

# Common Core State Standards for Geometry: An Alternative Approach

*Guershon Harel*

## Introduction

Forty-five states, including California, have adopted common core standards in mathematics for kindergarten through grade twelve. The CCSS (Common Core State Standards) [3] are designed to provide strong, shared expectations and, furthermore, allow the adopting states to collectively create and share tested tools such as assessments, curricula, and professional development programs. *Achieve* [1] has made this observation and further analyzed the CCSS in relation to the NAEP (National Assessment of Educational Progress) framework. Their findings suggest that the CCSS as a whole are mathematically more demanding than the current mathematics curricula. This is particularly true for the geometry standards due to their emphasis on mathematical proof, which, as has been widely established, is one of the most difficult concepts for students [8], [10].

Geometry often represents a high school student's first formal introduction to abstract mathematical reasoning. That is, the student is asked to (a) reason about such abstract concepts as points, lines, and triangles; (b) understand that certain geometric objects can be defined only in terms of their relation with each other; and (c) prove theorems about the Euclidean structure based on a small set of basic concepts and axioms. Not only is this type of reasoning fundamental for more advanced courses in mathematics, it is also prevalent in many areas of science and engineering where one reasons about simplified or ideal models. For example, in the applications of the theory of gravity, one thinks of mass being concentrated in a single point. Similarly, in the kinetics of motion,

solid objects are almost always considered as points at their “center of (inertial) mass”, and the most fundamental insight in all of kinetics is that objects proceed in straight line motion unless acted on by a force. The idea that *point*, *line*, *plane*, and *space* cannot be characterized independently of each other is analogous to situations where a physical quantity can be measured only by observing its interaction with other quantities (e.g., spin can be measured only by observing its interaction with other magnetic fields). Finally, the hypothetico-deductive method is undoubtedly the basis of all science. The practice of proving theorems on the basis of a small set of axioms serves as a cognitive precursor to the vital scientific practice of deriving one law from more fundamental laws; the derivation of Kepler's laws from Newton's laws—one of the greatest triumphs of calculus—is an example. Because high school geometry plays a key role in the development of our students' ability to reason about abstract models in mathematics, science, and engineering, we believe that it is absolutely essential that teachers and curriculum developers attend to the question of how to motivate abstract geometric concepts and reasoning to their students.

A close analysis of the narrative of the CCSS in high school geometry (hereafter, CCSS-Geometry) has revealed potentially serious problems with their future implementations. Our concerns were further validated by some of the initial curricular material developed, presentations given by teachers and curriculum developers in teacher conferences, and conversations with teachers who have participated in regional professional developments targeting the CCSS-Geometry. Collectively, these materials and activities represent a particular interpretation of the CCSS-Geometry that is pedagogically unsound. Given its current

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DOI: <http://dx.doi.org/10.1090/noti1070>

widespread use, we call this interpretation the *standard approach*.

Upon this realization, we developed an alternative interpretation of the CCSS-Geometry grounded in the *intellectual need* of the students. The notion of intellectual need was defined technically and discussed at length in [9] and will become clearer as this article unfolds. Generally speaking, intellectual need is an expression of a natural human behavior: When people encounter a situation that is incompatible with or presents a problem that is unsolvable by their existing knowledge, they are likely to search for a resolution or a solution and construct, as a result, new knowledge. Such knowledge is meaningful to them because it is a product of their personal need and connects to their prior experience. This human nature is the basis for what we call the *necessity principle*: *For students to learn what we intend to teach them, they must have a need for it, where "need" refers to intellectual need, not social or economic needs* [7], [9]. Pedagogically, this principle translates into three concrete instructional steps: (1) Recognize what constitutes an intellectual need for a particular population of students (high school students in our case), relative to the concept to be learned. (2) Present the students with a problem or sequence of problems that corresponds to their intellectual need and from whose solutions the concepts can be elicited. (3) Help students elicit the concepts from solutions to these problems.

### The Standard Approach

The CCSS-Geometry consists of goals together with narrative. We agreed with the goals (e.g., *reason abstractly, construct viable arguments, use appropriate tools strategically, attend to precision, look for and make use of structure, and look for and express regularity in repeated reasoning*), and we are particularly pleased with the new emphasis on geometry proof and construction (*prove geometric theorems, make geometric constructions*). We are concerned, however, with the narrative and especially with how it is currently being interpreted. Our characterization of the standard approach is, thus, based on this narrative and other resources, such as test items, presentations given in teacher conferences about the CCSS-Geometry, and some of the Common Core-based texts currently under examination for adoption by school districts. The ultimate goal of this article is to point out potential pitfalls with this approach and offer an alternative one.

Our main concerns can be summarized as (1) lack of attention to students' intellectual needs, (2) premature introduction to and overemphasis on plane transformations, and (3) lack of clarity about the importance of separating the analytic study and the synthetic study of Euclidean geometry.

### Lack of Attention to Students' Intellectual Needs

It is hard to imagine any educator disagreeing with the CCSS goal: "During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs." The question of critical importance is, How do we bring students to see an intellectual need to formalize their informal geometry experiences into formal definitions and to reason deductively when justifying or refuting claims? That is to say, how do we problematize students' current experiences so that they come to understand and appreciate precise definitions and careful proofs? The conceptual basis for this question is Piaget's theory of learning (see, for example, [13]). In essence, according to this theory, the means—the only means—of learning is problem solving. In Piaget's terms, a failure to assimilate results in a disequilibrium, which in turn leads the mental system to seek equilibrium to reach a balance between the structure of the mind and the environment. Learning transpires when such balance occurs. While Piaget himself concentrated on the development of mathematical knowledge at the early stages, his theory was extended and applied to advanced topics going into undergraduate mathematics.

The question of intellectual necessity does not seem to be part of the repertoire of considerations by current Common Core-based curricula. The transition from informal to formal is done largely descriptively by merely restating concepts and assertions assumed to be understood intuitively into formal definitions, theorems, and proofs. Basically, this is precisely the approach that has been in use for decades, which, as we now know from status studies on students' conceptions of geometry and proof, has failed miserably. In aggregate, these studies show that "students accept examples as verification, do not accept deductive proofs as verification, do not accept counterexamples as refutation, accept flawed deductive proofs as verification, accept arguments on bases of other than logical coherence, offer empirical arguments to verify, cannot write correct proofs" ([10], p. 59).

