

# ED'S STRATEGY

## FOR SOLVING DIVISION PROBLEMS

Guershon Harel and Meryn Behr

**P**rior to formal instruction, children possess considerable understanding of arithmetic and the strategies for solving basic arithmetic problems. Carpenter and Moser (1982), for example, have found that many children can successfully solve word problems that involve addition and subtraction before they receive any instruction in such problems. Kouba (1989) identified fifty-six appropriate strategies and several inappropriate strategies used by first-, second-, and third-grade pupils to solve multiplication and division problems. Knowledge can be constructed by pupils in various ways and at various times. Probably more often than we realize it, pupils have some conceptions about many topics on which they have not had school instruction. For example, before a pupil receives any formal instruction on division, he or she has had experience with sharing. A sharing problem might be solved by a one-by-one distribution of the objects to be shared or by multiple distribution (see, e.g., Kouba [1989]). Pupils also develop sophisticated strategies that go beyond modeling an intuitive or informal interpretation of an operation (Resnick 1986). This article

discusses one such strategy for solving division problems.

Ed is a second grader who was reported by his teacher to be slightly above average in mathematics and average in other subjects. By the time we interviewed him, Ed's formal instruction had included addition and subtraction without regrouping. In addition, Ed was informally exposed to the basic notion of multiplication as repeated addition and to the meaning of division as sharing equally, but he was never taught any division

strategy. Ed was orally given the problem "How much is forty-two divided by

seven?" His answer was, "Forty divided by ten is four; three and three and three are twelve; twelve plus two is fourteen; fourteen divided by two is seven; two plus four is six." Ed gave this same solution and answer each time he was asked to explain his solution. When asked where the "3" came from, he responded, "To make the ten a seven [apparently meaning 'subtract three from ten to yield seven']."

To make sure that Ed had a general strategy and didn't get the answer "by accident," we orally gave him another question similar to the first one, "How much is seventy-two divided by eight?" Interestingly, the structure of his solution was exactly the same as for the first one: "Seventy divided by ten is seven; seven times two is fourteen; fourteen plus two is sixteen; sixteen divided by two is eight; two plus seven is nine. The answer is nine."

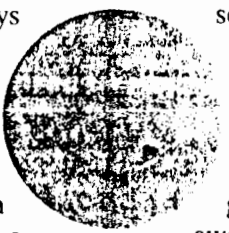
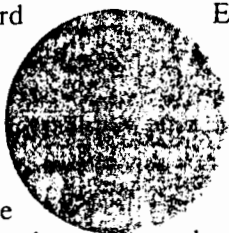
It was very difficult to get more information from Ed. Again, when asked about

where the "2" came from, he responded, "The two from the ten."

Ed was orally given a third problem: "How much is fifty-six divided by eight?" He was urged to go slowly and explain what he was doing. His solution had the same structure as the other two, "Fifty divided by ten is five; five times two is ten; ten plus six is sixteen; sixteen divided by two is eight; two plus five is seven. The answer is seven." This time we were able to capture more of his solution process: "The twos come from the ten. They have to give back two, so I have two and two and two and two and two, which is ten. And the six that is left, that makes sixteen." When asked who has to give back two, he answered, "Those who got ten, they need to get only eight."

Amazed at what Ed was doing, we were still not sure that the solution strategy was generally valid. We examined the validity of the strategy by looking carefully into Ed's responses, from which we were able to identify the conceptual base for this solution process. Let us look at the solution of 42 divided by 7 in terms of drawings to represent what might be the pupil's mental model and show how the solution might be demonstrated with manipulative objects.

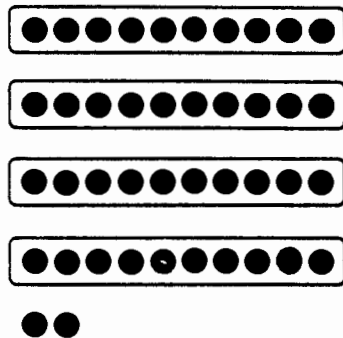
Ed's solution uses a measurement rather than partitive interpretation of division; that is, forty-two objects are put into groups of seven, and the answer to the division is then the number of groups. His solution starts with forty-two objects. In some manner he mentally distributed forty of these objects into four sets of ten, leaving two objects not distributed, as shown in **figure 1a**. Remembering that he was to make groups of only seven, he takes back three objects from each group of ten ("three and three and three and three are twelve"). These twelve elements and the two left from the forty-two ("twelve plus two is fourteen") must be distributed



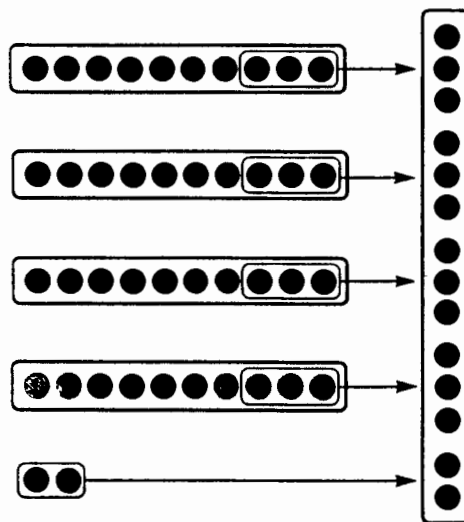
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**FIGURE 1**

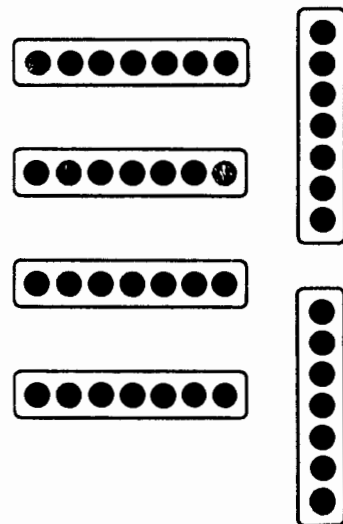
(a) Forty-two objects form sets of ten, leaving two objects undistributed.



