of a Lie algebra; here the structure is cleaner or the same as in the classical commutative theory. The verdict so far on noncommutative Nullstellensätze is mixed. In a free algebra it goes through so smoothly that no radical ideal is required. This leaves us short of the remarkable perfection we see in the Stengle -Tarski - Seidenberg commutative landscape. Readers will be overjoyed to hear that the proofs needed above are mostly known to them already: just as in earlier sections, non-negative functionals on the sums of squares cone in a $*$-algebra can be put in correspondence with tuples of non-commuting operators, and this carries most of the day.

This noncommutative semi-algebraic foundation underlies a rigid structure (at least) for free $*$-algebras which has recently become visible. A noncommutative polynomial p has second derivative p'' which is again a polynomial and if p'' is positive, then our forthcoming free $*$-algebra Positivstellensatz tells us that p'' is a sum of squares. It is a bizarre twist that this and the derivative structure are incompatible, so together imply that a "convex polynomial" in a free $*$- algebra has degree 2 or less; see §9. The authors suspect that this is a harbinger of a very rigid structure in a free $*$-algebra for "irreducible varieties" whose curvature is either nearly positive or nearly negative; but this is a tale for another (likely distant) day. Some of the material in this section on higher derivatives and the next is new.

A final topic on semi-algebraic geometry in a free $*$- algebra is applications to engineering, §10. Arguably the main practical development in systems and control through the 1990's was the reduction of linear systems problems to Linear Matrix Inequalities, LMIs. For theory and numerics to be highly successful something called "Convex Matrix Inequalities", henceforth denoted in short CMIs, will do nicely. Most experts would guess that the class of problems treatable with CMIs is much broader than with LMIs. But no, as we soon see, our draconian free $*$ convexity theorems suggest that for systems problems fully characterized by performance criteria based on L^2 and signal flow diagrams (as are most textbook classics), convex matrix inequalities give no greater generality than LMIs.

These systems problems have the key feature that their statement does not depend on the dimension of the systems involved. Thus we summarize our main engineering contention:

Dimension free convex problems are equivalent to an LMI

This and the next sections tells the story we just described but there is a lot it does not do. Our focus in this paper has been on inequalities, where various noncommutative equalities are of course a special and often well developed case. For example, algebraic geometry based on the Weyl algebra and corresponding computer algebra implementations, for example, Gröbner basis generators for the Weyl algebra are in the standard computer algebra packages such as Plural/Singular.

A very different and elegant area is that of rings with a polynomial identity, in short PI rings , e.g. $N \times N$ matrices for fixed N. While most PI research concerns identities, there is one line of work on polynomial inequalities, indeed sums of squares, by Procesi-Schacher [PS76]. A Nullstellensatz for PI rings is discussed in [Amit57].

8.1. **Sums of squares in a free $*$-algebra.** Let $\mathbb{R}\langle x, x^*\rangle$ denote the polynomials with real numbers as coefficients in variables $x_1, ..., x_g, x_1^*, ..., x_g^*$. These variables do not commute, indeed they are free of constraints other than * being an anti-linear involution:

$$(fq)^* = q^* f^*, \qquad (x_j)^* = x_j^*.$$

Thus $\mathbb{R}\langle x, x^*\rangle$ is called the *real free* $*-$ *algebra* on generators x, x^*.

Folklore has it that analysis in a free $*$-algebra gives results like ordinary commutative analysis in one variable. The SoS phenomenon we describe in this section is consistent with this picture, but convexity properties in the next section do not. Convexity in a free algebra is much more rigid.

We invite those who work in a free algebra (or their students) to try NCAlgebra, the free free-$*$ algebra computer package [HSM05]. Calculations with it had a profound impact on the results in §8 and 9; it is a very powerful tool.

The cone of sums of squares is the convex hull:

$$\Sigma^2 = \mathrm{co}\{f^* f;\ f \in \mathbb{R}\langle x, x^*\rangle\}.$$

A linear functional $L \in \mathbb{R}\langle x, x^*\rangle'$ satisfying $L|_{\Sigma^2} \geq 0$ produces a positive semidefinite bilinear form

$$\langle f, q\rangle = L(q^* f)$$

on $\mathbb{R}\langle x, x^*\rangle$. We use the same construction introduced in section 3, namely, mod out the null space of $\langle f, f\rangle$ and denote the Hilbert space completion by H, with $\mathcal{D}$ the dense subspace of H generated by $\mathbb{R}\langle x, x^*\rangle$. The separable Hilbert space H carries the multiplication operators $M_j : \mathcal{D} \longrightarrow \mathcal{D}$:

$$M_j f = x_j f, \quad f \in \mathcal{D},\ 1 \leq j \leq n.$$

One verifies from the definition that each M_j is well defined and

$$\langle M_j f, q\rangle = \langle x_j f, q\rangle = \langle f, x_j^* q\rangle, \quad f, q \in \mathcal{D}.$$

Thus $M_j^* = M_{x_j^*}$. The vector 1 is still $*$-cyclic, in the sense that the linear span $\vee_{p\in\mathbb{R}\langle x,x^*\rangle} p(M, M^*)1$ is dense in H. Thus, *mutatis mutandis*, we have obtained the following result.

Lemma 8.1. *There exists a bijective correspondence between positive linear functionals, namely*

$$L \in \mathbb{R}\langle x, x^*\rangle' \quad \textit{and} \quad L_{|\Sigma^2} \geq 0,$$

and g-tuples of unbounded linear operators T with a star cyclic vector ξ, established by the formula

$$L(f) = \langle f(T, T^*)\xi, \xi\rangle, \qquad f \in \mathbb{R}\langle x, x^*\rangle.$$

We stress that the above operators do not commute, and might be unbounded. The calculus $f(T, T^*)$ is the non-commutative functional calculus: $x_j(T) = T_j,\ x_j^*(T) = T_j^*$.

An important feature of the above correspondence is that it can be restricted by the degree filtration. Specifically, let $\mathbb{R}\langle x, x^*\rangle_k = \{f;\ \deg f \leq k\}$, and similarly, for a quadratic form L as in the lemma, let $\mathcal{D}_k$ denote the finite dimensional subspace of H generated by the elements of $\mathbb{R}\langle x, x^*\rangle_k$. Define also

$$\Sigma_k^2 = \Sigma^2 \cap \mathbb{R}\langle x, x^*\rangle_k.$$

Start with a functional $L \in \mathbb{R}\langle x, x^*\rangle'_{2k}$ satisfying $L|_{\Sigma_{2k}^2} \geq 0$. One can still construct a finite dimensional Hilbert space H, as the completion of $\mathbb{R}\langle x, x^*\rangle_k$ with respect to the inner product $\langle f, q\rangle = L(q^* f),\ f, q \in \mathbb{R}\langle x, x^*\rangle_k$. The multipliers

$$M_j : \mathcal{D}_{k-1} \longrightarrow H,\ M_j f = x_j f,$$

are well defined and can be extended by zero to the whole H. Let

$$N(k) = \dim \mathbb{R}\langle x, x^*\rangle_k = 1 + (2g) + (2g)^2 + \dots + (2g)^k = \frac{(2g)^{k+1} - 1}{2g - 1}.$$

In short, we have proved the following specialization of the main Lemma.

Lemma 8.2. *Let $L \in \mathbb{R}\langle x, x^*\rangle'_{2k}$ satisfy $L|_{\Sigma^2_{2k}} \geq 0$. There exists a Hilbert space of dimension $N(k)$ and an g-tuple of linear operators M on H, with a distinguished vector $\xi \in H$, such that*

$$L(p) = \langle p(M, M^*)\xi, \xi\rangle, \quad p \in \mathbb{R}\langle x, x^*\rangle_{2k-2}. \tag{8.1}$$

Following the pattern of the preceding section, we will derive now a Nichtnegativstellensatz.

Theorem 8.3 ([H02])**.** *Let $p \in \mathbb{R}\langle x, x^*\rangle_d$ be a non-commutative polynomial satisfying $p(M, M^*) \geq 0$ for all g-tuples of linear operators M acting on a Hilbert space of dimension at most $N(k)$, $2k \geq d + 2$. Then $p \in \Sigma^2$.*

Proof. The only necessary technical result we need is the closedness of the cone Σ^2_k in the Euclidean topology of the finite dimensional space $\mathbb{R}\langle x, x^*\rangle_k$. This is done as in the commutative case, using Carathédodory's convex hull theorem. More exactly, every element of Σ^2_k is a convex combination of at most $\dim \mathbb{R}\langle x, x^*\rangle_k + 1$ elements, and on the other hand there are finitely many positive functionals on Σ^2_k which separate the points of $\mathbb{R}\langle x, x^*\rangle_k$. See for details [HMP04a].