A case in point is the ability to characterize objects and prove assertions in terms of mathematical definitions, what might be called *definitional reasoning*. Evidence exists to indicate that this way of thinking is difficult to acquire. For example, in Van Hiele's model [14], only those high school students who reach the highest stage of geometric reasoning can reason in terms of definitions (see [2]). College students too experience difficulty reasoning in terms of definitions. For example, asked to define "invertible matrix", many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a nonzero determinant", "a square matrix with full rank", etc.) rather than a definition. The fact

that they provided more than one such property is an indication they are not definitional reasoners [6]. The standard approach seems to take for granted definitional reasoning. For example, taking the CCSS-Geometry's statement, "Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc" [3] literally in geometry curricula, as some current Common Core-based textbooks do, is likely to be unproductive for most students. Just telling students that a particular term is undefined does not guarantee that they will consider it as such and use it as a basis for defining other concepts.

Over a century ago, a great mathematician, with deep pedagogical sensitivity, pointed to the challenge of definitional reasoning: "What is a good definition? For the philosopher or the scientist, it is a definition which applies to all objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils..." (Poincaré, 1952, p. 117).

#### **Premature Introduction to and Overemphasis on Plane Transformations**

Central to the CCSS-Geometry is the concept of *plane transformation*. In the standard approach the transition from middle school geometry to high school geometry is to be carried out through the rigid motions of *translation*, *reflection*, and *rotation* and the motion of *dilation*. In middle school geometry, these motions are delivered informally, and in high school they are defined as functions on the plane. In both levels, the motions are merely described, not intellectually necessitated through problems the students understand and appreciate, for example, by helping high school students see the power of reasoning in terms of plane transformations when solving geometry construction problems, as we will see in the next section.

This intellectual-need-free pedagogy has been dominant in current mathematics curricula. One of its characteristics is that it takes certain difficult conceptualizations for granted. Relevant to our discussion here is the overwhelming evidence that students have enormous difficulties with the concept of function and that the process of acquiring this concept is inevitably challenging (see, for example, [5], [12]). Yet, plane transformations are central in the CCSS-Geometry. The formal definitions of plane transformations require the application of advanced conceptualizations of functions, right at the start of the introduction of deductive geometry. And not just any functions! Students must understand transformations as functions from the plane to the plane. Thompson [12] makes a strong argument that this is a tremendous intellectual achievement for students. They think that transformations act only on the points you happen

to have represented and that any transformation moves at most a few points, but of course this is not the case. For example, when one envisions rotating a triangle in a plane, one must be able to think of the rotation as a function acting on the plane, that is, rotating the entire plane, not just the triangle.

Why is there a focus on plane transformations? one might ask. An answer to this question can be found in [3]: namely, that rigid motions create continuity from middle school geometry to high school geometry. Once the motions of translation, reflection, and rotation are defined formally, they provide the foundation for the concept of superposition and, in turn, for *congruence* and congruence criteria; and once the motion of *dilation* is defined formally, it provides the foundation for *similarity* and the similarity criteria.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. ...During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures. ...Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent. [3]

Thus, the main purpose of the focus on plane transformations throughout middle school and high school is merely to establish the concept of congruence and similarity and the criteria for triangle congruence and triangle similarity. Support for this conclusion can be found in [15]:

One cannot overstate the fact that the CCSS do not pursue "transformational geometry" per se. Transformations are merely a means to an end: they are used in a strictly utilitarian way to streamline the existing school geometry curriculum. One can see from the high school geometry standards of the CCSS that, once translations, rotations,

reflections, and dilations have contributed to the proofs of the standard triangle congruence and similarity criteria (SAS, SSS, etc.), the development of plane geometry can proceed along traditional lines if one so desires.

It didn't seem to bother Euclid to define congruence through the image of picking a figure up and laying it on top of another, and it is unlikely that the same action would bother high school students right at the start of their geometry course. The amount of "curricular space" needed to establish congruence and similarity through plane transformations is substantial, as can be seen from the following data: The CCSS-Geometry for high school Euclidean geometry (not including trigonometry) appears under eight headings, three of which (close to 40 percent) are devoted to transformations in the plane. (This does not include the preparation expected in the middle school geometry!) Those eight headings are: (1) Experiment with transformations in the plane (154 words), (2) Understand congruence in terms of rigid motions (98 words), (3) Prove geometric theorems (129 words), (4) Make geometric constructions (77 words), (5) Understand similarity in terms of similarity transformations (126 words), (6) Prove theorems involving similarity (47 words), (7) Understand and apply theorems about circles (88 words), and (8) Find arc lengths and areas of sectors of circles (31 words). This word count shows that the narrative devoted to transformational geometry occupies over 50 percent of the entire narrative allocated to Euclidean geometry.

The standard approach, thus, would require enormous effort and time to be spent on plane transformations—their definitions, compositions, and properties—which will inevitably shift the attention from deductive reasoning, the main objective of the CCSS-Geometry.

#### **Lack of Clear Distinction between Analytic Geometry and Synthetic Geometry**

The CCSS-Geometry indicates the importance of studying Euclidean geometry both analytically and synthetically and, further, points to the role of analytic geometry as a tool to connect algebra and geometry. The CCSS-Geometry, however, is silent about the sequencing of and relative emphasis on the two studies. The result is that, in the name of *integrated math*, some current Common Core-based textbooks blend analytic geometry with synthetic geometry, putting emphasis on the former and giving limited attention to the latter. While both geometries are important, synthetic geometry has special roles in school mathematics. Beyond the problem-solving skills one develops from studying Euclidean geometry synthetically, one also develops a crucial way of thinking: *the desire to know what makes a theorem true, not just that it is true*. Compare, for example, the insight

one gets from a synthetic proof of a concurrency theorem (e.g., "The three medians in a triangle are concurrent") to the insight one gets from an analytic proof of the same theorem. In this respect, the distinction between the two geometries is analogous to, for example, the distinction between bijective proofs and generating function proofs of combinatorial identities and formulas for counting various classes of combinatorial objects. In many cases, bijective proofs provide more insight and understanding of the theorem at hand than proofs by manipulating formal series.

