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ORIGINAL ARTICLE

A *DNR* perspective on mathematics curriculum and instruction. Part II: with reference to teacher's knowledge base

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Abstract Two questions are on the mind of many mathematics educators; namely: What is the mathematics that we should teach in school? and how should we teach it? This is the second in a series of two papers addressing these fundamental questions. The first paper (Harel, 2008a) focuses on the first question and this paper on the second. Collectively, the two papers articulate a pedagogical stance oriented within a theoretical framework called *DNR*-based instruction in mathematics. The relation of this paper to the topic of this Special Issue is that it defines the concept of *teacher's knowledge base* and illustrates with authentic teaching episodes an approach to its development with mathematics teachers. This approach is entailed from *DNR*'s premises, concepts, and instructional principles, which are also discussed in this paper.

Keywords DNR · Way of understanding · Way of thinking · Duality principle · Necessity principle · Repeated reasoning principle · Intellectual need · Psychological need · Teacher's knowledge base

1 Introduction

This is the second in a series of two papers, the goal of which is to contribute to the debate on a pair of questions that are on the mind of many mathematics educators—teachers, teacher leaders, curriculum developers, and researchers who study the processes of learning and teaching—namely:

1. What is the mathematics that we should teach in school?
2. How should we teach it?

Clearly, a pair of papers is not sufficient to address these colossal questions, which are inextricably linked to other difficult questions—about student learning, teacher knowledge, school culture, societal need, and educational policy, to mention a few. My goal in these two publications is merely to articulate a pedagogical stance on these two questions. The stance is not limited to a particular mathematical area or grade level; rather, it encompasses the learning and teaching of mathematics in general.

This stance is oriented within a theoretical framework, called *DNR-based instruction in mathematics* (*DNR*, for short). *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts*—constructs defined and oriented within these premises; and *claims*—statements formulated in terms of the *DNR* concepts, entailed from the *DNR* premises, and supported by empirical studies. These claims include *instructional principles:* assertions about the potential effect of teaching actions on student learning. Not every instructional principle in the system is explicitly labeled as such. The system states three foundational principles: the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle;* hence, the acronym *DNR*. The other principles in the system are derivable from and organized around these three principles. Figure 1 depicts this structure at this level of elaboration. Additional figures with further elaboration will be presented as the paper unfolds.

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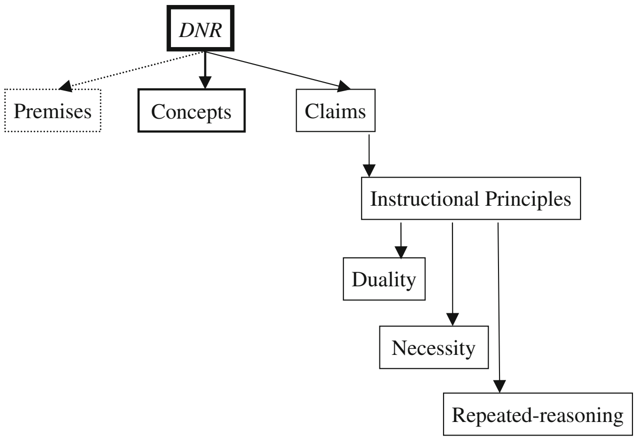


Fig. 1 *DNR* structure: elaboration 1

This paper begins, in Sect. 1, with *DNR* premises. Section 2 is a synopsis of Paper I (Harel, 2008a).[1] The reader is strongly encouraged to read Paper I before reading this paper. The two papers correspond roughly to the above two questions: Paper I focuses on elements of mathematics curricula, and this paper on elements of mathematics teaching. However, since one's approach to teaching necessarily depends on one's views of learning, this paper devotes a portion of its space to the *DNR*'s definition of learning. Learning is discussed in Sect. 3. What does all this have to do with the topic of this Special Issue: *empirical research on mathematics teachers and their education*? The answer to this question will be become clearer in Sects. 4 and 5. Section 4 formulates the three *DNR* foundational instructional principles—*duality, necessity* and *repeated reasoning*. Section 5 defines the concept of *teacher's knowledge base* in terms of the elements of learning, teaching, and curriculum laid out in the preceding sections and in Paper I, and it also illustrates *DNR*'s approach to advancing teachers' knowledge base.

The paper concludes, in Sect. 6, with a summary and questions entailed from the conceptual framework put forth in the preceding sections.

2 DNR's premises

A major effort was made to state the *DNR* underlying assumptions explicitly. These assumptions, called *DNR* premises, were not conceived a priori before *DNR* was formulated, but instead emerged in the process of reflection on and exploration of justifications for the *DNR* claims. Collectively, these premises underlie *DNR*'s philosophy of mathematics and the learning and teaching of mathematics.

[1] Throughout this paper, the reference Paper I rather than Harel (2008a) will be used.

DNR has eight premises. They are loosely organized in four categories:

1. Mathematics
   - Mathematics: Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history.
2. Learning
   - Epistemophilia: Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience.
   - Knowing: Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.
   - Knowing-Knowledge Linkage: Any piece of knowledge humans know is an outcome of their resolution of a problematic situation.
   - Context Dependency: Learning is context dependent.
3. Teaching
   - Teaching: Learning mathematics is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance.
4. Ontology
   - Subjectivity: Any observations humans claim to have made is due to what their mental structure attributes to their environment.
   - Interdependency: Humans' actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions.

As the reader might have recognized, these premises—with the exception of the Mathematics Premise, which has been discussed broadly in Harel (2008) and in Paper I—are taken from or based on known theories. Briefly, the Epistemophilia Premise follows from Aristotle (Lawson-Tancred, 1998); the Adaptation Premise is the nucleus of Piaget's theory of equilibration (Piaget, 1985); the Learning-Knowledge Linkage Premise, too, is inferable from Piaget, and is consistent with Brousseau's claim that for every piece of knowledge there exists a fundamental situation to give it an appropriate meaning (Brousseau, 1997);

the Context Dependency Premise is consistent with modern cognitive theories of knowledge acquisition, according to which learning is contextualized; the Teaching Premise derives from Vygotsky's (1978) known zone of proximal development (ZPD) idea; the Subjectivity Premise and the Interdependency Premise follow from both Piaget's constructivism theory (see, for example, von Glasersfeld, 1983) as well as from information processing theories (see, for example, Chiesi, Spilich, & Voss, 1979; Davis, 1984).

