Variations in Linear Algebra Content Presentations

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In preparation for developing a linear algebra program for upper-level high school students [Harel, 1985] 32 linear textbooks were analyzed to examine existing approaches to teaching this topic. Since linear algebra textbooks at the high school level are virtually non-existent, all the textbooks analyzed, but two, are at the college level. The textbooks varied in their approaches to: (a) sequencing of content; (b) generality of levels of vector space models; (c) introductory material; (d) embodiment; and (e) symbolization. In this paper, we characterize these approaches and suggest some pedagogical considerations and questions that arise.

Sequencing of content

Two approaches to sequencing linear algebra content were observed. In one approach, computational techniques appear before abstract ideas (computation-to-abstraction approach). The other approach is the reverse: abstract ideas appear before computational techniques (abstraction-to-computation approach). Frequently, the former approach is used by elementary textbooks, whereas the latter approach is used by the more advanced textbooks.

Linear systems, linear transformations, matrix arithmetic, and vector spaces form the core content of linear algebra. Mathematically, these topics can be sequenced in any order provided vector spaces precede linear transformations. The computation-to-abstraction approach starts with matrix arithmetic and linear systems, and concludes with vector spaces and linear transformations [e.g., Anton, 1981; Kolman, 1979; Staib, 1969]. In the abstraction-to-computation approach, matrix arithmetic and linear systems follow vector spaces and linear transformations [e.g., Fisher, 1970; Krause, 1970; Pillis, 1969; Thompson, 1970].

These two approaches are based on different conceptions. The computation-to-abstraction approach starts with matrix arithmetic and linear systems to enable the student to learn the new language and the new reasoning gradually while moving toward more abstract material; it prepares the student for understanding the major concepts of linear algebra, vector spaces and linear transformations. Examples that support the conceptual basis of this approach include the following:

1. Starting with matrix arithmetic and linear systems provides multiple embodiments of the vector space structure (e.g. $m \times n$ matrices and solutions of homogeneous linear systems), and useful techniques (e.g. row reduction of matrices) for solving problems concerning dependence and independence of vectors.
2. The matrix representation of a linear system (i.e. $Y = AX$, where A is the coefficient matrix) illustrates the notion of linear transformation.
3. The concept of dimension is based on the idea that an independent set of vectors cannot be "overly large" (i.e. if $S = \{v_1, v_2, \ldots, v_n\}$ spans a vector-space V, then every set in V that contains more than n vectors is linearly dependent). This idea can be established by using either the Gaussian elimination process, which is a simple algorithm for solving linear systems [see e.g. Anton, 1981], or the Steinitz Replacement Lemma [see e.g. Nering, 1970] which involves a deep mathematical induction. It is apparent, therefore, that the former alternative is easier than the latter one — particularly for a beginning student.

The abstraction-to-computation approach suggests that, by starting with vector spaces and linear transformations, the ideas would be absorbed through mathematical structures, and computations will be well-motivated. Examples supporting the conceptual basis of this approach include the following:

1. Relationships within and between solution sets of linear systems are more easily observed through the vector-space structure.
2. Linear transformations motivate the study of matrices and operations on matrices. This can be done by presenting the matrix as a code of linear transformation and the operations as sufficient conditions for isomorphizing the structure of linear transformations to the set of their matrix representations. [For a complete discussion, see Krause, 1970]
3. Fundamental facts concerning systems of linear equations can be easily retrieved if the system is viewed as a linear transformation from one vector space to another. Thus, the important statement, "if $m < n$, then an $m \times n$ system of homogeneous linear equations has a non-trivial solution," can be easily verified using the properties, "a linear transformation T is non-singular if and only if T is onto," and "rank(T) + nullity (T) = dim$V$, where V is the domain of T."