Assume that $p \notin \Sigma^2$ and let $k \geq (d+2)/2$, so that $p \in \mathbb{R}\langle x, x^*\rangle_{2k-2}$. Once we know that Σ^2_{2k} is a closed cone, we can invoke Minkowski separation theorem and find a functional $L \in \mathbb{R}\langle x, x^*\rangle'_{2k}$ providing the strict separation:

$$L(p) < 0 \leq L(f), \quad f \in \Sigma^2_{2k}.$$

According to Lemma 8.2 there exists a tuple M of operators acting on a Hilbert space H of dimension $N(k)$ and a vector $\xi \in H$, such that

$$0 \leq \langle p(M, M^*)\xi, \xi\rangle = L(p) < 0,$$

a contradiction. ■

When compared to the commutative framework, this theorem is stronger in the sense that it does not assume a strict positivity of p on a well chosen "spectrum". Variants with supports (for instance for spherical tuples M : $M_1^*M_1 + ... + M_g^*M_g \leq I$) of the above result are discussed in [HMP04a].

We state below an illustrative and generic result, from [HM04a], for sums of squares decompositions in a free $*$-algebra.

Theorem 8.4. *Let $p \in \mathbb{R}\langle x, x^*\rangle$ and let $q = \{q_1, ..., q_k\} \subset \mathbb{R}\langle x, x^*\rangle$ be a set of polynomials, so that the non-commutative quadratic module*

$$QM(q) = \text{co}\{f^* q_k f;\ f \in \mathbb{R}\langle x, x^*\rangle,\ 0 \le i \le k\},\ q_0 = 1,$$

contains $1 - x_1^* x_1 - ... - x_g^* x_g$. If for all tuples of linear bounded Hilbert space operators $X = (X_1, ..., X_g)$ subject to the conditions

$$q_i(X, X^*) \ge 0,\ 1 \le i \le k,$$

we have

$$p(X, X^*) > 0,$$

then $p \in QM(q)$.

Notice that the above theorem covers relations of the form $r(X, X^*) = 0$, the latter being assured by $\pm r \in QM(q)$. For instance we can assume that we evaluate only on commuting tuples of operators, in which situation all commutators $[x_i, x_j]$ are included among the (possibly other) generators of $QM(q)$.

Some interpretation is needed in degenerate cases, such as those where no bounded operators satisfy the relations $q_i(X, X^*) \ge 0$, for example, if some of q_i are the defining relations for the Weyl algebra; in this case, we would say $p(X, X^*) > 0$, since there are no X. Indeed $p \in QM(q)$ as the theorem says.

Proof Assume that p does not belong to the convex cone $QM(q)$. Since the latter is archimedean, by the same Minkovski principle there exists a linear functional $L \in \mathbb{R}\langle x, x^*\rangle'$, such that

$$L(p) \le 0 \le L(f), \quad f \in QM(q).$$

Define the Hilbert space H associated to L, and remark that the left multipliers M_{x_i} on $\mathbb{R}\langle x, x^*\rangle$ give rise to linear bounded operators (denoted by the same symbols) on H. Then

$$q_i(M, M^*) \geq 0, \quad 1 \leq i \leq k,$$

by construction, and

$$\langle p(M, M^*)1, 1\rangle = L(p) \leq 0,$$

a contradiction.

The above statement allows a variety of specialization to quotient algebras. Specifically, if I denotes a bilateral ideal of $\mathbb{R}\langle x, x^*\rangle$, then one can replace the quadratic module in the statement with $QM(q) + I$, and separate the latter convex cone from the potential positive element on the set of tuples of matrices X satisfying simultaneously

$$q_i(X, X^*) \geq 0, \; 0 \leq i \leq k, \quad f(X) = 0, \quad f \in I.$$

For instance, the next simple observation can also be deduced from the preceding theorem.

Corollary 8.5. *Let J be the bilateral ideal of $\mathbb{R}\langle x, x^*\rangle$ generated by the commutator polynomial $[x_1 + x_1^*, x_2 + x_2^*] - 1$. Then $J + QM(1 - x_1^*x_1 - ... - x_g^*x_g) = \mathbb{R}\langle x, x^*\rangle$.*

Proof Assume by contradiction that $J+QM(1-x_1^*x_1-...-x_g^*x_g) \neq \mathbb{R}\langle x, x^*\rangle$. By our basic separation lemma, there exists a linear functional $L \in \mathbb{R}\langle x, x^*\rangle'$ with the properties:

$$L_{J+QM(1-x_1^*x_1-...-x_g^*x_g)} \geq 0, \quad \text{and} \quad L(1) > 0.$$

Then the GNS construction will produce a tuple of linear bounded operators X, acting on the associated non-zero Hilbert space H, satisfying $X_1^*X_1 + ... + X_g^*X_g \leq I$ and

$$[X_1^* + X_1, X_2^* + X_2] = I.$$

The latter equation is however impossible, because the left hand side is anti-symmetric while the right hand side is symmetric and non-zero.

■

Similarly, we can derive following the same scheme the next result.

Corollary 8.6. *Assume, in the condition of the above Theorem, that* $p(X, X^*) > 0$ *for all* COMMUTING *tuples* X *of matrices subject to the positivity constraints* $q_i(X, X^*) \geq 0, 0 \leq i \leq k$. *Then*

$$p \in QM(q) + I,$$

where I *is the bilateral ideal generated by all commutators* $[x_i, x_j], [x_i, x_j]^*,\ 1 \leq i, j \leq g$.

With similar techniques (well chosen, separating, $*$-representations of the free algebra) one can prove a series of Nullstellensätze. We state for information one of them, see for an early version [HMP04b].

Theorem 8.7. *Let* $p_1(x), ..., p_m(x) \in \mathbb{R}\langle x \rangle$ *be polynomials not depending on the* x_j^* *variables and let* $q(x, x^*) \in \mathbb{R}\langle x, x^* \rangle$. *Assume that for every* g *tuple* X *of linear operators acting on a finite dimensional Hilbert space* H, *and every vector* $v \in H$, *we have:*

$$(p_j(X)v = 0,\ 1 \leq j \leq m) \ \Rightarrow\ (q(X, X^*)v = 0).$$

Then q *belongs to the left ideal* $\mathbb{R}\langle x, x^* \rangle p_1 + ... + \mathbb{R}\langle x, x^* \rangle p_m$.

Again, this proposition is stronger than its commutative counterpart. For instance there is no need of taking higher powers of q, or of adding a sum of squares to q.

We refer the reader to [HMP06] for the proof of Proposition 8.7. However, we say a few words about the intuition behind it. We are assuming

$$p_j(X)v = 0, \forall j \quad \Longrightarrow \quad q(X, X^*)v = 0.$$

On a very large vector space if X is determined on a small number of vectors, then X^* is not heavily constrained; it is almost like being able to take X^*

to be a completely independent tuple Y. If it were independent, we would have

$$p_j(X)v = 0, \forall j \quad \Longrightarrow \quad q(X,Y)v = 0.$$

Now, in the free algebra $\mathbb{R}\langle x, y\rangle$, it is much simpler to prove that this implies $q \in \sum_j^m \mathbb{R}\langle x, y\rangle\, p_j$, as required. We isolate this fact in a separate lemma.

Lemma 8.8. *Fix a finite collection $p_1, ..., p_m$ of polynomials in non-commuting variables $\{x_1, \ldots, x_g\}$ and let q be a given polynomial in $\{x_1, \ldots, x_g\}$. Let d denote the maximum of the* $\deg(q)$ *and* $\{\deg(p_j) : 1 \le j \le m\}$.

There exists a real Hilbert space $\mathcal{H}$ of dimension $\sum_{j=0}^d g^j$, such that, if

$$q(X)v = 0$$

whenever $X = (X_1, \ldots, X_g)$ is a tuple of operators on $\mathcal{H}$, $v \in \mathcal{H}$, and

$$p_j(X)v = 0 \text{ for all } j,$$

then q is in the left ideal generated by $p_1, ..., p_m$.

Proof (of Lemma). We sketch a proof based on an idea of G. Bergman, see [HM04a].

Let $\mathcal{I}$ be the left ideal generated by $p_1, ..., p_m$ in $F = \mathbb{R}\langle x_1, ..., x_g\rangle$. Define $\mathcal{V}$ to be the vector space $F/\mathcal{I}$ and denote by $[f]$ the equivalence class of $f \in F$ in the quotient $F/\mathcal{I}$.

Define X_j on the vector space $F/\mathcal{I}$ by $X_j[f] = [x_j f]$ for $f \in F$, so that $x_j \mapsto X_j$ implements a quotient of the left regular representation of the free algebra F.

If $\mathcal{V} := F/\mathcal{I}$ is finite dimensional, then the linear operators $X = (X_1, \ldots, X_g)$ acting on it can be viewed as a tuple of matrices and we have, for $f \in F$,

$$f(X)[1] = [f].$$

In particular, $p_j(X)[1] = 0$ for all j. If we do not worry about the dimension counts, by assumption, $0 = q(X)[1]$, so $0 = [q]$ and therefore $q \in \mathcal{I}$. Minus the precise statement about the dimension of $\mathcal{H}$ this establishes the result when $F/\mathcal{I}$ is finite dimensional.