Why are these premises needed? *DNR* is a conceptual framework for the *learning* and *teaching* of *mathematics*. As such, it needs lenses through which to see the realities of the different actors involved in these human activities—mathematicians, students, teachers, school administrators, etc.—particularly the realities of the students as learners in different stages in their conceptual development. *DNR* also needs a stance on the nature of the targeted knowledge to be taught—mathematics—and of the learning and teaching of this knowledge.

Starting from the end of the premises list, the two Ontology Premises—Subjectivity and Interdependency—orient our interpretations of the actions and views of students and teachers. Implications of the ontological positions expressed by these premises to mathematics education are not new. Scholars such as von Glasersfeld, Leslie Steffe, Patrick Thompson, Paul Cobb, Jere Confrey, and Ed Dubinsky were notable pioneers in offering and implementing research and curricular programs rooted in these positions (see, for example, Steffe, Cobb, & Glasersfeld, 1988; Steffe & Thompson, 2000; Confrey, 1990; Dubinsky, 1991). These scholars articulated essential implications to mathematics curriculum and instruction: that students' realities *are* their actual experiences, not what we speak of as observers; that when we describe our observations of students' experiences we merely offer a model describing our conception of what we have observed; that for these models to be effective pedagogically, they should include students' actions—what we see and hear—as well as their possible causes. All these elements are integral parts of *DNR.* As was discussed in Harel (2008), this subjective stance is already present in the definitions of "way of understanding" and "way of thinking," and hence in the conceptualization of the Mathematics Premise.

The Mathematics Premise comprises its own category; it concerns the nature of the mathematics knowledge—the targeted domain of knowledge to be taught—by stipulating that ways of understanding and ways of thinking are the constituent elements of this discipline, and therefore instructional objectives must be formulated in terms of both these elements, not only in terms of the former, as currently is largely the case (see Paper I).

Each of the four Learning Premises—Epistemophilia, Knowing, Knowing-Knowledge Linkage, and Context Dependency—attend to a different aspect of learning: The Epistemophilia Premise is about humans' propensity to know, as is suggested by the term "epistemophilia:" love of episteme. Not only do humans desire to solve puzzles in order to construct and impact their physical and intellectual environment, but also seek to be puzzled. The term "puzzle" should be interpreted broadly: it refers to problems intrinsic to an individual or community, not only to recreational problems, as the term is commonly used. Such problems are not restricted to a particular category of knowledge, though here we are solely interested in the domain of mathematics. The Epistemophilia Premise also attends to another significant issue. It claims that *all* humans are capable of learning if they are given the opportunity to be puzzled, create puzzles, and solve puzzles. While it assumes that the propensity to learn is innate, it rejects the view that individual differences reflect innate basic capacities that cannot be modified by adequate experience (social, emotional, psychological, and intellectual).

The Knowing Premise is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. A failure to assimilate results in a disequilibrium, which, in turn, leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. In essence, this premise is the basis for the position, held by many scholars (e.g. Brownell, 1946; Davis, 1992; Hiebert, 1997; Thompson, 1985), that problem solving is the only means of learning.

The Context Dependency Premise is about contextualization of learning. The premise does not claim that learning is entirely dependent on context—that knowledge acquired in one context is not transferrable to another context, as some scholars (Lave, 1988) seem to suggest. This claim, which implies, for example, that abstraction is of little use, is obviously not true, as is convincingly argued by Anderson, Reder, & Simon (1996). Instead, the Context Dependency Premise holds that ways of thinking belonging to a particular domain are best learned in the context and content of that domain. Consider, for example, reification—the way of thinking where one reconceptualizes processes as conceptual entities, objects the mental system can reason about in a direct way (Greeno, 1980).[2] Beginning at infancy and throughout life, humans form conceptual entities through interaction with their physical and social environments. They effortlessly reason directly about perceptual and social concepts, such as "texture" and

[2] Reification is a way of thinking because it is a cognitive characteristic of the mental act of abstracting. In Piaget's terms, reification is one kind of reflective abstraction. (For an excellent analysis of Piaget's notion of abstraction, see Dubinsky, 1991).

"color" and "friendship" and "justice," directly without a need to unpack or replay the experiences that led to their construction. The ease in which reification is applied in one's daily life does not, however, guarantee its successful application in other domains, particularly mathematics. It has been well documented that reifying processes into conceptual entities in mathematics is difficult. For example, reconceptualizing the concept of function from a mapping process into an object—as an element of a group or a vector space, for example—is difficult even for college students (Dubinsky, 1991; Sfard, 1991); reconceptualizing the concept of fraction from a multiplicative relation into a single number is difficult for middle-grade students (Behr, Wachsmuth, & Post, 1984); and reconceptualizing the process of counting into a number name is far from being trivial for early-grade students (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Context dependency exists even within subdisciplines of mathematics, in that each mathematical content area is characterized by a unique set of ways of thinking (and ways of understanding). For example, the set of ways of thinking that characterizes combinatorics is different from that which characterizes topology. Even within the same domain, say Euclidian geometry, the ways of thinking that characterize plane geometry, for example, are not identical to those that characterize spatial geometry.

Finally, the Teaching Premise asserts that expert guidance is indispensible in facilitating learning of mathematical knowledge. This premise is particularly needed in a framework oriented within a constructivist perspective, like *DNR,* because one might minimize the role of expert guidance in learning by (incorrectly) inferring from such a perspective that individuals are responsible for their own learning or that learning can proceed naturally and without much intervention (see, for example, Lerman, 2000). The Teaching Premise rejects this claim, and, after Vygotsky, insists that expert guidance in acquiring scientific knowledge—mathematics, in our case—is indispensable to facilitate learning.[3] The Teaching Premise leads naturally to questions concerning the necessary knowledge that an expert guide—a teacher—must possess and the nature of effective teaching practices that can bring about learning. The *DNR* constituent elements of teaching (Sect. 4) coupled with the definition of *teacher's knowledge base* (Sect. 5) attend to these issues.