#### **A Nonstandard Approach**

This leaves us with a challenging question: What might be an alternative approach to the standard approach that would address all these concerns and achieve the ultimate goals of the CCSS-Geometry for all students? Our answer to this question was to develop, implement, and test a yearlong geometry unit in deductive geometry that positions the mathematical soundness of the content taught and intellectual need of the student at the center of the instructional effort. For reasons which will become clear shortly, we call this approach a *non-standard approach* and the unit that is based on it *Conversations with Euclid*.

Many features differentiate the *Conversations with Euclid* unit from the curricula that follow the standard approach. We begin with four features as advance organizers to the presentation of the unit: (a) *The Conversations with Euclid* unit intellectually necessitates the abstract nature of geometric objects, including the so-called undefined terms such as *point*, *line*, and *plane* and attends to precise definitions and proofs without dwelling on axioms and postulates; (b) only when these critical skills are advanced significantly does it elicit plane transformations, and it does so by bringing the students to see their power to simplify solutions to geometry problems; (c) only when this power is sufficiently realized by the students does the unit use plane transformations to establish the congruence and similarity criteria; and (d) the unit studies Euclidean geometry synthetically. Following this presentation, we list other fundamental features of the *Conversations with Euclid* unit.

#### **The Conversations with Euclid Unit: A Synopsis**

We piloted the *Conversations with Euclid* unit during the summers of 2012 and 2013 with forty-two in-service secondary school teachers. At the center of the unit is a thought experiment, the goal of which is the process of gradually necessitating the abstract nature of geometric objects and with it the notion of geometric proof. The thought experiment, if extended, can be thought of as an allegory of the historical development of geometry—from Euclid (323–283 BC) to Hilbert (1862–1943). However, the stage of the thought experiment that is relevant to high school geometry



is limited in scope, but it serves as a pedagogical tool and mathematical foundation for the entire Euclidean geometry.

In what follows we present a few sporadic segments from the unit's lessons to illustrate how the *necessity principle* was implemented. These segments alone do not capture the logical flow of the lessons nor do they give a sufficient portrait of the rich classroom debate that was generated as the unit was taught. They do, we hope, provide an image for the nonstandard approach we are advocating.

Due to space limitations, it was necessary to omit most of the narrative within the segments. The parts that are direct quotes from the units appear in italics to separate them from the rest of the discussion.

### **Necessitating the Abstract Nature of Geometric Concepts**

The first lesson of the unit begins with a story about a group of mathematicians engaging in a certain project, and the students are asked to participate in the project's dilemmas and resolutions.

*Imagine an intelligent alien who possesses none of our visual, kinesthetic, or tactile senses and, therefore, does not share any of the images we have for our physical world. A team of four mathematicians, Natalie, Jose, Eli, and Evelyn, began a thought experiment of communicating with such an alien, who they named Euclid.<sup>1</sup>*

*The mathematicians began with two questions: (1) What aspects of humans' life should they begin describing to Euclid? And (2) What symbols and language to use to communicate to Euclid these aspects of life? After some discussion they decided that, at this stage of the thought experiment, it is best to focus only on the first question. And as to the second question, they decided that for now they should proceed as if Euclid were a person who spoke English and could think logically.*

*But humans' life is very complex—where should they start? Again, after some discussion, they narrowed their focus to humans' physical surroundings. But even this turned out to be very complex; for example, imagine telling someone who is blind and has no movement or touching sensations what a tree is. So the mathematicians decided to narrow their thought experiments to four questions:*

- 1. What are the most basic physical objects of our surroundings?*
- 2. What are humans' images for these objects?*
- 3. What is a useful way to imagine these objects when doing math?*
- 4. How can we share with Euclid the way we think of these objects when we do math?*

*You might ask, what is meant by the term "image"? An image of an object is [a discussion*

<sup>1</sup>The names are used in dialogues among the four mathematicians debating various questions.

*of this term appears here.]...we share no physical experiences with Euclid, so the communication with him is expected to be very challenging. This is the reason the mathematicians decided to focus on very simple objects as a start. ...*

### **ACTIVITY: WHILE THE MATHEMATICIANS ARE THINKING...**

**Question:** *What are some of the basic objects of our surroundings that you recommend the team should start with?*

*To the teacher:*

*As students make suggestions, keep reminding them that we are only interested in the spatial description of the objects, not their functions. It is not necessary for the students to come up with and agree upon a particular set of objects. What is important is that the students realize how difficult the assignment is. This will prepare them for the next mathematicians' decision to focus on the simplest objects of humans' physical surroundings: point, line, and plane.*

*The mathematicians decided to start with what seemed to them the most elementary objects of our physical world: planes, lines, and points. Once this decision was made, they turned to the next two questions in their list:*

*2. What are humans' images of point, line, and plane?*

*3. What is a useful way to imagine point, line, and plane when we do math?*

*It was necessary for the mathematicians to answer these two questions before engaging in the fourth question.*

*4. How can we share with Euclid the way we think of point, line, and plane when we do math?*

### **ACTIVITY: WHILE THE MATHEMATICIANS ARE THINKING ...**

*Discuss in your small group Questions 2 and 3.*

*To the teacher:*

*It should not be difficult for the students to agree that humans' image of a plane is a flat surface extending indefinitely in its four directions: north, south, east, and west; that humans' image of line is a thin thread stretched tight and extending indefinitely in its two directions; and that humans' image of point is a dot.*

*The goal of the discussion of Question 3 is merely to prepare students for the dialogue that follows. In this dialogue, students will learn an explanation as to why it is necessary and useful to go one step further by imagining point, line, and plane as objects with no thickness.*

*Although students should not see this dialogue until after this discussion, the teacher can use the ideas in the dialogue to navigate the discussion.*

*At this point the students are presented with a debate among the four mathematicians*

justifying the need and usefulness of thinking of *point*, *line*, and *plane* as objects with no thickness. The students are also asked to add their own justifications. Following this, the students are asked to suggest ways to communicate to Euclid these images. The goal is to bring them to experience the difficulty of this task so they appreciate the mathematicians' idea to describe the spatial relations among these objects rather than each object individually. After this discussion (in our pilot, this was an intense discussion among the teachers), the students are presented with the mathematicians' ideas. For example, to convey to Euclid the images (1) "Points and lines do not have any thickness", (2) "There are as many lines as pairs of points", (3) "Lines do not bend" and (4) "Planes are flat", the mathematicians chose the following statements, respectively: (1) "Through any point, infinitely many lines can be drawn", (2) "A line can be drawn between any two points", (3) "Only one line can be drawn between two points", and (4) "If a line shares two points with a plane, the line must be on the plane".