Now we treat the general case where we do not assume finite dimensionality of the quotient. Let $\mathcal{V}$ and $\mathcal{W}$ denote the vector spaces

$$\mathcal{V} := \{[f] : \ f \in F, \ \deg(f) \le d\},$$

$$\mathcal{W} := \{[f] : \ f \in F, \ \deg(f) \le d-1\}.$$

Note that the dimension of $\mathcal{V}$ is at most $\sum_{j=0}^{d} g^j$. We define X_j on $\mathcal{W}$ to be multiplication by x_j. It maps $\mathcal{W}$ into $\mathcal{V}$. Any linear extension of X_j to the whole $\mathcal{V}$ will satisfy: if f has degree at most d, then $f(X)[1] = [f]$. The proof now proceeds just as in the part 1 of the proof above. ∎

With this observation we can return and finish the proof of Theorem 8.7 Since X^* is dependent on X, an operator extension with properties stated in the lemma below gives just enough structure to make the above free algebra Nullstellensatz apply; and we prevail.

Lemma 8.9. *Let $x = \{x_1, \ldots, x_m\}$, $y = \{y_1, \ldots, y_m\}$ be free, non-commuting variables. Let H be a finite dimensional Hilbert space, and let X, Y be two m-tuples of linear operators acting on H. Fix a degree $d \ge 1$.*

Then there exists a larger Hilbert space $K \supset H$, an m-tuple of linear transformations $\tilde{X}$ acting on K, such that

$$\tilde{X}_j|_H = X_j, \quad 1 \le j \le g,$$

and for every polynomial $q \in \mathbb{R}\langle x, x^* \rangle$ of degree at most d and vector $v \in H$,

$$q(\tilde{X}, \tilde{X}^*)v = 0 \ \Rightarrow q(X, Y)v = 0.$$

For the matrical construction in the proof see [HMP06].

We end this subsection with an example, see [HM04a].

Example 8.10. Let $p = (x^*x + xx^*)^2$ and $q = x + x^*$ where x is a single variable. Then, for every matrix X and vector v (belonging to the space where X acts), $p(X)v = 0$ implies $q(X)v = 0$; however, there does not exist a positive integer m and $r, r_j \in \mathbb{R}\langle x, x^* \rangle$, so that

$$q^{2m} + \sum r_j^* r_j = pr + r^* p. \tag{8.2}$$

Moreover, we can modify the example to add the condition $p(X)$ is positive semi-definite implies $q(X)$ is positive semi-definite and still not obtain this representation. $\square$

Proof Since $A := XX^* + X^*X$ is self-adjoint, $A^2v = 0$ if and only if $Av = 0$. It now follows that if $p(X)v = 0$, then $Xv = 0 = X^*v$ and therefore $q(X)v = 0$.

For $\lambda \in \mathbb{R}$, let

$$X = X(\lambda) = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

viewed as an operator on $\mathbb{R}^3$ and let $v = e_1$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$.

We begin by calculating the first component of even powers of the matrix $q(X)$. Let $Q = q(X)^2$ and verify,

$$Q = \begin{pmatrix} \lambda^2 & 0 & \lambda \\ 0 & 1+\lambda^2 & 0 \\ \lambda & 0 & 1 \end{pmatrix}. \tag{8.3}$$

For each positive integer m there exist a polynomial q_m so that

$$Q^m e_1 = \begin{pmatrix} \lambda^2(1+\lambda q_m(\lambda)) \\ 0 \\ \lambda(1+\lambda q_m(\lambda)) \end{pmatrix} \tag{8.4}$$

which we now establish by an induction argument. In the case $m = 1$, from equation (8.3), it is evident that $q_1 = 0$. Now suppose equation (8.4) holds for m. Then, a computation of $QQ^m e_1$ shows that equation (8.4) holds for $m + 1$ with $q_{m+1} = \lambda(q_m + \lambda + \lambda q_m)$. Thus, for any m,

$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} < Q^m e_1, e_1 >= \lim_{\lambda\to 0}(1 + \lambda q_m(\lambda)) = 1. \tag{8.5}$$

Now we look at p and get

$$p(X) = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & (1+\lambda^2)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\,(< r(X)^*p(X)e_1, e_1 > + < p(X)r(X)e_1, e_1 >) = 0.$$

If the representation of equation (8.2) holds, then apply $< \cdot\ e_1, e_1 >$ to both sides and take λ to 0. We just saw that the right side is 0, so the left side is 0, which because

$$< \sum r_j(X)^* r_j(X)e_1, e_1 > \geq 0$$

forces

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2} < Q^m e_1, e_1 > \leq 0$$

a contradiction to equation (8.5). Hence the representation of equation (8.2) does not hold.

The last sentence claimed in the example is true when we use the same polynomial p and replace q with q^2. ■

There are more Positivstellensätze in a free *-algebra which fill in more of the picture. The techniques proving them are not vastly beyond what we illustrated here. For example, Klep-Schweighofer [KS05] do an analog of Stengle's Theorem 5.3(a), while Theorem 3.9 is faithfully made free in [HM04a]. In spite of the above results we are still far from having a full understanding (à la Stengle's Theorem) of the Null- and Positiv-stellensätze phenomena in the free algebra.

8.2. **The Weyl algebra.** Weyl's algebra, that is the enveloping algebra of the Heisenberg group is interesting because, by a deep result of Stone-von Neumann, it has a single irreducible representation; and that is infinite dimensional. Thus, to check on the spectrum the positivity of an element, one has to do it at a single point. The details were revealed by Schmüdgen in a very recent article [S05]. We reproduce from his work the main result.

Fix a positive integer g and consider the unital $*$-algebra $W(g)$ generated by $2g$ self-adjoint elements $p_1, ..., p_g, q_1, ..., q_g$, subject to the commutation relations:

$$[p_k, q_j] = -\delta_{kj}(i \cdot 1), \quad [p_k, p_j] = [q_j, q_k] = 0, \quad 1 \leq j, k \leq g.$$

The unique irreducible representation Φ of this algebra is given by the partial differential operators

$$\Phi(p_k)f = -i\frac{\partial f}{\partial x_k}, \quad \Phi(q_k)f = x_k f,$$

acting on Schwartz space $\mathcal{S}(\mathbb{R}^g)$. Via this representation, the elements of $W(g)$ are identified with linear partial differential operators with polynomial coefficients (in g variables). These operators can be regarded as densely defined, closed graph operators from $\mathcal{S}(\mathbb{R}^g)$ to $L^2(\mathbb{R}^g)$. The set

$$W(g)_+ = \{f \in W(g); \ \ \langle \Phi(f)\xi, \xi\rangle \geq 0, \ \xi \in \mathcal{S}(\mathbb{R}^g)\}$$

consists of all symmetric, non-negative elements, with respect to the representation Φ.

Define

$$a_k = \frac{q_k + ip_k}{\sqrt{2}}, \quad a_{-k} = \frac{q_k - ip_k}{\sqrt{2}},$$

so that $a_k^* = a_{-k}$. Fix a positive number α which is not an integer, and let

$$N = a_1^* a_1 + ... + a_g^* a_g;$$

denote by $\mathcal{N}$ the set of all finite products of elements $N + (\alpha + n)1$, with $n \in \mathbb{Z}$.

The algebra $W(g)$ carries a natural degree, defined on generators as

$$\deg(a_k) = \deg(a_{-k}) = 1.$$

Every element $f \in W(g)$ can be decomposed into homogeneous parts f_s of degree s:

$$f = f_m + f_{m-1} + ... + f_0.$$

We can regard f_k as a homogeneous polynomial of degree k, in the variables $a_{\pm 1}, ..., a_{\pm g}$. The principal symbol of f is the polynomial $f_m(z_1, ..., z_g, \overline{z}_1, ..., \overline{z}_g)$, where a_j was substituted by z_k and a_{-k} by $\overline{z_k}$.

Theorem 8.11. [S05] *Let $f \in W(g)$ be a self-adjoint element of even degree $2m$, and let $P(z, \overline{z})$ be its principal symbol. If*

a). There exists $\varepsilon > 0$ such that $f - \varepsilon \cdot 1 \in W(g)_+$, *b). $P(z, \overline{z}) > 0$ for $z \neq 0$,*

then, if m is even there exists $b \in \mathcal{N}$ such that $bfb \in \Sigma^2 W(g)$; if m is odd, there exists $b \in \mathcal{N}$ such that $\sum_{j=1}^{g} ba_j f a_{-j} b \in \Sigma^2 W(g)$.

For examples and details see [S05].

Already mentioned and annotated was our serious omission of any description of the Nullstellensatz in a Weyl Algebra.