Levels of generality

Each of the textbooks analyzed, even those which were designated for high school students [e.g. SMSG, 1965], presented the general definition of a vector space. These textbooks differed from one another in the generality levels of the vector space models in which they apply the theory of linear algebra. Four levels of generality were observed:

> Level 1: A vector space whose dimension is a specified number (usually 1, 2, or 3) and with defined elements (e.g. directed segments or polynomials).
> Level 2: A vector space whose dimension is a specified number and with undefined elements.
> Level 3: A vector space whose dimension is a parameter (i.e. n) and whose elements are defined (e.g. $\mathbf{R}_n$ as the space of n-tuples $(x_1, x_2, \ldots, x_n)$, where x_i's are real numbers).
> Level 4: A vector space whose dimension is a parameter and whose elements are not defined.

Concerning the complexity of these levels, it is likely that levels 1 and 4 are, respectively, the lowest and the highest levels, whereas an ordering between levels 2 and 3 is open to speculation. Dealing with linear algebra in levels 2 and 4 requires an understanding that the derived results in the system of linear algebra depend solely on the axioms of a vector space, not on the definition of its elements. Our experience in teaching linear algebra suggests that academic high school students and even first year college students find this idea difficult to grasp.

Introductory material

Textbooks provide introductory material in an attempt to motivate the student and to integrate new mathematics ideas with material previously studied. The pedagogical and cognitive aspects of introductory material are well known through the theory of advanced organizers developed by Ausubel [1968]. In linear algebra textbooks, four main strategies to introduce new material have been identified: analogy, abstraction, isomorphization, and postponing.

Analogy. An analogy describes similarities between new ideas to be learned and familiar ones that are outside the content area of immediate interest [Reigeluth, 1983]. Two kinds of analogies were identified. The first is an analogy to some non-mathematical content. For example, some textbooks introduce economics or game problems and solve them by applying operations similar to those on matrices. The second is an analogy to a mathematical content. For example, Amitsor [1970] introduces the definition of self-adjoint operator by analogy to the characterization of real numbers in the field of complex numbers.

Analogy is an important strategy in instruction; it has both motivational effects [Keller, 1983] and cognitive benefits [Reigeluth, 1983]. However, an analogy relating mathematical content to non-mathematical content has many nonanalogous aspects because the student has to abstract the analogic situation in order to distinguish between relevant and irrelevant features. This might weaken the anticipated motivational effect. In some cases the analogy strategy may cause confusion of concepts. For example, in some textbooks the definition of inner product is motivated by the idea of computing the total price of items $x_1, x_2, \ldots, x_n$ whose prices are $p_1, p_2, \ldots, p_n$, respectively, by the operation:

$$(x_1, x_2, \ldots, x_n)\cdot(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} x_i p_i$$

The student may erroneously think that this operation is an inner product. In fact, this operation is *not* an inner product but a bilinear form on two spaces, the space of items and its dual space.

Abstraction. Abstraction is a strategy for preparing students to abstract ideas by first introducing the students to these ideas in particular situations. For example, general concepts (such as independence, dependence, and span), and general theorems such as the dimensions theorem $(\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2))$ are generated from particular cases in vector space models (e.g. the n-real-tuples space and the directed line-segments space).

It is important to note that abstraction and analogy are different strategies. In abstraction, in contrast to analogy, the concept presented to the student is an instance of the general concept to be learned. In analogy, the two concepts are different, but there are some points of similarity between them.

Two approaches to the abstraction process were found: one is standard and used by most of the elementary textbooks; the other, suggested by Pedoe (1969), is unique. Both warrant attention. In the standard approach, a definition or the ovem is generated from a single particular case, often a geometrical one. The definition of vector space, however, is not treated uniformly by textbooks that use this approach. Some textbooks (e.g. Lange, 1968) abstract axioms for a vector space from *one* specific model, usually ata the lowest level of generality. Others (e.g. Fisher, 1970; Pillis, 1969) begin with a *vareity* of models, at generality levels 1 and 3 (e.g. directed-segments, n-tuples, matrices, polynomial functions, and solutions to linear equations), and generate from their common propertctor space, however, is not treated uniformly by textbooks that use this approach. Some textbooks [e.g. Lange, 1968] abstract axioms for a vector space from *one* specific model, usually at the lowest level of generality. Others [e.g. Fisher, 1970; Pillis, 1969] begin with a *variety* of models, at generality levels 1 and 3 (e.g. directed-segments, n-tuples, matrices, polynomial functions, and solutions to linear equations), and generate from their common properties the vector space axioms. It is likely that the latter approach has a stronger motivational effect than the former one because experiencing different situations with common properties enables the student to understand the necessity for representing these situations by a general concept.