In sum, the central theme behind the above activities was that these statements convey an idealized (perfect) version of the points, lines, and planes we experience in our surroundings. Part of the summary of Lesson 1 concerning points and lines is:

*First, the points and lines we draw on a sheet of paper, for example, have width, no matter how small we draw them. The points and lines we conveyed through these statements have no thickness. Second, while we can imagine drawing infinitely many lines through a point, no one can do so with a pencil on a sheet of paper, no matter how sharp the pencil is. Third, we can imagine extending a segment indefinitely in both directions, but in reality any line is of a limited length. Fourth, we can imagine segments of any length. When communicating with Euclid, we should always remember that, so far, these are the only images he has for our physical world; if we don't, he wouldn't understand what we say to him about objects made of points, lines, and segments.*

We eventually call statements such as statements 1–4 postulates. However, the goal is not to provide complete rigorous axiomatic foundations to geometry. Rather, the goal is to necessitate the idea that in geometry we deal with idealized physical objects, not the actual objects in our surroundings, and so geometric figures we draw on a sheet of paper are merely signs representing our memory images of spatial perceptions. As can be seen in the dialogues below, this theme is central

in the development of the unit. The theme is absolutely essential to the development of geometric proofs for the crucial reason that the incorporation of geometric figures in proofs is possible only if one understands the axioms that underlie these figures. (More on this point on page 36.)

### Necessitating Geometric Proofs and Constructions

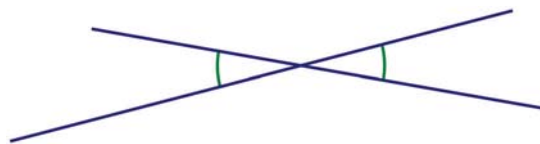
To necessitate the idea behind geometric proofs, the following question is raised: Can Euclid think logically as we do? The following activity, which appears in the unit as the 2nd Conversation with Euclid (in Lesson 2), illustrates the general approach taken to advance the concept of proof and abstract definitions of geometric objects.

#### 2nd Conversation with Euclid

**We:** *Euclid, we are going to define for you a new concept, called vertical angles. We wonder if you observe any property of these angles.*

**Euclid:** *I am ready.*

**We:** *When two lines cross, four angles are formed. The pairs of angles that do not share a side are called vertical. For us, vertical angles look like this.*



**Euclid:** *First, you know I am blind, so I have no idea how it looks to you. Second, I don't understand your definition. You have never told me what an angle is!*

**We:** *Oh, yes—you are right—we forgot. An angle is the figure formed by two rays which begin at the same point; this point is called the vertex of the angle.*

**Euclid:** *Again, you are using a geometry term I am not familiar with: What do you mean by ray?*

**We:** *Sorry, Euclid. We keep forgetting you don't know many of the terms we know. After this lesson we will define for you a collection of terms which will be in use for some time in our conversations. These are terms we learned in middle school. As to ray, take a segment and extend it in one direction. What you get is called a ray. And just for our record, to us a ray looks like this:*



**Euclid:** *Now I know three new terms: ray, angle, and vertical angles. You are asking me what I observe about vertical angles—right?*

**We:** *Right.*

**ACTIVITY: WHILE EUCLID IS THINKING...**

Predict what Euclid might come up with. (You may experiment with the geometry software available to you.)

**Euclid:** I say that any two vertical angles are equal.

**We:** What makes this true?

**ACTIVITY: PREDICT EUCLID'S PROOF**

**Euclid:** Imagine two lines intersecting; they create two pairs of vertical angles. Take any one of these angles and one of the angles that is adjacent to it. Now...

**We:** Wait a second, Euclid—we need to draw these lines, so we can follow you. We are humans, remember? Unlike you, we sometimes need to draw figures to assist us in our thinking.

**Euclid:** I can't see these figures.

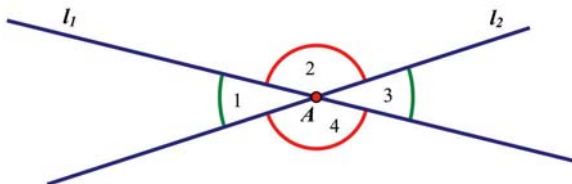
**We:** Yes, we know; these figures are just for us. So for example, when you say "line" or "point," we imagine a line and a point, just like the way you do, and at the same time we draw them on a piece of paper for ourselves.

**Euclid:** Okay.

**We:** We drew two lines,  $l_1$  and  $l_2$  through a point A.

**Euclid:** Good—you can do that because of Postulate 2. Since one can draw unlimited number of lines through a point, one can definitely draw two lines through a point.

[The following figure was described to Euclid and then he continued.]



**Euclid:**  $\sphericalangle 1 + \sphericalangle 2 = 180^\circ$  because  $\sphericalangle 1$  and  $\sphericalangle 2$  are adjacent.  $\sphericalangle 2 + \sphericalangle 3 = 180^\circ$  because  $\sphericalangle 2$  and  $\sphericalangle 3$  are adjacent. From here,  $\sphericalangle 1 = 180^\circ - \sphericalangle 2$  and  $\sphericalangle 3 = 180^\circ - \sphericalangle 2$ . Hence  $\sphericalangle 1 = \sphericalangle 3$ .