8.3. **Sums of squares modulo cyclic equivalence.** A still open, important conjecture in the classification theory of von Neumann algebras was recently reduced by F. Radulescu to an asymptotic Positivstellensatz in the free algebra. We reproduce from his preprint [Radul04] the main result. We do not explain below the standard terminology related to von Neumann algebras, see for instance [Tak02].

The following conjecture was proposed thirty years ago in [Connes76]:

Every type II_1 factor can be embedded into an ultraproduct of the hyperfinite factor.

There are presently quite a few reformulations or reductions of this conjecture. The one of interest for this survey can be formulated as follows.

Let $F = \mathbb{C}\langle x_1, ..., x_g \rangle$ be the free algebra with anti-linear involution $x_j^* = x_j,\ 1 \leq j \leq g$. We complete F to the algebra of convergent series

$$\hat{F} = \{\sum_w a_w w;\ \sum_w |a_w| r^{|w|} < \infty,\ \forall r > 0\},$$

where w runs over all words in F and $a_w \in \mathbb{C}$. The resulting Fréchet space $\hat{F}$ carries a natural weak topology denoted $\sigma(\hat{F}, \hat{F}^*)$.

A trace τ in a von-Neumann algebra M is a linear functional which has by definition the cyclic invariant property $\tau(a_1...a_n) = \tau(a_2 a_3 ... a_n a_1)$. Two series $f_1, f_2 \in \hat{F}$ are called *cyclically equivalent* if $f_1 - f_2$ is the weak limit of a linear combination of elements $w - w'$, where $w \in F$ is a word and w' is a cyclic permutation of it.

The following asymptotic Positivstellensatz holds.

Theorem 8.12. [Radul04] *Let $f \in \hat{F}$ be a symmetric series with the property that for every separable, type II_1 von Neumann algebra (M, τ) and every g-tuple of self-adjoint elements X of M we have $\tau(f(X)) \geq 0$. Then f is cyclically equivalent to a weak limit of sums of squares s_n, $s_n \in \Sigma^2 F$.*

It is not known whether one can replace the test II_1 algebras by finite dimensional algebras, but an answer to this querry would solve Connes conjecture.

Corollary 8.13. *Connes embedding conjecture holds if and only if for every symmetric element $f \in \hat{F}$ the following assertion holds:*

f is cyclically equivalent to a weak limit of sums of squares s_n, $s_n \in \Sigma^2 F$, if and only if for any positive integer d and g-tuple of self-adjoint $d \times d$ matrices X one has $\mathrm{trace} f(X) \geq 0$.

The proofs of Radulescu's theorem and the corollary follow the same pattern we are by now familiar with: a convex separation argument followed by a GNS construction. See for details [Radul04], and for a last minute refinement [KS06].

9. Convexity in a free algebra

Convexity of functions, domains and their close relative, positive curvature of varieties, are very natural notions in a $*$-free algebra. A shocking thing happens: these convex functions are so rare as to be almost trivial. This section illustrates a simple case, that of convex polynomials, and we see how in a free algebra the Nichtnegativtellensätze have extremely strong consequences for inequalities on derivatives. The phenomenon has direct qualitative consequences for systems engineering as we see in §10. The results of this section can be read independently of all but a few definitions in §8, and the proofs require only a light reading of it.

This time $\mathbb{R}\langle x \rangle$ denotes the free $*$-algebra in indeterminates $x = (x_1, ..., x_g)$, over the real field. There is an involution $x_j^* = x_j$ which reverses the order of multiplication $(fp)^* = p^* f^*$. In this exposition we take symmetric variables $x_j = x_j^*$, but in the literature we are summarizing typically x_j can be taken either free or symmetric with no change in the conclusion, for example, the results also hold for symmetric polynomials in $\mathbb{R}\langle x, x^*\rangle$.

A symmetric polynomial $p, p^* = p$, is *matrix convex* if for each positive integer n, each pair of tuples $X = (X_1, \dots, X_g)$ and $Y = (Y_1, \dots, Y_g)$ of symmetric $n \times n$ matrices, and each $0 \le t \le 1$,

$$p(tX + (1-t)Y) \le tp(X) + (1-t)p(Y). \tag{9.1}$$

Even in one-variable, convexity in the noncommutative setting differs from convexity in the commuting case because here Y need not commute with X. For example, to see that the polynomial $p = x^4$ is not matrix convex, let

$$X = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and compute

$$\frac{1}{2}X^4 + \frac{1}{2}Y^4 - (\frac{1}{2}X + \frac{1}{2}Y)^4 = \begin{pmatrix} 164 & 120 \\ 120 & 84 \end{pmatrix}$$

which is not positive semi-definite. On the other hand, to verify that x^2 is a matrix convex polynomial, observe that

$$\begin{aligned} tX^2 + (1-t)Y^2 \quad &- \quad (tX + (1-t)Y)^2 \\ &= \quad t(1-t)(X^2 - XY - YX + Y^2) = t(1-t)(X-Y)^2 \ge 0. \end{aligned}$$

Theorem 9.1. [HM04b] *Every convex symmetric polynomial in the free algebra $\mathbb{R}\langle x\rangle$ or $\mathbb{R}\langle x, x^*\rangle$ has degree two or less.*

As we shall see convexity of p is equivalent to its "second directional derivative" being a positive polynomial. As a matter of fact, the phenomenon has nothing to do with order two derivatives and the extension of this to polynomials with k^{th} derivative nonnegative is given later in Theorem 9.4.

Yet stronger about convexity is the next local implies global theorem.

Let $\mathcal{P}$ denote a collection of symmetric polynomials in non-commutative variables $x = \{x_1, \cdots, x_g\}$. Define the matrix nonnegativity domain $\mathcal{D}_{\mathcal{P}}$

associated to $\mathcal{P}$ to be the set of tuples $X = (X_1, \cdots, X_g)$ of finite dimensional *real* matrices of all sizes, except 0 dimensions, making $p(X_1, \cdots, X_g)$ a positive semi-definite matrix.

Theorem 9.2. [HM04b] *Suppose there is a set $\mathcal{P}$ of symmetric polynomials, whose matrix nonnegativity domain $\mathcal{D}_{\mathcal{P}}$ contains open sets in all large enough dimensions. Then every symmetric polynomial p in $\mathbb{R}\langle x\rangle$ or in $\mathbb{R}\langle x, x^*\rangle$ which is matrix convex on $\mathcal{D}_{\mathcal{P}}$ has degree two or less.*

The first convexity theorem follows from Theorem 8.3, and we outline below the main ideas in its proof. The proof of the more general, order k derivative, is similar and we will return to it later in this section. The proof of Theorem 9.2 requires different machinery (like that behind representation (10.5)) and is not presented here.

At this point we describe a bit of history. In the beginning was Karl Löwner who studied a class of real analytic functions in one real variable called matrix monotone, which we shall not define here. Löwner gave integral representations and these have developed beautifully over the years. The impact on our story comes a few years later when Löwner's student Klaus [K36] introduced matrix convex functions f in one variable. Such a function f on $[0, \infty] \subset \mathbb{R}$ can be represented as $f(t) = tg(t)$ with g matrix monotone, so the representations for g produce representations for f. Modern references are [OSTprept], [U02]. Frank Hansen has extensive deep work on matrix convex an monotone functions whose definition in several variables is different than the one we use here, see[HanT06]; for a recent reference see [Han97].

For a polynomial $p \in \mathbb{R}\langle x\rangle$ define the *directional derivative*:

$$p'(x)[h] = \frac{d}{dt} p(x + th)_{|t=0}.$$

It is a linear form in h. Similarly, the k^{th} derivative

$$p^{(k)}(x)[h] = \frac{d^k}{dt^k} p(x + th)_{|t=0}$$

is homogeneous of degree k in h.

More formally, we regard the directional derivative $p'(x)[h] \in \mathbb{R}\langle x, h\rangle$ as a polynomial in $2g$ free symmetric (i.e. invariant under *) variables $(x_1, \ldots, x_g, h_1, \ldots, h_g)$; In the case of a word $w = x_{j_1}x_{j_2}\cdots x_{j_n}$ the derivative is:

$$w'[h] = h_{j_1}x_{j_2}\cdots x_{j_n} + x_{j_1}h_{j_2}x_{j_3}\cdots x_{j_n} + \ \ldots\ + x_{j_1}\cdots x_{j_{n-1}}h_{j_n}$$

and for a polynomial $p = p'(x)[h] = \sum p_w w$ the derivative is

$$p'(x)[h] = \sum p_w w'[h].$$

If p is symmetric, then so is p'.

For g-tuples of symmetric matrices of a fixed size X, H, observe that the evaluation formula

$$p'(X)[H] = \lim_{t\to 0} \frac{p(X+tH) - p(X)}{t}$$

holds. Alternately, with $q(t) = p(X + tH)$, we find.

$$p'(X)[H] = q'(0).$$

Likewise for a polynomial $p \in \mathbb{R}\langle x\rangle$, the *Hessian* $p''(x)[h]$ of $p(x)$ can be thought of as the formal second directional derivative of p in the "direction" h. Equivalently, the Hessian of $p(x)$ can also be defined as the part of the polynomial

$$r(x)[h] := p(x + h) - p(x)$$

in the free algebra in the symmetric variables that is homogeneous of degree two in h.