Pedoe's approach to the abstraction process is unique in that it offers an intermediate stage between the particular and abstract. Pedoe's abstraction process, which seems compatible with the "spiral curriculum" of Bruner [1960] and the "learning cycle" of Dienes and Golding [1971], consists of three phases. In the first phase, central concepts (such as linear combination, dependence, independence, basis, and dimension) are defined and applied in vector space models of generality level 1 — coordinate geometry of two and three dimensions. In the second phase, these central concepts are redefined in a vector space model of generality level 2: — $\mathbf{R}_n$. Finally, these concepts are defined a third time in the general vector space with the highest level of generality (i.e. level 4).

The way in which Pedoe presents linear algebra concepts in the first phase is also unique. The emphasis is on inquiry into aspects of analytic geometry through problem solving in terms of linear algebra. The cognitive benefit of this approach is clear: Due to the problem solving approach, students form deep geometrical representations of linear algebra concepts before they encounter these concepts in other spaces at higher levels of generality.

Isomorphization. Isomorphization is a strategy used to motivate definitions of mathematical operations. If the student does not see the rationale for a definition, the concept being defined seems arbitrary. This has a negative motivational effect on the learning of the definition. In order to eliminate that arbitrariness from definitions of mathematical operations some authors impose an isomorphism on two mathematical structures where one of these structures is familiar to the student. For example, if the ordinary operations of linear transformations are familiar to the student, the operations on matrices are presented as those conditions on the function associating a linear transformation with its matrix representation that make it an isomorphism [see Krause, 1970]. Similarly, if $\mathbf{R}_n$ is familiar to the student, and if operations on polynomials are to be defined, a natural isomorphism between $\mathbf{R}_n$ and $\mathbf{P}_{n-1}$ (the space of polynomials with degree less or equal to $n-1$) is imposed.

Postponing. In some textbooks the introductory material merely consists of statements concerning the necessity, importance, and centrality of the ideas to be learned. We call this a postponing strategy since the need and importance of these ideas are not obvious. For example, Hoffman and Kunze [1975] introduce the linear transformations topic by a single statement: "We shall now introduce linear transformation, the object which we shall study in most of the remainder of this book" (p. 67). Frequently, the postponing strategy is reserved for use in more advanced textbooks.

Embodiment

Dienes [1960, 1964] identified the principle of multiple embodiment as an instructional tool for enhancing the understanding of concepts and for retaining mathematical structures. In linear algebra this principle is found in the process of translating general definitions and theorems in terms of given situations. Showing that the set of directed line segments is a vector space and showing that a set of n polynomials of degree less or equal to n is linearly dependent are examples of this process. The embodiment process, in constrast to the abstraction process, comes after the general concept is presented.

We examined the embodiment process with respect to familiarity and mode of representation (algebraic vs. geometrical) aspects. It was found: (a) most of the textbooks use algebraic embodiments rather than geometrical ones, and none of the embodiments are consistently used; (b) most of the textbooks pay scant attention to familiarizing the student with the embodied situations despite the fact that most of these situations cannot be expected to be familiar to a beginning student of linear algebra.

Familiarity and mode of representation in the embodiment process affect the formation of the student's concept representation: important components in the mental representation of a concept are its external, physical referents; sources for forming this component are concrete, or at least semi-concrete, embodiments. Since in the abstract system of linear algebra concrete embodiments are complex physical systems which are not within the scope of undergraduate curricula, it is the semi-concrete embodiments (i.e. the geometrical mode of representations) that must contribute to forming physical representations of concepts. On the other hand, it is obvious that the embodiment process can have no constructive cognitive effect if the situation being embodied is not fully understood by the student.

