The State of High School Textbooks

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Do textbooks matter? According to Education Market Research [1], in 2001–2002, K–12 school districts spent more than $4 billion on textbooks, and in mathematics an estimated $1.95 billion was spent nationwide in 2008 on mathematics instructional materials—a robust 25.8 percent increase from 2005. Tyson-Bernstein & Woodward [3] found that, while textbooks are a dominant part of teaching and learning in all subjects, in mathematics the reliance on textbooks is even greater. Textbooks are the primary source for planning daily mathematics instruction by teachers, according to Weiss et al. [4].

Given the high expenditure on mathematics textbooks and the central role they play in mathematics instruction, evaluating their quality and effectiveness would seem necessary. Recently, we had the opportunity to review the high school textbook series from four different publishers [2], [5]. We found the experience an enlightening but depressing one. We found an interesting combination of mathematical errors, obliviousness to basic foundational work, and a deliberate avoidance of symbolic manipulation in algebra and of a coherent postulate-based approach to geometry.

Our examination focused on two topics in algebra, forms of linear functions and equations and forms of quadratic functions and equations, and one topic in geometry, parallel lines and the triangle sum theorem. These topics were chosen because they are viewed as central to the high school curriculum. The examination was to ensure that they are coherently developed, completely covered, and mathematically correct, and provide solid foundation for further study in mathematics.[1]

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Linear Functions

The algebraic concepts and skills associated with linear functions are crucial for the rest of the study of algebra and beyond. Appropriate definitions and justifications for concepts such as slope provide the basis for understanding linear functions and equations. These issues were carefully examined, as were the presence of all forms of linear functions and equations, how these are connected to each other, and the opportunities given to apply them to solve problems.

Understanding linear functions is fundamental to a good Algebra 1 course. The connection between the graph of a linear function and the algebraic version is important. We were disappointed. No program produced the basics here. Slope, although defined, is never shown to be well defined. It is never shown that the graph of an algebraic linear function really is a line in the coordinate plane, and it is never shown that a line in the coordinate plane really is the graph of an algebraic linear function. The worst aspect of this was that it seemed the textbook authors were unaware that something was missing. There are no

[1] *We are, of course, not the first to notice mathematical problems with high school texts. For example, in the National Mathematics Advisory Panel final report in 2008,* Foundations for Success, *in Appendix B of the Conceptual Knowledge and Skills Task Group Report, Chapter 3, pages 63–65, there is a very nice summary of problems in high school algebra texts.*

comments such as "it can be shown." They don't always even state these things clearly. A typical sequence is:

"A function whose graph forms a straight line is called a linear function."

"A function is linear if it is described by a linear equation. A linear equation is any equation that can be written in the standard form shown below:

$$Ax + By = C$$

where A, B, and C are real numbers and A and B are not both 0."

Ignoring the little detail that if $B = 0$ we don't have a function, there is the rather large leap of faith: straight line graphs come from linear equations. There is no attempt to show this. For that matter, there seems to be no awareness that it should be shown. The worst part is that this is as good as it gets, because they actually tell you that functions that give lines come from linear equations.

From another book that is less sure of itself, we have: "A **linear equation** is an equation that forms a line when it is graphed. Linear equations are *often* written in the form $Ax + By = C$." (Our emphasis.) Again, no attempt to show that the equation gives rise to a line or to explain what equations give lines "not so often". But, it gets worse when they try to be more specific:

"The **standard form of a linear equation** is $Ax + By = C$, where $A \geq 0$, A and B are not both zero, and A, B, and C are integers with greatest common factor of 1."

Gone are real numbers. However, just when you think things can't get worse, open up another book and, after using a calculator to recursively plot points, you get:

"The points you plotted in the example showed a **linear relationship** between floor numbers and their heights."

This is it for the definition of linear relationship for this program.

For our final example, we have "**linear functions**—those with straight-line graphs, data patterns showing a constant rate of change in the dependent variable, and rules like $y = a + bx$." They explain why they use $y = a + bx$ instead of $y = mx + b$: "Statisticians prefer the general form $y = a + bx$." Mathematicians use $y = mx + b$. This is made explicit in the text and that is the explanation for the choice.

At best the relationship between a linear equation and the graph of a linear function as a line is assumed. At worst, it is murky. A real mathematical connection is never made, though.

Slope

In general, the slope of a line in the coordinate plane is defined as the change in y divided by the change in x, or the rise divided by the run. The problem, of course, is that the definition only involves two points on the line, and you really want to get the same answer (slope) if you use two different points. This is shown to be the case using similar triangles. However, texts never seem to see the need to show that slope is well defined, so we never see a proof. Obviously this must get pointed out to some authors, but they don't always know what to make of it. So an alternative definition of slope is that it is the rise divided by the run for "any two points on the line". They didn't quite get the message, though. Now you have to show that this is the same number for all pairs of points, just like before.