If $p'' \neq 0$, that is, if degree $p \geq 2$, then the degree of $p''(x)[h]$ as a polynomial in the $2g$ variables $x_1, \ldots, x_g, h_1 \ldots, h_g$ is equal to the degree of $p(x)$ as a polynomial in $x_1, \ldots, x_g$.

Likewise for k^{th} derivatives.

Example 9.3. 1. $p(x) = x_2x_1x_2$

$$p'(x)[h] = \frac{d}{dt}[(x_2+th_2)(x_1+th_1)(x_2+h_2)]_{|t=0} = h_2x_1x_2 + x_2h_1x_2 + x_2x_1h_2.$$

2. One variable $p(x) = x^4$. Then

$$p'(x)[h] = hxxx + xhxx + xxhx + xxxh$$

Note each term is linear in h and h replaces each occurrence of x once and only once:

$$p''(x)[h] =$$

$$hhxx + hhxx + hxhx + hxxh +$$

$$hxhx + xhhx + xhhx + xhxh +$$

$$hxxh + xhxh + xxhh + xxhh,$$

which yields

$$p''(x)[h] = 2hhxx + 2hxhx + 2hxxh + 2xhhx + 2xhxh + 2xxhh.$$

Note each term is degree two in h and h replaces each pair of x's exactly once. Likewise

$$p^{(3)}(x)[h] = 6(hhhx + hhxh + hxhh + xhhh)$$

and $p^{(4)}(x)[h] = 24hhhh$ and $p^{(5)}(x)[h] = 0$.

3. $p = x_1^2 x_2$

$$p''(x)[h] = h_1^2 x_2 + h_1 x_1 h_2 + x_1 h_1 h_2.$$

□

The definition of a convex polynomial can be easily adapted to domains. Then one remarks without difficulty that, in exact analogy with the commutative case, a polynomial p is convex (in a domain) if and only if the Hessian evaluated at the respective points is non-negative definite. Because of this Theorem 9.1 is an immediate consequence of the next theorem restricted to $k = 2$.

Theorem 9.4. *Every symmetric polynomial p in the free algebra $\mathbb{R}\langle x\rangle$ or $\mathbb{R}\langle x, x^*\rangle$ whose k^{th} derivative is a matrix positive polynomial has degree k or less.*

Proof (when the variables x_j are symmetric).

Assume $p^{(k)}(x)[h]$ is a matrix positive polynomial, so that, in view of Theorem 8.3 we can write it as a sum of squares:

$$p^{(k)}(x)[h] = \sum f_j^* f_j;$$

here each $f_j(x,h)$ is a polynomial in the free algebra $\mathbb{R}\langle x, h\rangle$.

If $p^{(k)}(x)[h]$ is identically equal to zero, then the statement follows. Assume the contrary, so that $p^{(k)}(x)[h]$ is homogeneous of degree k in h, and there are tuples of matrices X, H and a vector ξ in the underlying finite dimensional Hilbert space, so that

$$\langle p^{(k)}(X)[H]\xi, \xi\rangle > 0.$$

By multiplying H by a real scalar t we find

$$t^k \langle p^{(k)}(X)[H]\xi, \xi\rangle = \langle p^{(k)}(X)[tH]\xi, \xi\rangle > 0,$$

whence $k = 2\mu$ is an even integer.

Since in a sum of squares the highest degree terms cannot cancel, the degree of each f_j is at most ν in x and μ in h, where 2ν is the degree of $p^{(k)}$ in x.

Since $p^{(k)}$ is a directional derivative, it must have a highest degree term of the form $h_{i_1} \cdots h_{i_k} m(x)$ where the monomial $m(x)$ has degree equal to degree $p^{(k)} - k$; also h_{i_j} is allowed to equal h_{i_ℓ}. Thus some product, denote it $f_J^* f_J$, must contain such a term. (Note the the order of the $h's$ vs. the $x's$ matters.) This forces f_J to have the form

$$f_J = c_1(h_{i_{\mu}+1} \cdots h_{i_k})m(x) \ + \ c_2(h_{i_1} \cdots h_{i_\mu}) \ + \ \ldots \ ,$$

the c_j being scalars.

To finish the proof use that $f_J^* f_J$ contains

$$c^2 \ m(x)^*(h_{i_{\mu}+1} \cdots h_{i_k})^*(h_{i_{\mu}+1} \cdots h_{i_k})m(x)$$

and this can not be cancelled out, so

$$\deg p^{(k)} \ = \ k + 2(\deg p^{(k)} - k) = 2 \deg p^{(k)} - k.$$

Solve this to find $\deg p^{(k)} = k$. Thus p has degree k. ■

We use a previous example in order to illustrate this proof when $k = 2$.

Example 9.5. *Example $p = x^4$ is not matrix convex; here $x = x^*$.*
Calculate that

$$p''(x)[h] = 2hhxx + 2hxhx + 2hxxh + 2xhhx + 2xhxh + 2xxhh.$$

Up to positive constants some polynomial $f_J^* f_J$ contains a term $hhxx$, so $f_J = hxx + h + \ldots$.

So $f_J^* f_J$ contains $xxhhxx$. This is a highest order perfect square so can be cancelled out. Thus is appears in p'', which as a consequence has degree 6. This a contradiction. □

We call the readers attention to work which goes beyond what we have done in several directions. One [HMVprept] concerns a noncommutative rational function r and characterizes those which are convex near 0. It is an extremely small and rigidly behaved class, for example, r is convex on the entire component of the "domain of r" which contains 0. This rigidity is in analogy to convex polynomials on some "open set" having degree 2 or less and this implying they are convex everywhere. Another direction is the classification of noncommutative polynomials whose Hessian $p''(x)[h]$ at most k "negative noncommutative eigenvalues" In [DHMprept] it is shown that this implies

$$\deg p \quad \leq 2k + 2.$$

Of course the special case we studied in this section is exactly that of polynomials with $k = 0$.

10. Dimension free engineering: LMIs vs. CMIs

This section demonstrates the need for real algebraic geometry (in the broad sense) aimed at convexity over a free or nearly free $*$- algebra. From this viewpoint the theory in this survey goes in an essential direction but much more is called for in order to do general engineering problems. Hopefully the brief description in this section will give a little feel for where we now stand. We are aiming at one of the major issues in linear systems theory:

Which problems convert to a convex matrix inequality, CMI? How does one do the conversion?

To be in line with the engineering literature, we use below a slightly different notation than the rest of the article. For instance A^T denotes the transpose of a (real entries) matrix, and A^T replaces in this case the involution A^* we have encountered in the previous sections. The inner product of vectors in a real Hilbert space will be denoted $u \cdot v$.

10.1. **Linear systems.** A *linear system* $\mathfrak{F}$ is given by the linear differential equations

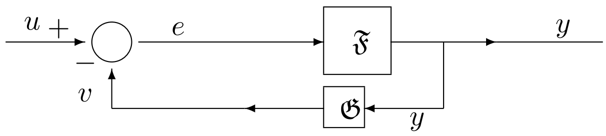
$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

with the vector

- $x(t)$ at each time t being in the vector space $\mathcal{X}$ called the *state space*,
- $u(t)$ at each time t being in the vector space $\mathcal{U}$ called the *input space*,
- $y(t)$ at each time t being in the vector space $\mathcal{Y}$ called the *output space*,

and A, B, C being linear maps on the corresponding vector spaces.

10.2. **Connecting linear systems.** Systems can be connected in incredibly complicated configurations. We describe a simple connection and this goes along way toward illustrating the general idea. Given two linear systems $\mathfrak{F}$, $\mathfrak{G}$, we describe the formulas for connecting them as follows.



Systems $\mathfrak{F}$ and $\mathfrak{G}$ are respectively given by the linear differential equations

$$\frac{dx}{dt} = Ax + Be, \qquad \frac{d\xi}{dt} = a\xi + bw,$$

$$y = Cx, \qquad v = c\xi.$$

The connection diagram is equivalent to the algebraic statements

$$w = y \quad \text{and} \quad e = u - v.$$

The *closed loop system* is a new system whose differential equations are

$$\frac{dx}{dt} = Ax - Bc\xi + Bu,$$

$$\frac{d\xi}{dt} = a\xi + by = a\xi + bCx,$$

$$y = Cx.$$

In matrix form this is

$$\frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A & -Bc \\ bC & a \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u, \tag{10.1}$$

$$y = (C\ 0)\begin{pmatrix} x \\ \xi \end{pmatrix},$$

where the state space of the closed loop systems is the direct sum '$\mathcal{X} \oplus \mathcal{Y}$' of the state spaces $\mathcal{X}$ of $\mathfrak{F}$ and $\mathcal{Y}$ of $\mathfrak{G}$. The moral of the story is:

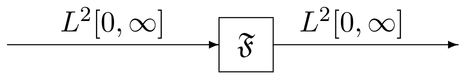
System connections produce a new system whose coefficients are matrices with entries which are polynomials in the coefficients of the component systems.