Harel [1985] described the difficulties students have with algebraic embodiments dealing with mathematical systems whose elements are collections of numbers. One of the main reasons for these difficulties is that these systems do not have an easily accessible geometrical or other visual representation to describe their operations and relations. When beginning students are presented with these systems, they encounter, probably for the first time, mathematical systems different in nature from the number system. They have difficulties accepting the idea that a collection of numbers, such as a matrix, or a function, is a mathematical entity. That is, a mathematical object within a system that has its own structure and its own operations (addition and scalar multiplication).

Symbolization

A definition of a mathematical object consists of variables. Some of these variables are encoded in the symbol of the object, others are not. Those which are not encoded in the symbol are, usually, fixed throughout the discussion of the object. For example, the "Matrix of the Linear Transformation T Relative to the Pair of Ordered Bases ϕ and μ" can be represented by $[T]_{\phi\mu}$. If during the discussion of this concept the bases ϕ and μ are fixed, the concept can be represented by the simpler symbol $[T]$. (See Note [1]) On the other hand, to indicate that the matrix representation of transformation T depends on the relationship between the bases in its range and in its domain, the symbol $[T]_{\phi\rightarrow\mu}$ might be used.

Textbooks in linear algebra lack uniformity in this aspect of symbolization. Consider, for example, the following additional symbols assigned to the above concept:

> We define the matrix representing ∂ with respect to the bases A and B to be the matrix $[a_{ij}]$.
> [Nering, 1970, p. 38]

$$A = \begin{matrix}\text{matrix of } T \text{ with}\\ \text{respect to the}\\ \text{bases } B \text{ and } B'\end{matrix} = [[T(u_1)]_{B'} : [T(u_2)]_{B'} : \ldots : [T(u_n)]_{B'}$$

[Anton, 1981, p. 248]

In the former symbolization, the non-fixed variable "dependency between ∂ and the bases A and B" is not encoded, whereas in the latter symbol, the superfluous variables $u_1, u_2 \ldots u_n$ are encoded. It is likely that a symbol is better understood and remembered if it expresses the main and salient variables in the concept it represents. On the other hand, a symbol that excludes non-fixed variables or includes superfluous variables would not correspond adequately to the meaning of the concept it represents, and consequently it becomes difficult to encode and to recognize.

Summary

Considerable evidence shows that topics and emphases included in a mathematics textbook have an effect on the students' understanding and on mathematics educational goals [Begle, 1979]. However, most textbooks do not express explicitly their conceptions and approaches; these have to be inferred from the content presentation. The analysis provided here identifies conceptions and approaches of textbooks in linear algebra with respect to the content presentation of 5 variables. Only one variable — "levels of generality" — is specific to the linear algebra content; the other four are general and independent of a specific content. Since many instructional topics in mathematics involve these four variables, our analysis can be used as a scheme for analyzing textbooks in other areas, for examples, group theory. This scheme may help both instructors and textbook authors to consider the content presentation aspects discussed in this paper.

Some general questions have emerged from this analysis:

1. Is one of the sequencing approaches — computation-to-abstraction and abstraction-to-computation — pedagogically more effective than the other?
2. Dieudonné's textbook [1969] is the only one which consistently deals with linear algebra at generality level 2. Is Dieudonné's teaching approach more apt, as he claims, to lead to increased achievement in linear algebra?
3. What should be done to get students to absorb the important idea that the derived results in the system of linear algebra depend solely on the axioms of vector space, not on the definitions of specific elements? (An answer to this question was discussed in Harel [1985]).
4. Do the strategies of analogy, abstraction, isomorphization, and postponing affect differently the students' understanding of new material?
5. Is one of the embodiment representations — geometrical and algebraic — pedagogically more effective than the other? (An answer to this question was discussed in Harel [1985]).
6. To what extent does the encoding of symbols (with respect to the aspect discussed earlier) affect the understanding and remembering of symbols?

Note

[1] Although the variable T is arbitrary and is therefore fixed during the discussion of the concept, it is represented in the symbol. This phenomenon is common in symbolizing functions: thus the symbol $f(x)$ indicates that x denotes items from the function's domain; similarly, the symbol $[T]$ indicates that T denotes items in the domain of the function [ ].

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