Figure 2 adds to Fig. 1 the eight premises of *DNR.*

[3] The potential inconsistency between Piaget and Vygotsky regarding the source of meaning, which according to Piaget it comes from the individual's actions and operations and according to Vygotsky from communication among individuals, will not be addressed in this paper.

3 Constituent elements of mathematics curriculum: a synopsis of paper I

Paper I addresses the question, "What is the mathematics that we should teach in school?" *DNR*'s position on this question, based on its first premise, is that the constituent elements of mathematics, and therefore of desirable mathematics curricula, are ways of understanding *and* ways of thinking. A way of understanding is a product of a mental act, whereas a way of thinking is a characteristic of ways of understanding associated with that act. The triad "mental act, way of understanding, and way of thinking" is central in *DNR.* It is a generalization of the triad "proving, proof, and proof scheme," which emerged in investigations concerning the learning and teaching of mathematical proof (see, for example, Harel & Sowder, 1998, 2008c). The generalization was necessitated in part by the realization that the processes of learning and teaching mathematical proof involve numerous mental acts, such as "interpreting," "connecting," "modeling," "generalizing," "abstracting," "searching," and "symbolizing" and so attention to proving alone is insufficient to identify and communicate classroom and clinical observations. Examples of categories of ways of thinking include problem-solving approaches, proof schemes, and beliefs about mathematics. Of these, the proof schemes category was taxonomized on the basis of students' work and historical development (Harel and Sowder, 1998). The taxonomy consists of three classes—*external conviction, empirical,* and *deductive*—each with subclasses.

In a nutshell, Paper I argues that the answer to the first question in the opening of this paper should be driven by desirable[4] ways of understanding *and* ways of thinking, not only by the former, as is currently the case. Section 4 deals with pedagogical tools needed for effectively teaching such curricula. These tools hinge, in part, upon *DNR*'s definition of *learning,* which is discussed in the next section.

4 Constituent elements of mathematics learning

One's view of learning might be informed and formed by a scholarly-based theory—such as behaviorism, information processing, and constructivism—or by informal experience. In *DNR,* the definition of mathematics learning follows from the *DNR* premises. It follows from the Learning-Knowledge Linkage Premise that problem solving is the means—the only means—to learn. When one encounters a problematic situation, one necessarily experiences phases of disequilibrium, often intermediated by phases of equilibrium. Disequilibrium, or perturbation, is a

[4] For the particular meaning of the term "desirable," see Harel (2008b).

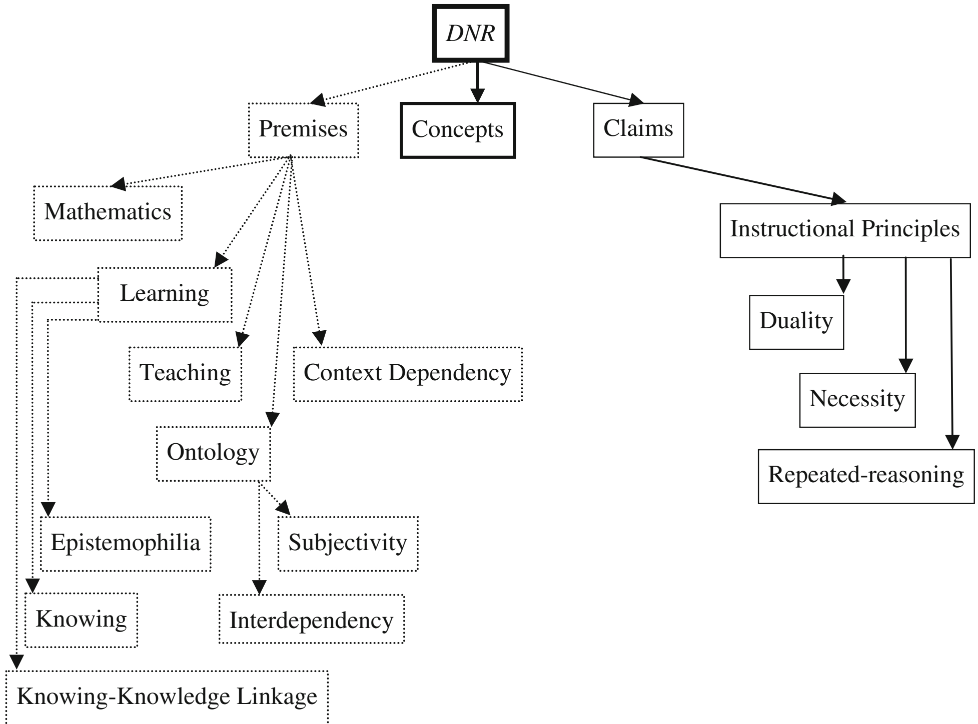


Fig. 2 *DNR* structure: elaboration 2

state that results when one encounters an obstacle. Its cognitive effect is that it "forces the subject to go beyond his current state and strike out in new directions" (Piaget, 1985, p. 10). Equilibrium is a state when one perceives success in removing such an obstacle. In Piaget's terms, it is a state when one modifies her or his viewpoint (accommodation) and is able, as a result, to integrate new ideas toward the solution of the problem (assimilation). But what constitutes perturbation? More relevant to this paper, what constitutes perturbation in mathematical practice? *DNR* defines perturbation in terms of two types of human needs: *intellectual need* and *psychological need*, both of which are discussed below. *DNR*'s definition of learning, thus, incorporates these two needs and, consistent with the Subjectivity Premise, it also incorporates the knowledge currently held and newly produced during the learning process. Further, since our interest is restricted to mathematics learning, this knowledge is defined in terms of ways of understanding and ways of thinking, by the Mathematics Premise. Thus, *DNR*'s definition of *learning* is:

> *Learning* is a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases.