Complicated signal flow diagrams give complicated matrices of polynomials. Note in what was said the dimensions of vector spaces and matrices never entered explicitly; the algebraic form of (10.1) is completely determined by the flow diagram. We have coined the term *dimension free* for such problems.

10.3. **Energy dissipation.** We have a system $\mathfrak{F}$ and want a condition which checks whether

$$\int_0^\infty |u|^2 dt \geq \int_0^\infty |\mathfrak{F}u|^2 dt, \qquad x(0) = 0,$$

holds for all input functions u, where $\mathfrak{F}u = y$ in the above notation . If this holds $\mathfrak{F}$ is called a *dissipative system*



This is analysis but it converts to algebra because of the following construction. Hope there is a "potential energy" like function $V \geq 0,\ V(0) = 0$, on the state space; it should satisfy:

potential energy now + energy in $\geq$ *potential energy then + energy out.*

In mathematical notation this is

$$V(x(t_1)) + \int_{t_1}^{t_2} |u(t)|^2 \quad \geq \quad V(x(t_2)) + \int_{t_1}^{t_2} |y(t)|^2$$

and a $V \geq 0,\ V(0) = 0$, which satisfies this for all input functions u and initial states $x(t_1)$ is called a *storage function.* We can manipulate this integral condition to obtain first a differential inequality and then an algebraic inequality, as follows:

$$0 \geq \frac{V(x(t_2)) - V(x(t_1))}{t_2 - t_1} \quad + \quad \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |y(t)|^2 - |u(t)|^2,$$

$$0 \geq \nabla V(x(t_1)) \cdot \frac{dx}{dt}(t_1) \quad + \quad |y(t_1)|^2 - |u(t_1)|^2.$$

Use $\frac{dx}{dt}(t_1) = \ Ax(t_1) + Bu(t_1)$ to get

$$0 \ \geq \ \nabla V(x(t_1)) \cdot (Ax(t_1) + Bu(t_1)) \quad + \quad |Cx((t_1))|^2 - |u(t_1)|^2.$$

The system is dissipative if and only if this holds for all $u(t_1)$, $x(t_1)$ which can occur when it runs (starting at $x(0) = 0$). All vectors $u(t_1)$ in $\mathcal{U}$ can certainly occur as an input and if all $x(t_1)$ can occur we call the system *reachable* . Denote $x(t_1)$ by x and $u(t_1)$ by u

$$0 \ \geq \ \ \nabla V(x) \cdot (Ax + Bu) \ \ + \ \ |Cx|^2 - |u|^2, \tag{10.2}$$

and conclude:

Theorem 10.1. *The system A, B, C is dissipative if inequality (10.2) holds for all $u \in \mathcal{U}, x \in \mathcal{X}$. Conversely, if A, B, C is reachable, then dissipativity implies inequality (10.2) holds for all $u \in \mathcal{U}, x \in \mathcal{X}$.*

For a linear system we try V which is quadratic, so $V(x) = Px \cdot x$ with $P \geq 0$ and $\nabla V(x) = 2Px$. At this point there are two commonly pursued paths which constitute the next two subsections.

10.3.1. *Riccati inequalities.* Use that $\nabla V(x) = 2Wx$ in (10.2) to get

$$0 \geq 2Wx \cdot (Ax + Bu) + |Cx|^2 - |u|^2, \qquad \text{for all } u, x,$$

so

$$0 \geq \max_u \left([WA + A^T W + C^T C]x \cdot x + 2B^T Wx \cdot u - |u|^2 \right). \tag{10.3}$$

The maximizer in u is $u = B^T Wx$, hence

$$0 \geq 2Wx \cdot Ax + 2|B^T Wx|^2 + |Cx|^2 - |B^T Wx|^2.$$

Which in turn is

$$0 \geq [WA + A^T W + WBB^T W + C^T C]x \cdot x.$$

This is the classical *Riccati matrix inequality*

$$0 \geq \ WA + A^T W + WBB^T W + C^T C$$

which together with $W \geq 0$ insures dissipativity and is also necessary for it when the system is reachable.

10.3.2. *Linear Matrix Inequalities (LMI).* Alternatively we do not need to compute $\max_u$ but can express (10.3) as the inequality:

$$L(W) := \begin{pmatrix} WA + A^TW + C^TC & WB \\ B^TW & -I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \cdot \begin{pmatrix} x \\ u \end{pmatrix} \leq 0$$

for all $u \in \mathcal{U}, x \in \mathcal{X}$. That is the following matrix inequality holds:

$$\begin{pmatrix} WA + A^TW + C^TC & WB \\ B^TW & -I \end{pmatrix} \leq 0.$$

Here A, B, C describe the system and W is an unknown matrix. If the system is reachable, then A, B, C is dissipative if and only if $L(W) \leq 0$ and $W \geq 0$.

Recall that the *Schur complement* of a matrix is defined by

$$SchurComp \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix} := \alpha - \beta\gamma^{-1}\beta^T.$$

Suppose γ is invertible. The matrix is positive semi-definite if and only if $\gamma > 0$ and its Schur complement is positive semi-definite. Note that

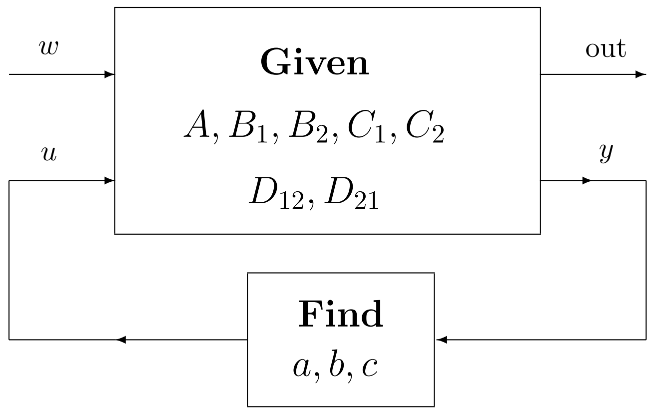
$$SchurComp\, L(W) \;=\; WA + A^TW + WBB^TW + C^TC$$

featuring the Riccati inequality we saw before. Indeed, $L(W) \leq 0$ if and only if $WA+A^TW+WBB^TW+C^TC \leq 0$, since this implies $WA+A^TW+C^TC \leq 0$. Thus the Riccati approach and the LMI approach give equivalent answers.

10.4. **Example: An H^∞ control problem.** Here is a basic engineering problem:

Make a given system dissipative by designing a feedback law.

To be more specific, we are given a signal flow diagram:



where the given system is

$$\begin{aligned}\frac{ds}{dt} &= As + B_1 w + B_2 u,\\ \text{out} &= C_1 s + D_{12} u + D_{11} w,\\ y &= C_2 s + D_{21} w,\end{aligned}$$

$$D_{21} = I, \qquad D_{12}D_{12}' = I, \qquad D_{12}' D_{12} = I, \qquad D_{11} = 0.$$

We want to find an unknown system

$$\frac{d\xi}{dt} = a\xi + b, \qquad\qquad u = c\xi,$$

called the *controller*, which makes the system dissipative over every finite horizon. Namely:

$$\int_0^T |w(t)|^2 dt \geq \int_0^T |out(t)|^2 dt, \qquad s(0) = 0.$$

So a, b, c are the critical unknowns.