Linear equations and functions can be written in many forms, for example $Ax+By = C$, $y = mx+b$, $y - y_0 = m(x - x_0)$, and $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$.

The letter "m" in $y = mx + b$ is assumed to be the slope, but this connection is rarely made mathematically.

Not all texts introduce all of these forms, and even when they do, they don't always do the general algebraic version. So a text might show how to find the equation for a line with slope 3 that goes through the point (2,1) but won't show how to find the equation for a line with slope "m" that goes through the point (x_0, y_0) or show how to rewrite $y - y_0 = m(x - x_0)$ or $Ax + By = C$, $B \neq 0$, in the form $y = mx + b$.

There seems to be a general tendency to avoid algebraic manipulation if at all possible. The closest any book comes to proving that slope is well defined is "You've probably noticed by now that the rate of change of a linear function is constant."

Quadratic Functions

The ability to put quadratic functions in vertex form allows access to symmetry and finding the maximum or the minimum of the function. This opens up a new world of problems that can be solved, namely max/min problems. The approach to max/min problems is examined for both the basic algebra and the conceptual development, which includes a coherent definition of a quadratic function and how the line of symmetry is explained and justified.

Core to Algebra 2 is a complete understanding of quadratic functions. Problems similar to those found for linear functions also occur here. The texts tend to love the symmetry of quadratic functions, and we'll now take a look at how they deal with it.

Our first book actually shows the symmetry for $y = x^2$. It then immediately states that "This shows that parabolas are symmetric curves. The **axis of symmetry** is the line through the vertex of a parabola that divides the parabola into two congruent halves." Of course this makes you want to back up a few pages and find out what a parabola is. We are told that the graph of $y = x^2$ is a curve called a parabola.

Next they show how to do vertical and horizontal translations of the function $y = x^2$ to get other quadratic functions. They could show that these new functions $y = (x - h)^2 + k$ are also symmetric, since they are just translations, but this is not mentioned. Next, functions are multiplied by a constant to reflect, stretch, and/or compress them. We now have the vertex form of a quadratic function: $y = a(x - h)^2 + k$, which this book uses for its definition of a quadratic. It is now described as a parabola, or, more accurately, the fact that the graph is a parabola is taken for granted in the following way that also gives us the **vertex** that we also wanted to look up: "If a parabola opens upward, it has a lowest point. If a parabola opens downward, it has a highest point. This lowest or highest point is the **vertex of a parabola**." Up to this point, we didn't even know these were parabolas. Neither symmetry nor the name was carried along through the transformations.

This particular text is as good as it gets. Next it mentions the **standard form** of a quadratic function as $f(x) = ax^2 + bx + c$ and then expands the vertex form and solves for b and c in terms of h and k, as well as h in terms of b and c. Consequently, they have computed the x-coordinate for the line of symmetry as $\frac{-b}{2a}$, assuming there was a line of symmetry. A good, knowledgeable teacher might help make sense of all of this.

Our second example takes for the definition of a **quadratic function**:

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0.$$

And we are told "The graph of a quadratic function is called a **parabola**." We are then shown a graph of one (1) example of a quadratic, $f(x) = 3x^2 - 12x + 6$, and the book concludes: "The **axis of symmetry** is a line through the graph of a parabola that divides the graph into two congruent halves." And we are given the equation for the axis of symmetry: $x = \frac{-b}{2a}$. There is no computation or justification either for the existence of the axis of symmetry or the value of x that gives it. It does have the advantage that it is clean, quick, and simple.

For our next example we again start with $y = x^2$ and look at vertical and horizontal translations. We are told that the shape of this graph is a **parabola** and that parabolas always have a **line of symmetry**. The proof that $y = x^2$ is symmetric is just $(-x)^2 = x^2$, but this is not done. Reflections and stretches (as transformations) are done in general, but not specifically for quadratic functions. A hundred and fifty pages later **quadratic functions** are defined as second-degree polynomial functions, and it is asserted that all quadratic functions come from transformations of $y = x^2$. In fact, they have shown no such thing. They have shown that quadratics in the vertex form come from such transformations, but they have not shown that all second-degree polynomials can be written in vertex form. In addition, they now call all quadratic functions parabolas. Symmetry is not mentioned as these transformations occur. Symmetry could be carried along, but they do not bother.

In our final example, "quadratic function" is used for several pages without definition. Eventually the general form is coughed up. We are told that the graphs of all quadratics are called parabolas. It is left to an exercise to show that $y = ax^2$ is symmetric, and this is generalized to $y = ax^2 + c$. The next step is harder. The authors just assume symmetry for $y = ax^2 + bx$. This makes it easy to compute the line of symmetry and to get the line of symmetry for the general form from that.