Harel (2008b) and Paper I discuss in length the two notions *way of understanding* and *way of thinking*. In this section we discuss the other two notions appearing in this definition: *intellectual need* and *psychological need.*

Let K be a piece of knowledge possessed by an individual or community. By the Knowing-Knowledge Linkage Premise, there exists a problematic situation S out of which K arose. S (as well as K) is subjective, by the Subjectivity Premise, in the sense that it is a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, her or his current knowledge. Such a problematic situation S, prior to the construction of K, is referred to as an individual's *intellectual need:* S is the need to reach equilibrium by learning a new piece of knowledge.[5] Perturbational states do not necessarily lead to knowledge construction—a person can remain in a state of disequilibrium either due to inability or lack of motivation. Here, however, we are talking about the case where a perturbational state S has led to the construction of K. In this case, if the individual also sees how K resolves S, then we say that the individual possesses an *epistemological justification for the creation of K.* Thus, epistemological justifications concern the genesis of knowledge, the perceived reasons for its birth in the eyes of the learner.

There is often confusion between *intellectual need* and *motivation.* The two are related but are fundamentally

[5] There more to say about the "intellectual" part of "intellectual need." Historical and epistemological analyses led to five categories of intellectual needs, which are briefly characterized in Sect. 6.

different. While intellectual need belongs to epistemology, motivation belongs to psychology. Intellectual need has to do with disciplinary knowledge being born out of people's current knowledge through engagement in problematic situations conceived as such by them. Motivation, on the other hand, has to do with people's desire, volition, interest, self determination, and the like. Indeed, before one immerses oneself in a problem, one must desire, or at least be willing, to engage in the problem, and once one has engaged in a problem, often persistence and perseverance are needed to continue the engagement. These characteristics are manifestations of *psychological needs*: motivational drives to initially engage in a problem and to pursue its solution. The existence of these needs is implied from the Epistemophilia Premise, which asserts that people desire to solve problems and to look for problems to solve—they do not passively wait for disequilibrium!

Psychological needs, thus, belong to the field of motivation, which addresses conditions that activate and boost—or, alternatively, halt and inhibit—learning in general. In contrast, intellectual needs refer to the epistemology of a particular discipline with an individual or community from the knowledge they currently hold. Of course, as human behaviors, the two categories of needs are related; in fact, they complement each other: On the one hand, knowledge of a discipline always stems from problematic situations unique to that discipline and understood as such by an individual or community studying the discipline. On the other hand, in suitable physical, emotional, and social environments, humans are ready to engage in these problematic situations and persevere in pursuing their solutions.

Research in mathematics education has offered useful models for learning trajectories of various mathematical concepts and ideas, but, as with *DNR*, psychological needs have not been a major focus. Examples of such models include: Fischebein's (1985) intuitive models for the concepts of multiplication and division; Tournaire's (1986) model for the concept of proportionality; Dubinsky's action-process-object-schema model for the concept of function (E. D. Dubinsky & McDonald, 2001); Schoenfeld's (1992) model for thinking mathematically. However, relative to the broad scope of our definition of learning, these models are largely partial. This is expected due to the enormous empirical and theoretical difficulties in building models that incorporate phases of disequilibrium-equilibrium, their utilized or resultant ways of understanding and ways of thinking, and the intellectual and psychological needs that result from and instigate the various phases. Such comprehensive models, however, are needed and hopefully will be constructible in the future.

Figure 3 depicts the *DNR* concepts discussed this far.

5 Constituent elements of mathematics teaching

This section discusses *DNR*'s three foundational instructional principles: *duality, necessity,* and *repeated reasoning*. The *duality principle* deals with the developmental

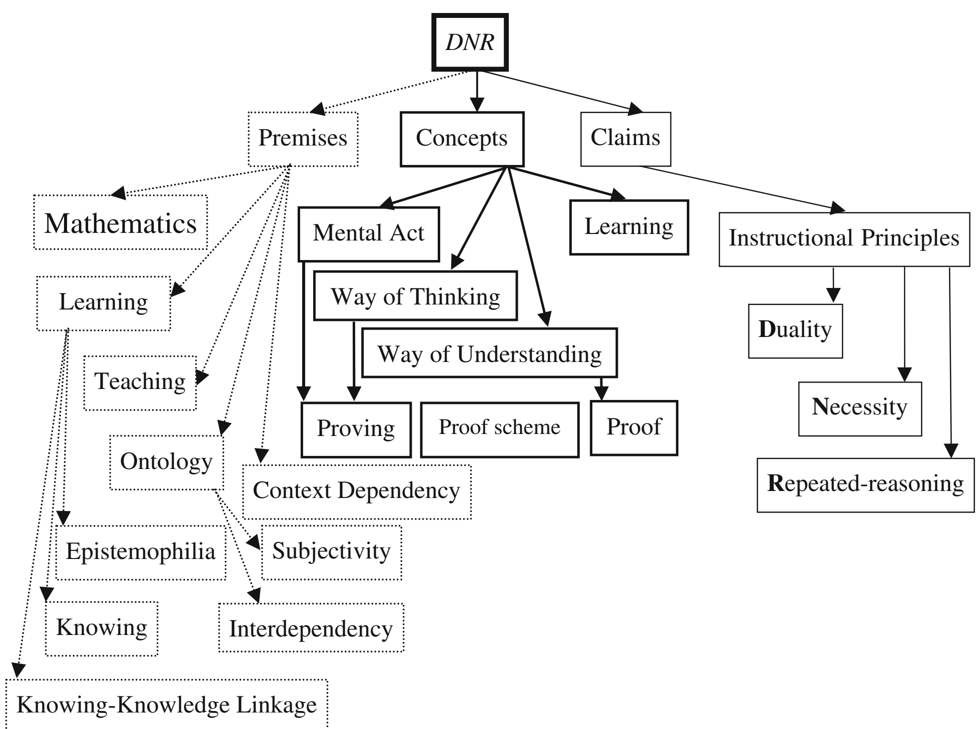


Fig. 3 *DNR* structure: elaboration 3

interdependency between ways of understanding and ways of thinking; the *necessity principle* with students' *intellectual need;* and the *repeated reasoning principle* with internalization, organization, and retention of knowledge. As we will see, the three principles are strongly linked. A single principle, considered individually and separately from the other two principles, is likely to be of lesser pedagogical value than if is considered in the context of the other two.