10.4.1. *Conversion to algebra.* The dynamics of the "closed loop" system has the form

$$\frac{d}{dt}\begin{pmatrix} s \\ \xi \end{pmatrix} = \mathcal{A}\begin{pmatrix} s \\ \xi \end{pmatrix} + \mathcal{B}w$$

$$out = \mathcal{C}\begin{pmatrix} s \\ \xi \end{pmatrix} + \mathcal{D}w$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are "2 × 2 block matrices" whose entries are polynomials in the $A's, B's, \cdots, a, b, c$ etc. The storage function inequality which corresponds to energy dissipation has the form

$$H := \mathcal{A}^T E + E\mathcal{A} + E\mathcal{B}\mathcal{B}^T E + \mathcal{C}^T\mathcal{C} \leq 0 \tag{10.4}$$

where E has the form

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}, \qquad E_{12} = E_{21}^T.$$

The algebra problem above in more detail is to solve inequality (10.4)

$$H = \begin{pmatrix} H_{ss} & H_{sy} \\ H_{ys} & E_{yy} \end{pmatrix} \leq 0, \qquad H_{sy} = H_{ys}^T,$$

where the entries of H are the polynomials:

$$H_{ss} = E_{11}\,A + A^T\,E_{11} + C_1^T\,C_1 + E_{12}^T\,b\,C_2 + C_2^T\,b^T\,E_{12}^T + E_{11}\,B_1\,b^T\,E_{12}^T +$$
$$E_{11}\,B_1\,B_1^T\,E_{11} + E_{12}\,b\,b^T\,E_{12}^T + E_{12}\,b\,B_1^T\,E_{11},$$
$$H_{sz} = E_{21}\,A + \frac{a^T\,(E_{21} + E_{12}^T)}{2} + c^T\,C_1 + E_{22}\,b\,C_2 + c^T\,B_2^T\,E_{11}^T +$$
$$\frac{E_{21}\,B_1\,b^T\,(E_{21} + E_{12}^T)}{2} + E_{21}\,B_1\,B_1^T\,E_{11}^T + \frac{E_{22}\,b\,b^T\,(E_{21} + E_{12}^T)}{2} + E_{22}\,b\,B_1^T\,E_{11}^T,$$
$$H_{zs} = A^T\,E_{21}^T + C_1^T\,c + \frac{(E_{12} + E_{21}^T)\,a}{2} + E_{11}\,B_2\,c + C_2^T\,b^T\,E_{22}^T + E_{11}\,B_1\,b^T\,E_{22}^T +$$
$$E_{11}\,B_1\,B_1^T\,E_{21}^T + \frac{(E_{12} + E_{21}^T)\,b\,b^T\,E_{22}^T}{2} + \frac{(E_{12} + E_{21}^T)\,b\,B_1^T\,E_{21}^T}{2},$$
$$H_{zz} = E_{22}\,a + a^T\,E_{22}^T + c^T\,c + E_{21}\,B_2\,c + c^T\,B_2^T\,E_{21}^T + E_{21}\,B_1\,b^T\,E_{22}^T +$$
$$E_{21}\,B_1\,B_1^T\,E_{21}^T + E_{22}\,b\,b^T\,E_{22}^T + E_{22}\,b\,B_1^T\,E_{21}^T.$$

Here A, B_1, B_2, C_1, C_2 are known and the unknowns are a, b, c and for E_{11}, E_{12}, E_{21} and E_{22}.

We very much wish that these inequalities are convex in the unknowns (so that numerical solutions will be reliable). But our key inequality above is not convex in the unknowns.

10.4.2. *The key question.* Is there is a set of noncommutative *convex* inequalities with an equivalent set of solutions?

This is a question in algebra not in numerics and the answer after a lot of work is yes. The path to success is:

(1) *Firstly, one must eliminate unknowns, change variables and get a new set of inequalities* $\mathcal{K}$.
(2) *Secondly, one must check that* $\mathcal{K}$ *is "convex" in the unknowns.*

This outline transcends our example and applies to very many situations. The second issue of this is becoming reasonably understood, for as we saw earlier, a convex polynomial with real coefficients has degree two or less, so these are trivial to identify. While the level of generality of the theory we have presented in this paper is less than we now require, to wit, polynomials with indeterminates as coefficients and matrices with polynomial entries; this does not add radically different structure, see discussion in §10.5. The first issue, changing variables, is still a collection of isolated tricks, with which mathematical theory has not caught up. For the particular problem in our example we shall not derive the solution since it is long. However, we do state the classical answer in the next subsection.

10.4.3. *Solution to the Problem.* The textbook solution is as follows, due to Doyle-Glover- Kargonekar-Francis. It appeared in [DGKF89] which won the 1991 annual prize for the best paper to appear in an IEEE journal. Roughly speaking it was deemed the best paper in electrical engineering in that year.

We denote

$$DGKF_X := (A - B_2C_1)'X + X(A - B_2C_1) + X(\gamma^{-2}B_1B_1' - B_2^{-1}B_2')X$$

$$DGKF_Y := A^{\times}Y + YA^{\times\prime} + Y(\gamma^{-2}C_1'C_1 - C_2'C_2)Y,$$

where $A^{\times} := A - B_1C_2$.

Theorem 10.2. [DGKF89] *There is a system solving the control problem if there exist solutions*

$$X \geq 0 \qquad and \qquad Y > 0$$

to inequalities the

$$DGKF_Y \leq 0 \ \ and \ \ DGKF_X \leq 0$$

which satisfy the coupling condition

$$X - Y^{-1} < 0.$$

This is if and only if provided $Y > 0$ is replaced by $Y \geq 0$ and Y^{-1} is interpreted correctly.

This set of inequalities while not usually convex in X, Y are convex in the new variables $W = X^{-1}$ and $Z = Y^{-1}$, since $DGKF_X$ and $DGKF_Y$ are linear in them and $X - Y^{-1} = W^{-1} - Z$ has second derivative $2W^{-1}HW^{-1}HW^{-1}$ which is non negative in H for each $W^{-1} = X > 0$. These inequalities are also equivalent to LMIs which we do not write down.

10.4.4. *Numerics and symbolics.* A single Riccati inequality is much more special than an LMI and numerical solvers for Riccatis are faster and handle bigger matrices. This survey obviously has not aimed at numerics, but at algebraic precursors to using numerics.

The mathematics here aims toward helping an engineer who writes a toolbox which other engineers will use for designing systems, like control systems. What goes in such toolboxes is algebraic formulas like the DGKF inequalities above with matrices A, B, C unspecified and reliable numerics for solving them when a user does specify A, B, C as matrices. A user who designs a controller for a helicopter puts in the mathematical systems model for his helicopter and puts in matrices, for example, A is a particular $R^{8\times 8}$ matrix etc. Another user who designs a satellite controller might have a 50 dimensional state space and of course would pick completely different A, B, C. Essentially any matrices of any compatible dimensions can occur and our claim that our algebraic formulas are convex in the ranges we specify must be true.

The toolbox designer faces two completely different tasks. One is manipulation of algebraic inequalities; the other is numerical solutions. Often the first is far more daunting since the numerics is handled by some standard package. Thus there is a great need for algebraic theory.

10.5. **Engineers need generality.** To make exposition palatable in this paper we have refrained from generality which does not have much effect on mathematical structure. However, to embrace linear systems problems we need more general theorems. A level of generality which most linear systems problems require is to work with polynomials p in two classes of variables $p(a, x)$ where we shall be interested in matrix convexity in x over ranges of the variable a. Describing this setup fully takes a while, as one can see in [CHSY03] where it is worked out. An engineer might look at [CHSprept], especially the first part which describe a computational noncommutative algebra attack on convexity, it seems to be the most intuitive read on the subject at hand. Here we try to indicate the idea. In private notes of Helton and Adrian Lim one shows that second derivatives of $p(a, x)$ in x determine convexity in x and that convexity in the x variable on some "open set" of a, x implies that p has degree 2 or less in x. From this we get

If $P(a, x)$ is a symmetric $d \times d$ matrix with polynomial entries $p_{ij}(a, x)$, then convexity in x for all X and all A satisfying some strict algebraic inequality of the form $g(A) > 0$, implies each p_{ij} has degree 2 or less.

We obtain this from the following argument. We shall test $P(a, x)$ by plugging in tuples A and X of $n \times n$ matrices for a and x. First note that matrix convexity of P in X through a range of A, X implies that the diagonals p_{ii} must have this property. Thus they have degree 2 or less in x. Consider how the Hessian $F(t) := P_{xx}(A, tX)[H]$ in x scales with a scalar parameter t. The matrix function being convex implies its diagonals are convex. Thus as we saw above, $F_{kk}(t)$ is independent of t for all k.

Apply $F(t)$ to the vectors $v = column\ (\pm v_1, \cdots, \pm v_d)$ in $\mathbb{R}^{n\times n}$ and use that $v^T F(t) v \geq 0$ for all t, to get that for each i, j the entries F_{ij} satisfy

$$v_i^T F_{ii}(t) v_i + v_j^T F_{jj}(t) v_j \quad \geq \quad \pm (v_i^T F_{ij}(t) v_j)^2.$$

This implies by letting $t \to \infty$ that the degree of $v_i^T F_{ij}(t) v_j$ in t is 0, which implies the same for $F_{ij}(t)$. To this point we have that all polynomials in $P_{xx}(a, x)[h]$ are independent of X whenever matrix tuples A from an open set $\{A : g(A) > 0\}$ are plugged in. This is independent of the size n of the matrices we plug in, so all polynomials in $P_{xx}(a, x)[h]$ are 0, algebraically speaking. Thus all polynomials in $P(a, x)[h]$ have degree 2 in x or less. The engineering conclusion from all of this is formulated below.