(b) Three objects are taken from each set of ten and put with the two left over.



(c) The fourteen elements on the right are distributed into two groups of seven.



(see fig. 1b). These fourteen elements were distributed into two groups of seven objects ("fourteen divided by two is seven," which apparently means that because  $14 \div 2 = 7$ , then  $14 \div 7 = 2$ ). Combining the four groups of seven with the two groups of seven gives the six groups of seven ("two plus four is six. The answer is six"). See figure 1c.

## Conclusions and Implications

Teachers put forth a great deal of effort to understand and evaluate pupils' knowledge, especially after instruction. But pupils' knowledge prior to instruction is also important (Mack 1988). Good teaching should build on the pupil's existing knowledge, which forms the basis for building new knowledge (Romberg and Carpenter 1986).

It is important for the teacher to recognize a pupil's attempts at a creative solution of a problem and to praise such inventiveness. Even when the invention does not lead to correct answers, often it originates in some intelligent reasoning process. A thorough discussion of pupils' inventiveness in several domains of mathematics is given by Resnick (1987).

Whether the invention is correct or incorrect, the teacher should capitalize on the knowledge constructed by the pupils when leading them to learn new knowledge. The existing knowledge should be recognized, not ignored. Resnick (1987) observes that pupils have the basic ability to decompose and recompose quantities to represent such numerical operations as subtraction with regrouping in division. The ability to compose and decompose quantities is evident in the strategy that Ed used to solve division problems. Another observation to be made about pupils' manipulation of representations of quantity is the flexibility with which they compose, decompose, and recompose these quantities.

In addition to such flexibility, the division strategy described in the discussion has three additional important features. First, the strategy can be represented with manipulative aids. Second, as was demonstrated earlier, the strategy is applicable to a general class of division problems. Finally, the solution of some problems by this strategy might involve recur-

sive thinking. For example, the solution of the problem  $136 \div 8$ , shown in the next section, includes the subproblem  $32 \div 8$ , which can in turn be solved by the same strategy.

## Examining Ed's Strategy in Algebraic Terms

Ed's strategy can be applied to solve division problems for which the divisor and the dividend are whole numbers, with the restriction that the divisor must be less than or equal to the dividend. Unfortunately, this strategy begins to exceed one's mental-arithmetic capabilities when two- and three-digit divisors are used, but then a calculator can be used if practice is desired or if a curious pupil just wants to try it out with big numbers.

We encoded Ed's solution in algebraic terms. Given any two-digit number  $10x + y$  (e.g.,  $42, 4 \times 10 + 2$ ) and a one-digit number  $z$  (e.g.,  $7$ ), it can be seen from Ed's responses that his solution process for dividing  $10x + y$  by  $z$  is as follows:

$$(10x \div 10) + \{(10 - z)x + y\} \div z,$$

that is,

$$(40 \div 10) + \{(10 - 7)4 + 2\} \div 7.$$

A simplification of this expression shows that it equals  $(10x + y) \div z$ , which proves that the pupil's strategy is mathematically valid for the whole class of division problems in which the dividend is a two-digit number and the divisor is a one-digit number. However, we don't know if Ed would use this strategy to solve problems in which the divisor is small (e.g.,  $42 \div 2$  or  $42 \div 4$ ). We give some examples that use bigger numbers to illustrate step by step the generalizability of Ed's process.

*Example 1:*  $136 \div 8$

$$130 \div 10 = 13$$

$$10 - 8 = 2$$

$$13 \times 2 = 26$$

$$136 - 130 = 6$$

$$26 + 6 = 32$$

$$32 \div 8 = 4$$

$$13 + 4 = 17$$

The answer is 17.

*Example 2:*  $192 \div 24$

$$180 \div 30 = 6$$

$$30 - 24 = 6$$

$$\begin{aligned} 6 \times 6 &= 36 \\ 192 - 180 &= 12 \\ 36 + 12 &= 48 \\ 48 \div 24 &= 2 \\ 6 + 2 &= 8 \end{aligned}$$

The answer is 8.

*Example 3:*  $930 \div 186$

$$\begin{aligned} 760 \div 190 &= 4 \\ 190 - 186 &= 4 \\ 4 \times 4 &= 16 \\ 930 - 760 &= 170 \\ 170 + 16 &= 186 \\ 186 \div 186 &= 1 \\ 4 + 1 &= 5 \end{aligned}$$

The answer is 5.

To extend this strategy from one-digit divisors to multidigit divisors, the pupil must first find the least multiple of 10 that is greater than the divisor then decompose the dividend into two parts of which one is a multiple of the least multiple of 10 greater than the divisor. From there on, the process is exactly the same as that

used by Ed to solve divisions with single-digit divisors.

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