5.1 The duality principle

Let us begin by taking a closer look at the formation of the empirical proof scheme (see Paper I). Recall that a person with this scheme proves—that is, removes doubts about the truth of an assertion—inductively: by relying on evidence from examples of direct measurements of quantities, substitutions of several numbers in algebraic expressions, etc, or perceptually: by relying on evidence from physiological senses such as visual or tactile perceptions. Research has shown that the empirical proof scheme is prevalent and persistent among students at all grade levels (Chazan, 1993; Goetting, 1995; Harel & Sowder, 2008). What might be causing the dominance of this scheme?

Students do not come to school as blank slates, ready to acquire knowledge independently of what they already know (Piaget, 1952, 1969, 1973, 1978). Rather, what students know now impacts what they will know in the future. This is true for all ways of understanding and ways of thinking associated with any mental act; the mental act of proving is no exception. In everyday life and in science, the means of justification available to humans are largely limited to empirical evidence. Since early childhood, when we seek to justify or account for a particular phenomenon, we are likely to base our judgment on similar or related phenomena in our past (Anderson, 1980). Given that the number of such phenomena in our past is finite, our judgments are typically empirical. Through such repeated experience, which begins in early childhood, our hypothesis evaluation becomes dominantly empirical; that is, the proofs that we produce to ascertain for ourselves or to persuade others become characteristically inductive or perceptual. If, during early grades, our judgment of truth in mathematics continues to rely on empirical considerations, the empirical proof scheme will likely dominate our reasoning in later grades and more advanced classes, as research findings clearly show (Harel & Sowder, 2008). While unavoidable, the extent of the dominance of the empirical proof scheme on people is not uniform. Children who are raised in an environment where sense making is encouraged and debate and argumentation are an integral part of their social interaction with adults are likely to have a smoother transition to deductive reasoning than those who are not raised in such an environment.

A simple, yet key, observation here is this: the proofs children produce to prove assertions and account for phenomena in everyday life impact the kind and robustness of the proof schemes they form. Proofs, as was explained earlier, are ways of understanding associated the mental act of proving, and proof schemes are ways of thinking associated with the same act. Hence, a generalization of this observation is: for any mental act, the ways of understanding one produces impact the quality of the ways of thinking one forms.

Of equal importance is the converse of this statement; namely: For any mental act, the ways of thinking one has formed impact the quality of the ways of understanding one produces. The latter statement is supported by observations of students' mathematical behaviors, for example, when proving. As was indicated earlier, the empirical proof scheme does not disappear upon entering school, nor does it fade away effortlessly when students take mathematics classes. Rather, it continues to impact the proofs students produce. It takes enormous instructional effort for students to recognize the limits and role of empirical evidence in mathematics and begin to construct alternative, deductively-based proof schemes. Even mathematically able students are not immune from the impact of the empirical proof scheme, as was demonstrated by Fischbein & Kedem (1982). Students' past mathematical experience, however, plays a critical role in the extent to which their empirical proof schemes impact the proofs they produce.

This analysis points to a reciprocal developmental relationship between ways of understanding and ways of thinking, which is expressed in the following principle:

> The *Duality Principle*: Students develop ways of thinking through the production of ways of understanding, and, conversely, the ways of understanding they produce are impacted by the ways of thinking they possess.

For easier reference, the first statement of the Duality Principle is denoted by T_{U} (indicating that ways of understanding serve a basis for the development of ways of thinking), and its converse by U_{T} (indicating that ways of thinking serve a basis for the production of ways of understanding).

The analysis preceding the Duality Principle, and hence the principle itself, is implied from the Interdependency Premise. To see this, one only need to recognize that a person's ways of thinking are part of her or his view of the world, and that a person's ways of understanding are manifestations of her or his actions. Specifically, the U_{T} statement is an instantiation of the premise's assertion that humans' actions are induced and governed by their views of the world, whereas the T_{U} statement is an instantiation of the premise's assertion that humans' views of the world are formed by their actions. Furthermore, the Context Dependency Premise adds a qualification to the T_U statement: ways of thinking belonging to a particular discipline best develop from or are impacted by ways of understanding belonging to the same discipline.

5.2 The necessity principle

There is a lack of attention to students' intellectual need in mathematics curricula at all grade levels. Consider the following two examples: After learning how to multiply polynomials, high-school students typically learn techniques for factoring (certain) polynomials. Following this, they learn how to apply these techniques to simplify rational expressions. From the students' perspective, these activities are intellectually purposeless. Students learn to transform one form of expression into another form of expression without understanding the mathematical purpose such transformations serve and the circumstances under which one form of expression is more advantageous than another. A case in point is the way the quadratic formula is taught. Some algebra textbooks present the quadratic formula before the method of completing the square. Seldom do students see an intellectual purpose for the latter method—to solve quadratic equations and to derive a formula for their solutions—rendering completing the square problems alien to most students (see Harel, 2008a for a discussion on a related way of thinking: *algebraic invariance*). Likewise, linear algebra textbooks typically introduce the pivotal concepts of "eigenvalue," "eigenvector," and "matrix diagonalization" with statements such as the following:

> The concepts of "eigenvalue" and "eigenvector" are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form XDX^{-1}, where D is diagonal. The latter factorization would provide important information about A, such as its rank and determinant.