**We:** Bravo, Euclid, you proved what you claimed to be true.

**ACTIVITY: COMPARE EUCLID'S PROOF TO YOUR PROOF.**

**7th Conversation with Euclid**

**We:** Let's now go back to the questions you asked in Conversation 3 (not appearing in this presentation).

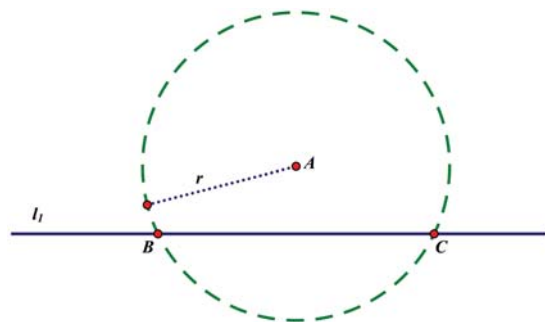
**Euclid:** Oh, yes: You taught me how to imagine constructing a line: I take two points and imagine a line between them, as stated in Postulate 3. You also taught me how to imagine constructing a circle: I take a point in the plane and imagine all the points in the plane that are equidistant from

that point. But how do I imagine constructing two perpendicular lines? You never taught me that. Nor did you ever teach me how to bisect an angle.

**We:** Okay. Let's start with how to construct two perpendicular lines:

1. Take any line. Call it  $l_1$ .
2. Take a point A not on the line.
3. Draw a circle whose center is at A and which intersects the line at two points, B and C. Call the radius of this circle r.

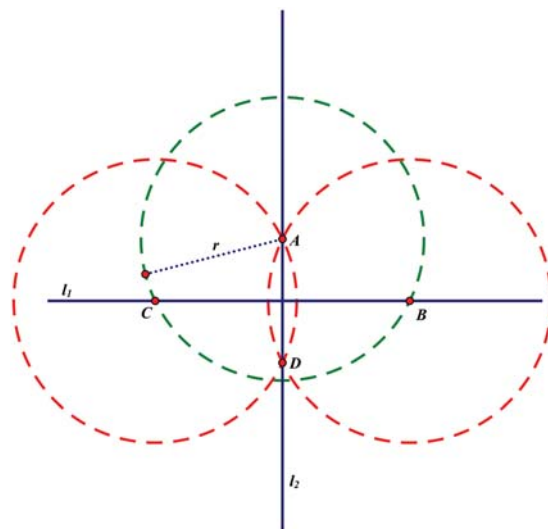
Euclid, pardon us; we have to pause for a few minutes. In order to keep track of our construction, we draw for ourselves each of the steps in the construction. We realize that the only two constructions you can perform are (1) creating a line through two given points and (2) creating a circle with given center and radius. We invented two mechanical instruments to draw these objects. One is called a straightedge, which we use to draw lines, and the other is called a compass, which is used to draw circles.



**Euclid:** Go ahead—I'll wait...

**We:**

4. Draw a circle with center B and radius r, and draw another circle with center C and radius r. These two circles intersect at a point; call it D.



5. Draw a line  $l_2$  between A and D. We claim that  $l_1$  is perpendicular to  $l_2$ .

**Euclid:** Why is that?

**We:** Take the triangles  $ABD$  and  $ACD$ . They are equal in three sides:  $AB=AC$ ,  $DB=DC$ , and  $AD$  is common. Hence, by the SSS Theorem they are congruent.

**Euclid:** Wow—this is exciting!! Can I do the next one (bisecting an angle)?

**We:** Go ahead.

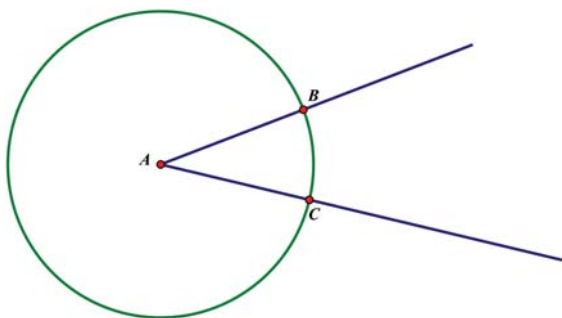
**ACTIVITY: WHILE EUCLID IS THINKING ...**

Predict what Euclid might come up with. (You may use geometry software available to you should you wish to do so.)

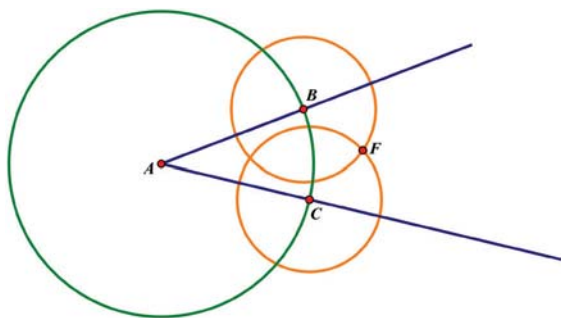
**Euclid:**

1. Take any angle and mark its vertex by  $A$ .
2. Construct a circle with center at  $A$ .
3. The circle intersects the sides of the angle at two points, say,  $B$  and  $C$ .

**We:** Hold on... We need to draw this:



**Euclid:** Now, construct a circle with center at  $B$ , and another circle with the same radius at  $C$ , so that the two circles meet at a point. Call this point  $F$ . I say  $FA$  is on the angle bisector of angle  $A$ .



**ACTIVITY: PREDICT EUCLID'S NEXT STEP**

Now, construct the segments  $BF$  and  $CF$ . The triangles  $ABF$  and  $ACF$  are congruent by the SSS Theorem, since by construction,  $AB=AC$ ,  $BF=CF$ , and  $AF$  is common to the two triangles. Hence  $\angle BAF = \angle FAC$ .

**We:** Done!

**Necessitating Transformational Geometry**

Our treatment of transformational geometry is completely different from that of the standard approach. While in the standard approach, plane transformations appear early in the curriculum, in the nonstandard approach they are defined at the completion of four chapters: congruence, triangle inequality, parallelograms, and circles, focusing extensively on proofs and Euclidean constructions. The following two conversations are five chapters apart: the first aims to raise the need for rigid motions, and the second is a dialogue demonstrating our approach to eliciting them.

**5th Conversation with Euclid**

**We:** The congruence criterion we stated earlier [in Conversation 3] about congruent triangles is true. Let's restate it: If two sides and the angle enclosed by them in one triangle are equal respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.

