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Comments on Boyce Diprima and Addtions

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The objective is to list some of the things mentioned in class which are not in the book. In addition some point will be made for emphasis. This is NOT at all a comprehensive treatment.

1. Ch3

1.1. **Linear algebraic equations.** Solve

$ax + by = 7$

$cx + dy = 2$

for x and y.

Such problems occur repeatedly in Ch 3, 7 and 6. They are best solved using Gauss Elimination. You should look this up. For a brief explanation see Ch 7 BD, bottom of page 368 and page 369 (Ex 2). I suggest avoiding Cramer's rule. However, you should look it up since the Wronskian (loved by BP) is based on it.

1.2. **Sec 3.2.** The book uses Wronskian's heavily to do some really simple things, better done other ways.

Two functions y_1 and y_2 are called **linearly independent** or **independent** for short provided one is a constant multiple of the other. In practice you can see if 2 functions are independent just by looking at them. Eg.

$2e^t$ and $\cos t$ are clearly independent.

A test is to compare values of $y_1(t)$ and $y_2(t)$ at two values of t. After all if $y_1 = 5y_2$, then

$$y_1(0) = 5y_2(0), \qquad \textit{and} \qquad y_1'(0) = 5y_2'(0).$$

If this fails, then we have independence. Note this a strong constraint to satisfy, so we usually have independence.

Lemma 1.1. *y_1 and y_2 are dependent iff the Wronskian of y_1 and y_2 is identically 0.*

Pf: Suppose $y_1(t) = ky_2(t)$.

$$wronk(t) := y_1(t)y_2'(t) - y_2(t)y_1'(t) = ky_2(t)y_2'(t) - y_2(t)ky_2'(t) = 0.$$

Conversely, $0 = wronk(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ for all t implies

$$\frac{y_1'(t)}{y_1(t)} = \frac{y_2'(t)}{y_2(t)}.$$

Integrate to get $ln|y_1(t)| = ln|y_2(t)| + c$, so $y_1 = \pm e^c y_2$. Thus $y_1 = const\ y_2$. □

The moral of the story is you never need to look at the Wronskian, (and most moderns don't). On the other hand linear independence sits at the heart of linear algebra, e.g. in M20F. Thru your career you will be dealing with it repeatedly.

1.3. **Midterm I.** Study hard.

sec:complexexp

1.4. **Sec 3.3 and Beyond.** Look at at the M20Bsupplement to review complex varibles, see website.

Consider

$$(DE) \qquad y'' + a_1 y' + a_0 y = g$$

with a_1, a_0 real and g a possibly complex function. Take the complex conjugate of (DE):

$$(\hat{DE}) \qquad \bar{y}'' + a_1 \bar{y}' + a_0 \bar{y} = \bar{g}$$

Add

$$\bar{y}'' + y'' + a_1(\bar{y}' + y') + a_0(\bar{y} + y) = \bar{g} + g$$

Divide by 2:

$$Re\ y'' \ + \ a_1\ Re\ y' \ + \ a_0 Re\ y = Re\ g$$

The point is if we have complex solutions it is easy to get real valued solutions to differential equations.

This comes in handy because calculations with exponentials (even complex exponentials) are easy.

Sec 3.3 deals only with $g = 0$. Starting in §3.5 g is not assumed to be 0.

Example with $g \neq 0$. Solve

$$(DE) \qquad r'' + a_1 r' + a_0 r = e^{\lambda t} cos \mu t$$

for real valued y.

The trick is set $w := \lambda + i\mu$. Then we must solve

$$r'' + a_1 r' + a_0 r = Re\ e^{wt}$$

which the general calculation above tells us can be done by solving the easier equation

$$y'' + a_1 y' + a_0 y = e^{wt}$$

and then we get

$$r := Re\ y.$$

□

1.5. **Summary after §3.4.** There is a general structure to solving all constant coefficient n^{th} linear homogeneous DE. We illustrate this on 3^{rd} order equations.

$$y'' + a_2 y'' + a_1 y' + a_0 y = 0$$

Its charachteristic polynomial is

$$p(z) = z^3 + a_2 z^2 + a_1 z + a_0 = 0$$

By the fundemental theorem of algebra p factors

$$p(z) = (z - z_1)(z - z_2)(z - z_3)$$

and $p(z) = 0$ has solutions z_1, z_2, z_3.

There are now cases:

If $z_1 \neq z_2$, $z_1 \neq z_3$, $z_2 \neq z_3$, then $e^{z_1 t}$, $e^{z_2 t}$, $e^{z_3 t}$ are linearly independent solutions.

If $z_1 \neq z_2$, $z_1 \neq z_3$, $z_2 = z_3$, then $e^{z_1 t}$, $e^{z_2 t}$, $te^{z_2 t}$ are linearly independent solutions.

If $z_1 = z_2$, $z_1 = z_3$, $z_2 = z_3$, then $e^{z_1 t}$, $te^{z_1 t}$, $t^2 e^{z_2 t}$ are linearly independent solutions.

Thus is each case we have built fundamental solns for the problem. A more modern name would be a **bases for the space of solutions to the DE**.

1.6. **Linear nonhomogeneous equations; §3.5.** There were two main additions to the book during the lecture.

Notation: To study

$$(DE_g) \qquad ay'' + by' + cy = g$$

it saves time to define L by

$$L(y) := ay'' + by' + cy$$

often called a linear operator. It maps functions y to functions. Then DE_g is $L(y) = g$.

Suppose a, b, c are real numbers (constants) . Let p_L denote

$$p_L(z) := az^2 + bz + c$$

the charachteristic polynomial of L.

1. Solve $L(y) = e^{wt}$ for y.

Try $y = Ae^{wt}$ and get $Ap_L(w)e^{wt} = e^{wt}$, so $A = \frac{1}{p_L(w)}$ provided $p_L(w) \neq 0$. Thus one solution to (DE_g) is

eq:1overp

$$Y_1(t) := \frac{e^{wt}}{p_L(w)} \qquad (1)$$

This link with the charachteristic polynomial turns out later to be very important and it works even if w is a complex number.

Exercise: Show if L is a 3^{rd} order lin const coeff differential operator. Does a solution of the form (1) [eq:1overp] solve $L(Y_1) = e^{wt}$?

2. Solving $L(r) = e^{\lambda t} cos \mu t$ using complex exponentials has the advantage that it scales up to more complicated problems more simply than fighting products of cos and exponentials. I leave it to you to do the HW either way, but if you are an engineer etc you will get a better foundation by using complex exponentials. So I did it that way in class.

First set $w = \lambda + i\mu$ the point being

$$e^{\lambda t} cos\mu t = Re\ e^{wt}.$$

Solve $L(y) = e^{wt}$ (where y will be complex valued). Then the fact in sec:complexexp §1.4 tells you that

$$L(r) = Re\ e^{wt} = e^{\lambda t} cos\mu t$$

has solution

$$r = Re\ y = Re\ \frac{e^{wt}}{p_L(w)}.$$

As a reminder: all solutions to (DE_g) with $g = e^{wt}$ have the form

$$Y(t) = \frac{e^{wt}}{p_L(w)} + c_1 e^{z_1 t} + c_2 e^{z_2 t}$$

where $p_L(z_1) = 0 = p_L(z_2)$.

1.7. **§3.5, 3.6.** You will need to read §3.5, §3.6 on your own more than usual.