Such introductory statements aim at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university instructor), it is likely to be alien to the students because a regular undergraduate student in an elementary linear algebra course is unlikely to realize from such statements the nature of the problem indicated, its mathematical importance, and the role the concepts to be taught ("eigenvalue," "eigenvector," and "diagonalization") play in determining its solution. What these two examples demonstrate is that the intellectual need element in (the *DNR* definition of) learning is largely ignored in teaching. The Necessity Principle attends to the indispensability of intellectual need in learning:

> *The Necessity Principle*: For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' here refers to intellectual need.

5.3 The repeated reasoning principle

Even if ways of understanding and ways of thinking are intellectually necessitated for students, teachers must still ensure that their students internalize, retain, and organize this knowledge. Repeated experience, or practice, is a critical factor in achieving this goal, as the following studies show: Cooper (1991) demonstrated the role of practice in organizing knowledge. DeGroot (1965) concluded that increasing experience has the effect that knowledge becomes more readily accessible: "[knowledge] which, at earlier stages, had to be abstracted, or even inferred, [is] apt to be immediately perceived at later stages." (pp. 33–34). Repeated experience results in fluency, or effortless processing, which places fewer demands on conscious attention. "Since the amount of information a person can attend to at any one time is limited (Miller, 1956), ease of processing some aspects of a task gives a person more capacity to attend to other aspects of the task (LaBerge and Samuels, 1974; Schneider and Shiffrin, 1977; Anderson, 1982; Lesgold et al., 1988)" (quote from Bransford, Brown, & Cocking, 1999, p. 32).

The emphasis of *DNR-based instruction* is on repeated reasoning that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization, where one is able to apply knowledge autonomously and spontaneously. The sequence of problems given to students must continually call for thinking through the situations and solutions, and problems must respond to the students' changing intellectual needs. This is the basis for the *repeated reasoning principle*.

> *The Repeated Reasoning Principle*: Students must practice reasoning in order to internalize desirable ways of understanding and ways of thinking.

6 Teacher's knowledge base (TKB)

An educational system can be thought of as a triad of agents together with an *action theory*. The agents are students, teachers, and institutions, such as school, school district, home, etc. The action theory consists of these agents' shared meanings for "knowledge," "learning," and "teaching" and shared perspectives on the social, cultural, behavioral, and emotional factors involved in the learning and teaching of particular knowledge. Usually, action theories are not explicit to their associated agents, though they determine the agents' conceptions and shape and govern their actions. For example, an action theory dictates the responsibility and school-related behaviors of students, teachers, school administrators, and parents. It includes conventions for how learning occurs and what facilitates or impedes learning. It also offers tools for measuring what students know and sets expectations for what they ought to know. Thus, action theories orient educational systems as to what, why, and how to carry out school-related actions. A crucial aspect of any action theory is its assumptions. In most cases, these assumptions are not explicit, not even to their agents. It is necessary, however, to understand action theories, for they can help us explain—perhaps even predict—educational systems' behaviors, which, in turn, can help us improve their efficacy.

In this paper we are particularly concerned with teachers' action theories, not those belonging to educational systems in general, though the two are strongly related. Some crucial components of a teacher's action theory comprise her or his *knowledge base.* Building on Shulman's (1986, 1987) work and consistent with the views of other scholars (e.g., Brousseau, 1997; Cohen & Ball, 1999, 2000), a teacher's knowledge base (TKB*)* was defined in Harel (1993) in terms of three components: *knowledge of mathematics*, *knowledge of student learning*, and *knowledge of pedagogy*.

Consistent with the Mathematics Premise, the first component (knowledge of mathematics) is defined in terms of both ways of understanding and ways of thinking. In this respect, teachers' knowledge of mathematics is not of a special kind, though it might be different in scope and depth from that of a professional mathematician. The mathematics that a teacher ought to know should be determined largely by the desirable ways of understanding and ways of thinking targeted by the mathematics curricula the teacher is expected to teach. This does not mean that a teacher needs to know just what he or she teaches. For example, although a second-grade teacher is not expected to teach geometry and algebra proofs, for her to judge the quality of the justifications her students provide and be able to help them gradually refine and advance them, she must herself be able to reason deductively. In *DNR* terms, a second-grade teacher must possess the transformational proof scheme but not necessarily the axiomatic proof scheme; the latter is beyond the cognitive reach of most second-grade students.

The second component of knowledge (knowledge of student learning) deals with two aspects. The first aspect is the teacher's view of the process of mathematics learning. Based on the *DNR* definition of learning, the teacher should understand that the process of learning often involves confusion and uncertainty (results of disequilibrium), that the trajectory of learning is impacted by the learner's background knowledge, and that both psychological and intellectual needs instigate the learning process. The second aspect deals with cognitive as well as epistemological issues involved in the learning of a particular piece of knowledge. Examples of cognitive issues include the teacher's understanding of the difficulty involved in conceptualizing a fraction as a number, of differentiating between a "variable" and a "parameter," of transitioning from additive reasoning to multiplicative reasoning and from empirical reasoning to deductive reasoning, etc. Epistemological issues refer the teacher's understanding of obstacles that are unavoidable—those that have to do with the meaning of the concept—as opposed to didactical obstacles, those that are the result of narrow instruction (see Brousseau, 1997).

Finally, the third component (knowledge of pedagogy) refers to a teacher's *teaching practices* and *instructional principles*, terms we will now define. The notion of teaching practice is based on two concepts: *teaching action* and *teaching behavior*. A teaching action refers to what teachers in a particular community or culture typically do in the classroom. For example, in Western cultures, teachers' functions include presenting new material, asking questions, responding to students' ideas, and evaluating performance. A teaching action can be thought of as a teaching function without qualification, without a description of *how* the function is carried out and without a value judgment about its quality. A teaching behavior, on the other hand, is a typical characteristic of a teaching action. A teaching behavior is always inferred from a multitude of observations; hence the adjective "typical." For example, answering students' questions is a teaching action, but the way a teacher typically chooses to answer students' questions determines his or her teaching behavior relative to this teaching action. For example, some teachers typically respond to students' questions directly by phrases such as "right," "wrong," "yes," "no," etc. without attempting to find a possible conceptual basis for the students' questions. Other teachers, in contrast, typically probe students' current understanding while responding to their questions. These are two different teaching behaviors associated with the teaching action of answering students' questions. Likewise, justifying assertions to the class is a teaching action, but the typical nature of the justifications presented is a teaching behavior associated with this teaching action. Thus, the notion of teaching action is intended to convey a neutral instructional activity carried out by teachers in a given community or culture, whereas a teaching behavior is one of its typical attributes. A teaching action is more observable than a teaching behavior in that the latter requires more analysis on the part of an observer. For example, one can directly observe a teacher responding to students' questions and ideas, but to determine a characteristic of this action—a teaching behavior—many observations and deeper levels of interpretation are needed.