10.6. **Conclusion.**

(1) Many linear systems problems which are "dimension free" readily reduce to noncommuting inequalities on $d\times d$ matrices of polynomials of the form $P(a, x) \leq 0$. These do so as in the §10.4 example, or even after simplifying solving and substituting they yield a matrix of polynomials.
(2) If such $P(A, X)$ is X-convex on the set of $n \times n$ matrix tuples A satisfying a strict polynomial inequality $g(A) > 0$ and on all X (regardless of dimension n), then $P(a, x)$ has degree 2 in x, as we saw in §10.5. Alas, P is surprisingly simple.
(3) Assume a $d \times d$ matrix of polynomials $P(a, x)$ has degree 2 in x. There are tests (not perfect) to see where in the a variable $P(X, A)$ is negative semi-definite for all X. Equivalently, to see where P is convex in x.
(4) Convexity and the degree 2 property imply $P(a, x) \leq 0$ can be expressed as an LMI. Often the LMI can be constructed with coefficients which are noncommutative polynomials (dimension free formula). See proof below.

This very strong conclusion is bad news for engineers and we emphasize that it does not preclude transformation to convexity or convexity for dimension dependent problems.

10.6.1. *Tests for convexity and the making of an LMI.* Here we shall sketch of the proof of Conclusions (3) and (4). We use methods not described earlier in this paper, but despite that restrict our presentation to be only a brief outline. For proofs in detail see [CHSY03] or more generally [HMPpos].

Suppose $q(a)[h]$ is a symmetric polynomial in a, h which is homogeneous of degree 2 in h, then q being quadratic in h, can be represented as

$$q(a)[h] = V(a)[h]^T M(a) V(a)[h] \tag{10.5}$$

where M is matrix of noncommutative polynomials in a, and V is a vector each entry of which is a monomial of the form $h_j m(a)$ where $m(a)$ is a monomial in a. We can choose the representation so that no monomial repeats. A key is Theorem 10.10 and Lemma 9.4 in [CHSY03] which imply

Lemma 10.3. *Let $q(a,h), g(a)$ be polynomials in the free algebra with q purely quadratic in h. Then $q(A)[H] \geq 0$ for $g(A) > 0$ and all H is equivalent to $M(A) \geq 0$ for $g(A) > 0$.*

We shall apply this by representing $P_{xx}(a,x)[h] = V(a)[h]^T M(a) V(a)[h]$. The x- Hessian being quadratic in x satisfies $P_{xx}(a,x)[h]$ is independent of x. From the lemma we have $P_{xx}(A,X)[H] \geq 0$ for $g(A) > 0$ and all X is equivalent to $M(A) \geq 0$ for $g(A) > 0$.

Two tests for positivity as mentioned in Conclusion (3) follow.

(1) The test in [CHSY03] is: take the symbolic noncommutative $M(a) = L(a)^T D(a) L(a)$ decomposition of $M(a)$. This gives $D(a)$ a matrix with diagonal or 2×2 block diagonal entries which are nc rational functions in a. $M(A) \geq 0$ if and only if $D(A) \geq 0$, so checking positivity of the functions on diagonal $D(a)$ is a test for where $p(a,x)$ is convex.

(2) Here is another test. If the Positivstellensatz holds (despite a failure of the strict positivity hypothesis), then

$$M(a,x) \in \Sigma^2 + \sum r_j^T g r_j + \sum t_{ij}^T (C^2 - x_j^T x_j) t_{ij}. \tag{10.6}$$

Computing the terms r_j, t_{ij} and the sums of squares component gives an algebraic test.

Conversion to LMIs, namely, Conclusion (4). Denote by $P^I(a,x)$ the terms in $P(a,x)$ with x degree exactly one, respectively $P^{II}(a,x)$with x degree exactly two.

(1) We now give quite a practical numerical algorithm for producing an LMI, under no offensive assumptions. However, we do not get formulas which are polynomials in the symbol a. Once $n \times n$ matrices A are given with $M(A)$ positive semi-definite we can compute numerically its Cholesky decomposition $M(A) = L(A)^T D(A)^{\frac{1}{2}} D(A)^{\frac{1}{2}} L(A)$ (actually any square root of $M(A)$ will do). Then we have

$$P^{II}(A,X) = V(A)[X]^T L(A)^T D(A)^{\frac{1}{2}} \ D(A)^{\frac{1}{2}} L(A) V(A)[X]$$

and taking

$$\mathcal{L}(A)[X] := \begin{pmatrix} p^0(A,X) + p^I(A,X) & D(A)^{\frac{1}{2}} L(A) V(A)[X] \\ V(A)[X]^T L(A)^T D(A)^{\frac{1}{2}} & -I \end{pmatrix}$$

which produces $\mathcal{L}(A)[X]$ whose Schur complement equals $P(A,X)$ and which produces a matrix inequality

$$\{X : P(A,X) \leq 0\} \ = \ \{X : \mathcal{L}(A)[X] \leq 0\}. \tag{10.7}$$

The entries of $\mathcal{L}(A)[X]$ are linear in scalar unknowns X_{lm} and have $n \times n$ matrix coefficients. This is standard input to the LMI numerical solvers prevalent today.

(2) Another recipe which produces algebraic formulas for solution the following. Continue with item (2) above. The terms $P^{II}(a,x)$ in $P(a,x)$ with x degree exactly two can be represented by $M(a)$ as in (10.5). From the Positivstellensätz (10.6) for $M(a)$ and the fact that linear terms are trivial to handle, we can easily build an algebraic

expression for a matrix $\mathcal{L}(a)[x]$ with polynomial entries which are linear in x whose Schur complement equals $P(a,x)$. Moreover, for any fixed A satisfying $g(A) > 0$, the solution sets to our favorite matrix inequality $P > 0$ and the LMI based on $\mathcal{L}$ are the same, as in (10.7). This completes the proof that, if the Positivstellensätz (10.6) for $P^{II}(a,x)$ exists, then a LMI which is polynomial in a exists.

As a side remark, for the degree 2 and other properties of matrix valued polynomials we could use weaker hypotheses allowing coupling of a and x (as in done in private Helton- Lim notes for polynomials), these probably work by the same argument, basically the argument in [HM04b]).

10.7. **Keep going.** This subject of noncommutative real algebraic geometry and its geometric offshoots on convexity is a child of the 21^{st} century. Understanding the relationship between Convex MIs and LMIs was a core motivation for its developments. When we look at the two basic techniques in §(10.4.2) what we have done in this paper bears successfully on issue 2. But nothing has been said about issue 1. Nick Slinglend's UCSD thesis in progress makes a start in that direction.

This physical section has focused on "dimension free" problems. What about dimension dependent ones? In these problems the variables commute. There the behavior is quite different; as we saw in §7 there is an extra constraint beyond convexity to have equivalence to an LMI.

11. A guide to literature

While classical semi-algebraic geometry has developed over the last century through an outpouring of seemingly countless papers, the thrust toward a noncommutative semi-algebraic geometry is sufficiently new that we have attempted to reference the majority of papers directly on the subject here in this survey. This non-discriminating approach is not entirely good news for the student, so in this section we provide some guidance to the more readable references.

The Functional Analysis book by Riesz and Nagy [RN90] is a class in itself. For a historical perspective on the evolution of the spectral theorem the reader can go directly to Hilbert's book [Hilb1953] or the German Encyclopedia article by Hellinger and Toeplitz [HT53]. Reading von Neumann in original [vN1] is still very rewarding.

The many facets of matrix positivity, as applied to function theory and systems theory, are well exposed in the books by Agler-McCarthy [AM02], Foias-Frazho [FF90] and Rosenblum-Rovnyak [RR97]. The monograph of Constantinescu [Constantinescu96] is entirely devoted to the Schur algorithm.

For the classical moment problem Akhiezer's text [Akh65] remains the basic reference, although having a look at Marcel Riesz original articles [MR23], Carleman's quasi-analytic functions [C26], or at the continued fractions monograph of Perron [Per50] might bring new insights. Good surveys of the multivariate moment problems are Berg [Berg87] and Fuglede [F83]. Reznick's memoir [R92] exploits in a novel and optimal way the duality between moments and positive polynomials.

For real algebraic geometry, including the logical aspects of the theory, we refer to the well circulated texts [BCR98, J89, M00] and the recent monograph by Prestel and Delzell [PD01]; the latter offers an elegant and full access to a wide selection of aspects of positive polynomials. For new results in algorithmic real (commutative) algebra see [BPR03]; all recent articles of Lasserre contain generous recapitulations and reviews of past articles devoted to applications of sums of squares and moments to optimization. Scheiderer's very informative survey [S03] is centered on sums of squares decompositions. Parrilo's thesis [ParThesis] is a wonderful exposition of many new areas of application which he discovered.

An account of one of the most systematic and elegant ways for producing LMIs for engineering problems is the subject of the book [SIG97]. The condensed version we heartily recommend is their 15 page paper [SI95].

Software:

Common semi-definite programming packages are [Sturm99]*SeDuMi* and *LMI Toolbox* [GNLC95].

Semi-algebraic geometry packages are *SOS tools* [PPSP04] and *GloptiPoly* [HL03].

For symbolic computation in a free ∗- algebra see *NCAlgebra* and *NCGB* (which requires Mathematica) [HSM05].

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