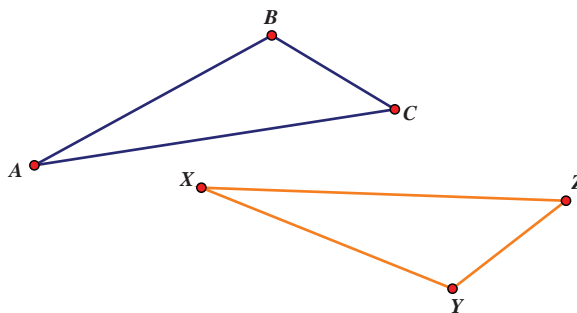
**Euclid:** Why would this be true?

**We:** Here is our first attempt to convince you of this congruence criterion. Tell us if it makes sense to you. Please bear with us, as we have to draw pictures to help us keep track of our description.

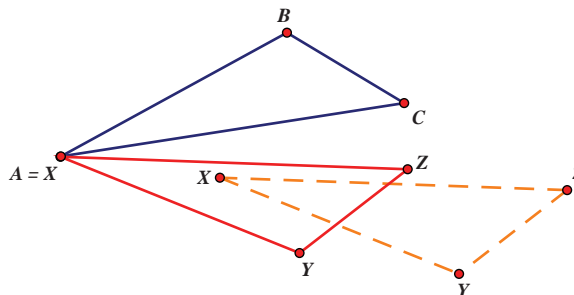
**Euclid:** Go ahead, I am getting used to your habits of mind...

**We:**

1. Say we are given two triangles  $ABC$  and  $XYZ$  with  $AB = XY$ ,  $AC = XZ$ , and  $\angle A = \angle X$



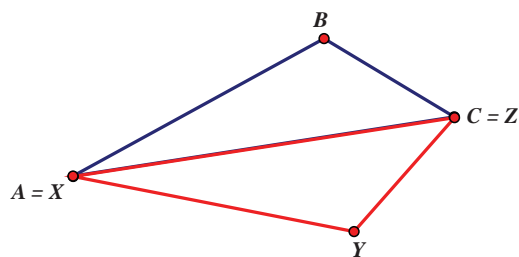
2. Slide the vertex  $X$  onto the vertex  $A$ .



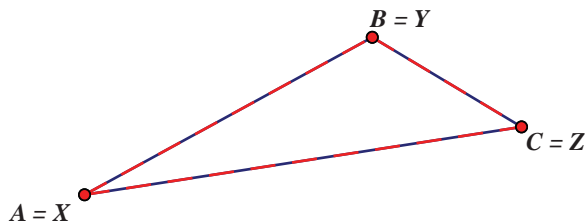
3. Rotate triangle  $ABC$  as needed so that ray  $\vec{AC}$  coincides with ray  $\vec{XZ}$ .



4. Since  $AC = XZ$ , the point  $Z$  coincides with the point  $C$ .



5. Reflect  $\triangle XYZ$  with respect to its side  $XZ$ .  
 6. Since  $\angle A = \angle X$ , ray  $AB$  coincides with ray  $XY$ .  
 7. Since  $AB = XY$ , the point  $Y$  coincides with the point  $B$ .



8. Hence,  $\triangle ABC \cong \triangle XYZ$ .

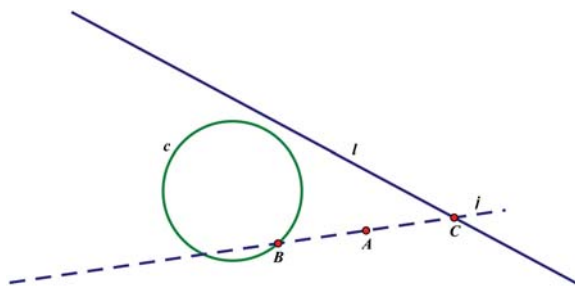
Does this make sense to you, Euclid?

**Euclid:** Not at all. I understood that somehow you made the two triangles coincide, but I didn't understand how you did that. The terms slide, rotate, and reflect are foreign to me.

**We:** You are right. We are planning to define these terms to you sometime in the future and show you more precisely how they can be used to prove this congruence criterion. For your information, Euclid, we have two additional criteria for congruent triangles. They too will be proven in the future.

Five chapters later, we begin to elicit plane transformations. This is done by bringing the students to experience the power and efficacy of reasoning in terms of plane transformations in solving geometry problems. We illustrate this approach with a segment from the lesson dealing with the *half-turn* motion, which follows the lesson on *translation*. The lesson opens with the following plane-geometry construction problem:

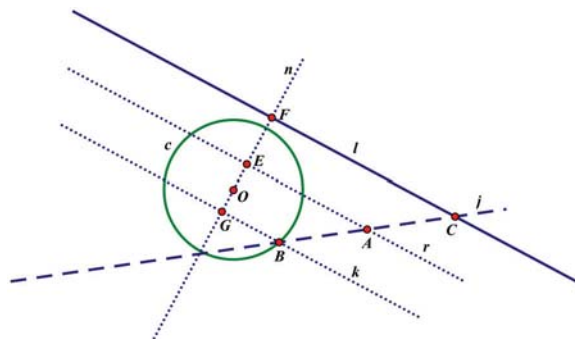
**Problem:** A line  $l$ , circle  $c$ , and point  $A$  are given. Construct a line  $j$  through  $A$  so that the following two conditions hold: (a)  $j$  intersects both  $l$  and  $c$  (call these intersections  $C$  and  $B$ , respectively) and (b)  $AB = AC$ .



Students work in small groups on this problem, and arrive (some with the teacher's help) at the following rather nontrivial solution.

**Solution:**

1. Through  $O$ , the center of the circle  $c$ , construct a line  $n$  perpendicular to line  $l$ , and let  $F$  be the intersection point of  $l$  and  $n$ .
2. Through the point  $A$ , construct a line  $r$  parallel to  $l$ , and let  $E$  be the intersection point of lines  $r$  and  $n$ .
3. Let  $G$  be a point on  $n$  so that  $EF = EG$ .
4. Through  $G$ , construct a line  $k$  parallel to  $l$ , and let  $B$  be the intersection of line  $k$  and circle  $c$ .
5. Let  $j$  be the line through  $B$  and  $A$ , and let  $C$  be the intersection of lines  $j$  and  $l$ .
6. Since  $BCFG$  is a trapezoid and  $AE$  is its mid-segment (why?), line  $j$  is the desired line.