As to the notion of *instructional principle*, consider the common conception "in sequencing mathematics instruction, start with what is easy." This conception might be interpreted as implying a cause-effect link between a teaching action—that of sequencing mathematics instruction—and student learning. The teaching action might be viewed as a likelihood condition: starting with what is easy for the students may help students learn. It might be viewed as a necessary condition: for students to learn, teachers must start with what is easy for them. Or it may be viewed as a sufficient condition: starting with what is easy for the students will help students learn. This example can be abstracted to define the notion of instructional principle:

An *instructional principle* is a conception about the effect of a teaching action on student learning. The teaching action may be conceived as a likelihood condition, necessary condition, or sufficient condition for the effect to take place.

In sum, thus, a TKB is defined in *DNR* as follows:

- *Knowledge of mathematics* refers to the mathematics a teacher knows—that is, the desirable ways of understanding and ways of thinking possessed by the teacher.[6]
- *Knowledge of student learning* refers to a teacher's view of "mathematics learning" and to her or his understanding of the cognitive and epistemological issues involved in learning particular mathematical ways of understanding and ways of thinking.
- *Knowledge of pedagogy* refers to a teacher's *teaching practices*, which are manifestations of the teacher's *instructional principles*.

Figure 4 depicts the *DNR* concepts discussed this far.

6.1 Application of DNR-based instruction to the development of TKB

The goal of this section is to illustrate our approach to advancing teachers' knowledge base. *DNR*-based activities with teachers aim at providing the teachers with an intense exposure to *DNR*-based instruction, involving roles as both learners of mathematics and as teachers reflecting upon the mathematics they are learning. Typically, a *DNR* activity begins with a problem, in some cases a problem that the teachers had worked on the previous day or that had been assigned to them as homework. The teachers are free to work on each problem individually or in small groups. In most cases, they spend some time individually establishing an initial solution approach before they get into their small working groups. Each classroom activity consists of a subset of the following teaching segments: teachers work on the problem individually and in small groups; representatives of the small groups present to the class the groups' solutions or attempts; teachers reflect on and discuss the thought processes of solution approaches, particularly difficulties the teachers encountered as they attempted to solve the problem; a whole-class discussion to discern differences and similarities of the different solutions; articulation of ways of understanding and ways of thinking necessitated by the problem; whole-class discussion of possible difficulties students might have with different elements in the problem or in its solution, including the sources of such difficulties; and discussion of actual work done by students in their class and possible instructional treatments to deal with students' difficulties.

In what follows, we will illustrate a few of these teaching segments specify the instructional objectives they intend to achieve, and point to their rationale in terms of *DNR*. To this end, we will discuss four actual classroom activities with teachers:

6.1.1 Advancing ways of understanding and ways of thinking

> *Problem*: Two pipes are connected to a pool. One pipe can fill the pool in 20 h, and the other in 30 h. How long will it take the two pipes together to fill the pool?

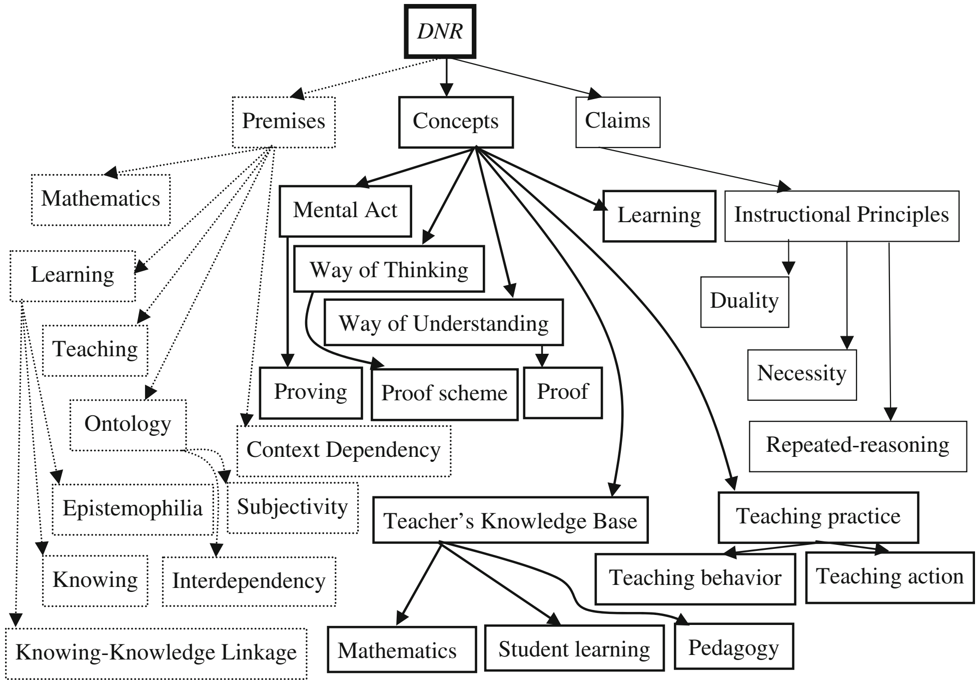
After solving this problem and discussing their solution approaches, the teachers were presented with the following actual four categories of solutions provided by a ninth-grade class, and were asked to analyze the possible conceptual basis for these solutions.