It isn't hard to show that $ax^2 + bx$ has a symmetric graph, but it isn't done. Is it skipped intentionally, or do the authors just not realize what they are doing?

The second author's early 1960s (small-town Kansas) Algebra 2 textbook shows that $x = \frac{-b}{2a}$ must give a maximum or a minimum for $y = ax^2 + bx + c$ by plugging in both $x = \frac{-b}{2a}$ and $x = \frac{-b}{2a} \pm k$ and showing that the difference between the two y values is ak^2. This also shows symmetry. None of the textbooks we looked at would dare venture into this level of algebraic manipulation.

To solve max/min problems for quadratic functions it is essential to be able to move back and forth between the standard form, $y = ax^2 + bx + c$, and the vertex form, $y = a(x - h)^2 + k$. Once this is done, symmetry is quite easy to see, but only if it is pointed out. It is not pointed out. As with linear functions, this is much more likely to be done with numbers, if at all, rather than for the general algebraic version.

Triangle Sum Theorem

The development and application of the triangle sum theorem (that the sum of the angles of a triangle is 180 degrees) was examined. Despite the simplicity of both the theorem and the proof, they depend on a great many preliminary results, postulates, and definitions. It is an excellent way to see if the material is structured properly. For example, the theorem depends on a solid understanding of parallel lines, the lines that cross them, and the angles associated with them all. The examination focused on the coherence and logical progression of the material leading up to the theorem.

Euclidean geometry is perhaps the only place in high school mathematics where a (relatively) complete and rigorous mathematical structure can be taught. However, deductive geometry can be treated in numerous ways and in different levels of rigor. Our examination was based on the view that an adequate level of rigor is necessary and possible in high school. Deciding what constitutes an "adequate level of rigor" is crucial. In dealing with this question, we used Euclid's *Elements* as a framework. In a program consistent with this framework, subtle concepts and axioms, such as those related to "betweenness" and "separation" are dealt with intuitively, but the progression from definitions and axioms to theorems and from one theorem to the next is coherent and logical and exhibits a clear mathematical structure.

Two of the geometry books we reviewed do not have a clear logical structure for the material taught, and there is no clear development of a demarcation line between empirical reasoning and deductive reasoning. In one of these books, all the material is presented through problems. There is nothing wrong with this approach, except that all the problems seem to be of equal "status". However, although some problems are essential to the development of a geometric structure, others are not. For a teacher to discern the essential mathematical progression, he or she must identify all the critical problems—many of which appear in the homework sections—and know in advance what the intended structure is. Missing one or two of these problems would result in an incomplete or deficient structure.

In this book, important theorems in geometry are not justified. Moreover, with the way the material is sequenced, some of these theorems cannot be justified. Specifically, the construction of perpendicular lines requires congruence of triangles, which appears in earlier courses but mostly in empirical, not deductive, forms. Congruence appears later, after parallel lines, and is based on similarity, which, in turn, is based on parallel lines. Thus the construction that is fundamentally needed for parallel lines can only be justified by results that are based on parallel lines! In other words, the argument is circular.

Also, due to the program's choice of starting with parallel lines rather than congruence, there is loss of an opportunity to convey a critical mathematical lesson about the role of postulates in the development of mathematical structures—that a whole constellation of theorems can be proved without the use of the parallel postulates. This lesson—a landmark in the historical development of mathematics—should be within the grasp of high school students.

In another book the approach amounts to empirical observations of geometric facts; it has little or nothing to do with deductive geometry. There is definitely a need for intuitive treatment of geometry in any textbook, especially one intended for high school students. But the experiential geometry presented in the first nearly 800 pages of the book is not utilized to develop geometry as a deductive system. Most, if not all, assertions appear in the form of conjectures, and most of the conjectures are not proved. It is difficult, if not impossible, to systematically differentiate which of the conjectures are postulates and which are theorems. It is difficult to learn from this text what a mathematical definition is or to distinguish between a necessary condition and a sufficient condition. Students are also expected to discover definitions given pictures as hints.

The development that leads up to the proof of the triangle sum theorem in the other two books does not include circular reasoning. However, there is repeated misuse of the concept of postulate, and some important theorems are stated without proof. In addition, this development is interrupted by two sections on analytic geometry, with theorems that are either incorrectly labeled as postulates or appear without proof. In the process of developing a deductive structure for synthetic geometry, the text introduced a "foreign object", analytic geometry, which does not belong to the development of this structure.

Conclusion

The texts discussed are not unusual. Middle and elementary school textbooks are no better, perhaps even worse. How can mathematics be viewed as logical when foundational work is missing, and, worse, it is not even pointed out as something that could or should be done?

The purpose is to inform the college mathematics teaching community about the sorry state of high school textbooks. There is much to complain about in college textbooks as well, but at least they are usually written by mathematicians with some sense of mathematical integrity.

References

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