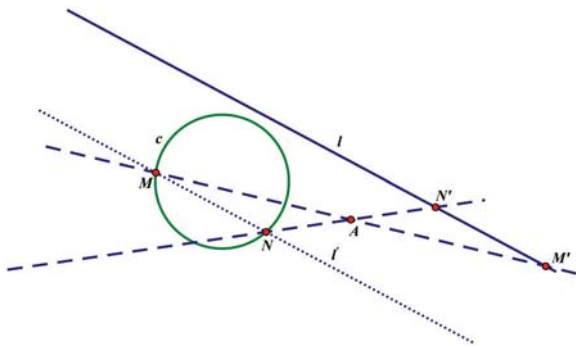


*QED*<sup>2</sup>

At this point, the teacher presents a different solution in the voice of an imaginary student. She tells the students that this same problem was solved in a similar way in Ms. Carlson's geometry class and when the discussion of the solution was ended, Alec, one of students in this class, said:

**Alec:** Nice solution, Ms. Carlson. But I have a different solution. Earlier we learned translation and proved that the translation of a line is a line and of a circle is a circle. Can we also rotate figures?

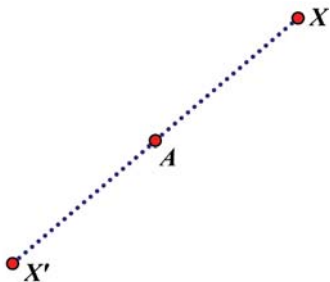
<sup>2</sup>The solution is followed by a group project to determine the number of solutions to the problem.



If yes, rotate line  $l$  about the point  $A$   $180^\circ$ ; we get a line  $l'$ . If  $l'$  intersects circle  $c$  in two points, say,  $M$  and  $N$ , then the lines  $MA$  and  $NA$  are the desired lines, simply because  $MA = AM'$  and  $NA = AN'$ .

**Teacher:** Wow! Let's clarify what it means to rotate a line  $180^\circ$  about a point.

When we rotate a point  $X$   $180^\circ$  about a point  $A$ , we get a point  $X'$ , where  $X, A, X'$  are collinear and  $XA = X'A$ .



Rotating a line  $l$   $180^\circ$  about a point  $A$  means rotating each of the points of the line  $180^\circ$  about a point  $A$ . We get a line  $l'$ .

Bravo, Alec.

**ACTIVITY: PREDICT EUCLID'S QUESTIONS ABOUT ALEC'S SOLUTION AND ANSWER THEM**

The kinds of questions students are expected to raise in the name of Euclid are similar to those Euclid raised in the previous lesson on *translation* (see below). As with the half-turn motion, the translation motion too emerged from a solution to a geometry construction problem involving a *translation* of a circle. After the definition of translation was stated to Euclid, the following conversation ensued.

***n*th Conversation with Euclid<sup>3</sup>**

**Euclid:** From what you have said so far, I conclude that you are claiming that the translation of a circle along a given vector is a circle congruent to the original circle—why?

**We:** Mmm... good question, Euclid. It is so obvious to us, but we understand why you need a proof for

<sup>3</sup>The exact number of this conversation is yet to be determined.

this fact. Give us some time to think how to answer your question.

**ACTIVITY: HOW SHOULD WE RESPOND TO EUCLID?**

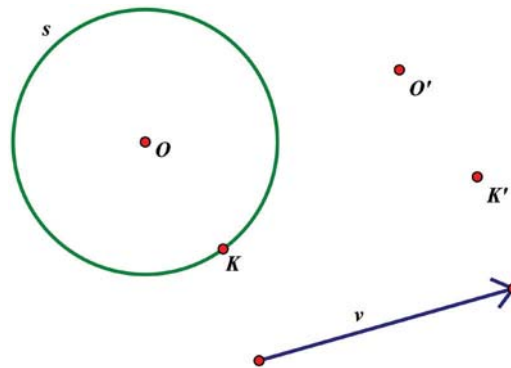
**Euclid:** While you were thinking how to answer my question, I came up with an answer of my own.

1. Let  $s$  be a circle with center  $O$  and radius  $r$ , and let  $v$  be any vector.
2. Translate  $s$  and  $O$  a distance  $|v|$  along the vector  $v$ , to get  $s'$  and  $O'$ , respectively. I will show that the distance of any point on  $s'$  from  $O'$  is  $r$ .
3. Let  $K'$  be any point on  $s'$ .  $K'$  is the translation of a point  $K$  on  $s$ .

By now you perhaps lost me. Go ahead and do your drawings, so you can follow what I say.

**We:** Thank you, Euclid, for being considerate.

**We:** We are ready.



**Euclid:**

4. Consider the quadrilateral  $OO'K'K$ . By the definition of translation, we have  $OO' \parallel KK' \parallel v$  and  $OO' = KK' = |v|$ , and therefore,  $OO'K'K$  is a parallelogram.
  5. Hence  $O'K' = OK = r$ .
- QED

**ACTIVITY: IS EUCLID'S PROOF COMPLETE?**

Finally, at the end of the chapter on plane transformations, we return to the 5th Conversation with Euclid to close what was left open by proving the congruence criteria using rigid motions.

**Characteristics of the Nonstandard Approach**

The nonstandard approach is intended to help students realize the need to state basic assertions about our idealized physical reality (axioms) as well as basic theorems, which they then use to solve geometry problems. These problems are designed to lead the students to derive more facts and theorems in a way that is logically coherent. The students are naturally led to make their arguments and definitions increasingly precise as time goes on so that they can effectively communicate with Euclid. The sample of lesson segments we have just presented raises several questions concerning the *Conversations with Euclid* unit and its implementation.

*What Can Be Assumed about Euclid's Knowledge?* Speaking to Euclid, an “alien” who does not share our taken-as-shared meanings may seem confusing for students, since it is not always clear what can be assumed about his knowledge and what might be considered a fair assumption. For many years I have been using the “game” of conversing with an “alien” as a pedagogical technique to advance students’ ability to formulate and formalize mathematical ideas in geometry. In each case, I find the experience pleasantly surprising. Not only do students enthusiastically engage in the “game” but also very quickly—in a lesson or two—develop a clear sense about “the rules of the game”, that this is a “mathematical game”, not a “language arts game”. For example, students seldom ask whether Euclid knows what is meant by terms such as “true”, “false”, “understand”, “sorry”, or any of the kinds of natural language phrases appearing in the above conversations with Euclid. Students do, however, often raise questions about Euclid’s knowledge of certain geometric facts known to them from previous classes (e.g., “The sum of the angles in a triangle being  $180^\circ$ ”). What is particularly pleasing is that students often raise questions of a subtle nature, for example: Does Euclid know what is meant by “different sides of a line”? “a point between two points”? “direction”? “algebra”? etc.

*What Is the Underlying Structure of the Conversations with Euclid Unit?* The issue of structure is particularly critical in the case of geometry. It is perhaps the only place in high school mathematics where a (relatively) complete and rigorous mathematical structure can be taught. Deductive geometry can be treated in numerous ways and at different levels of rigor. The *Conversations with Euclid* unit uses Euclid’s *Elements* as a framework. In a program consistent with this framework, subtle concepts and axioms, such as those related to “betweenness” and “separation”, are dealt with intuitively, but the progression from definitions and axioms to theorems and from one theorem to the next is coherent, logical, and exhibits a clear mathematical structure. Furthermore, the unit sequences its lessons so that neutral geometry—a geometry without the Parallel Postulate—precedes Euclidean geometry—a geometry with the Parallel Postulate.

Another central theme concerning structure is that the *Conversations with Euclid* unit puts more emphasis on the form of reasoning than on the content to which the reasoning is applied. This is done by continually drawing students’ attention to the cause of facts—what makes a fact to be the way it is—rather than to the fact itself. By this, we try to create and establish the norm that a claim such as “In an isosceles triangle the angles opposite the equal sides are congruent” is less interesting than understanding how the fact that a triangle is

isosceles causes the angles opposite its equal sides to be congruent. In other words, the focus is more on *form* than on *content*.

*How Does the Conversations with Euclid Unit Deal with the Role of Figures in Geometry?* The conversations with a fictitious “alien” who is devoid of visual, kinesthetic, and haptic perceptions has proven to be enormously effective in necessitating the role of figures in geometry proofs, namely, that the geometric figures we draw on a sheet of paper and incorporate in our proofs are mere signs of ideal spatial perceptions which can only be communicated with Euclid by means of agreements (axioms).

Beyond the realization of the need to define objects accurately and objectively and justify assertions logically, we want students to use figures to generate conjectures. This is critical because we do not want Euclid’s constraints to limit the students’ intuition and observations of their physical surroundings. The separation between Euclid’s images and students’ own images helps to combat one of the most prevalent difficulties students have with geometry proofs: namely, student’s reliance on actual figures in justifying mathematical assertion [8], [10], e.g., a geometric relation is true because it looks so.

*What Is the Nature of the Intellectual Need for the Conversations with Euclid?* The intellectual need underlying the *Conversations with Euclid* unit is primarily, but not limited to, the *need for communication*,<sup>4</sup> which comprises two reflexive acts: *formulating* and *formalizing*. *Formulating* is the act of transforming strings of spoken language into mathematical language. *Formalizing* is the act of externalizing the exact intended meaning of an idea or a concept or the logical basis underlying an argument. The two acts are reflexive in that as one formalizes a mathematical idea, it is often necessary to formulate it, and, conversely, as one formulates an idea, one often encounters a need to formalize it. The preconceptualization that orients us humans to these acts when we learn mathematics is the act of conveying, exchanging, and defending ideas by means of a spoken language and gestures, which are defining features of humans.

*What Is the Context of the Conversations with Euclid?* Typically, the conversations in the unit are presented on an overhead projector by the teacher or handed out on paper to the students. However, their distribution is carried out strategically to avoid introduction of material prematurely, before the students have fully realized the need for an idea. In the first few lessons when solving

<sup>4</sup>The need for communication is one of five intellectual needs (need for certainty, need for causality, need for communication, need for computation, and need for structure), which characterize mathematical practice (see [9]).

geometry problems in class, the teacher plays the role of Euclid, but gradually the students take on his persona.

### Final Observation

The idea of *Conversations with Euclid* was created years ago when I realized that my students in college geometry courses had serious difficulties dealing with geometric properties outside their own imaginative space. For example, they invoked imageries of “betweenness”, “infinite length”, “order”, etc., in finite geometries devoid of these properties. The *Conversations with Euclid* technique was instrumental in helping students separate their own imageries from those implied by the geometry at hand.

Teachers who participated in the pilot experiment with the *Conversations with Euclid* unit found it effective. Here are several of their responses.

Teacher A: “The point of geometry (and all other math) is to improve our understanding and deductive reasoning. From the lessons we received, this goal is obtainable. Because we started with nothing and were able to prove so much, it allowed me to use logic and reasoning to deductively prove many geometrical theorems I never previously understood.”

Teacher B: “I always knew that the way geometry is traditionally taught didn’t give students the opportunity to reason deductively. We typically move through endless theorems and expect students to apply them algebraically. However, the “why” is often left out. Students are terrible at proofs because we never teach ways of thinking and reasoning. I now have a better idea of how to teach deductive reasoning.”

Other teachers have been using the *Conversations with Euclid* unit in their own classes, and have reported success.

Teacher C reported that her students were so intrigued by the *Conversations with Euclid* game that they composed a rap song about his perceived personality.

Teacher D reported the results of a semi-formal study she conducted to compare students’ achievement on three types of curricula in her school: the *Conversations with Euclid* unit (137 students), a common district geometry unit (180 students), and an accelerated geometry unit (91 students). All the students took the same external district geometry end-of-course (EOC) exam at the end of fall of 2012. The EOC tested thirteen standards. Students with the *Conversations with Euclid* unit performed significantly better than those with the common geometry unit and about the same or better than those in the accelerated geometry curriculum.

When Teacher D had to resort to the common geometry unit in the spring of 2013 due to the institution’s curricular constraints, her students

protested and expressed their wish to return to their conversations with Euclid.

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