> *Solution 1*: Divide the pool into five equal parts. The first pipe would fill one part in 4 h, and the second pipe in 6 h. Hence, in 12 h the first pipe would fill 3/5 of the pool and the second pipe the remaining 2/5.
> *Solution 2*: It will take the two pipes 50 h to fill the pool.
> *Solution 3*: It will take the two pipes 10 h to fill the pool.
> *Solution 4*: It would take x hours. In 1 h, the first pipe will fill 1/20 of the pool whereas the second will fill 1/30. In x hours the first pipe would fill $x/20$, and the second $x/30$. Thus, $x/20 + x/30 = 1$ (which they then solved to obtain $x = 12$.)

The teachers discussed these solutions and concluded that they represent different ways of understanding the given problem and that the way of thinking "Look for a key word" accounts for the way of understanding expressed in Solution 2 due the word "together" in the problem

[6] It is important to note again here that the use of the term "desirable" is in the sense described in Harel (2008b).

Fig. 4 *DNR* structure: elaboration 4



text. They also hypothesized that although Solution 3 is incorrect, it might indicate a realization by the students who offered it that the time needed to fill the pool by the two pipes must be less than that the time needed to fill the pool by one pipe. Of particular interest to the teachers was Solution 1, which was given by only one student, G. The discussion among the teachers centered on a possible juxtaposition of ways of thinking that seemed to have driven G's solution, which may include "draw a diagram," "guess and check," and "look for relevant relationships among the given quantities."

This episode illustrates progress toward the goal of advancing the teachers' knowledge of student learning. Through such activities, teachers realize that the approach a student chooses to solve a problem depends on how he or she represents, or interprets, the problem statement. The ways of thinking, "a problem can have multiple solutions" and "it can be advantageous to solve a problem in different ways," are extended in our instructional interventions to concepts; namely, "a concept can have multiple interpretations" and "it can be advantageous to possess multiple interpretations of a concept." These ways of thinking, although essential in the learning and creation of mathematics, are often absent from teachers and students' repertoires of reasoning. To advance these ways of thinking, we engaged the teachers in activities aimed at promoting multiple ways of understanding concepts. For example, through their solutions to different problems the teachers learned that the concept of fraction, say 3/4, can be understood in different ways and it is advantageous to understand it in different ways. Such ways of understanding include: *unit fraction* (3/4 is the sum, 1/4 + 1/4 + 1/4), *partition* (3/4 is the quantity that results from dividing 3 units into 4 equal parts), *measurement* (3/4 is the measure of a 3 cm long segment with a 4-cm unit ruler), *solution to an equation* (3/4 is the solution to $4x = 3$), and *part-whole* (3/4 is 3 out of 4 units). Similarly, the teachers solved problems through which they learned multiple ways of understanding the string of symbols $y = f(x)$ and the significance of each interpretation. For example, they saw how one can understand $y = 6x^2 - 5$ in terms of a condition on the variables x and y—the set of all ordered pairs (x, y) for which y is equal to the quantity $6x^2 - 5$. While here $y = 6x^2 - 5$ is viewed as an equation, one can understand it as a function: for each input of x there corresponds the output $6x^2 - 5$. These newly acquired ways of understanding by the teachers were in turn utilized to enhance their knowledge of pedagogy and student learning. This was done by contrasting these mature interpretations with the interpretation commonly possessed by students. For example, through actual work of students, the teachers saw that for many students, symbols such as $y = 6x^2 - 5$ represent no quantitative reality, except possibly that the "equal" sign is understood as a "do something signal," where one side of the equation is reserved for the operation to be carried out and the other side for its outcome, as was documented by Behr, Erlwanger, & Nichols (1976).

6.1.2 Necessitating ways of understanding and ways of thinking

Problem: Take a fraction whose numerator and denominator are integers. Add a number different from zero to the numerator and divide the denominator by that number. When would the new fraction be greater than the original fraction?

In accordance with the necessity principle, the different interpretations of concepts were necessitated for the teachers by problems they solved, as the following discussion illustrates: The teachers discussed in small groups the approaches they took, their partial solutions, and their full solutions. After 35 min, one of the teachers presented her group's solution. Her solution process involved many quadratic inequalities. During her presentation, there were numerous questions and suggestions from the class regarding the solution process. Following this presentation, one of the teachers, K, suggested checking a few cases to see if they agree with the final answer. After about 5 min of group work, K indicated that he noticed a strange phenomenon: when different forms of the same fraction are used different results are obtained. He demonstrated his observation with the equivalent fractions −2/5 and −6/15. Indeed, for −2/5 any value of x works but for −6/15 only $x<3-\sqrt{3}$ or $x>3+\sqrt{3}$ work. This provided a great opportunity—intended and anticipated—for the instructor to point to the necessity to differentiate between "fraction" and "rational number." Although the ratio is the same, the fraction is not, and the solution with a different fraction, even if the ratio is the same, is also different. The solution for the fraction −2/5 is different than the solution for the fraction −6/15. This result was particularly amazing for the teachers.

6.1.3 Internalizing ways of understanding and ways of thinking

Problem: Prove the quadratic formula.

Prior to this problem, the teachers had repeatedly worked with many quadratic functions, finding their roots by completing the square. They abstracted this process to develop the quadratic formula. In doing so, they repeatedly transformed a given equation $ax^2+bx+c=0$ into an equivalent equation of the form $(x+T)^2=L$ for some terms T and L, in order to solve for x (as $-T+\sqrt{L}$ and $-T-\sqrt{L}$). To get to the desired equivalent form, they understood the reason and need for dividing through by a, bringing c/a to the other side of the equation, and completing the square. For these teachers, the symbolic manipulation process was goal oriented and conditioned by quantitative considerations; namely, transformations are applied with the intention to achieve a predetermined intrinsic goal. In this case, the teachers practiced the way of thinking of transforming an algebraic equation into a desired form without altering its solution set. This way of thinking—which is one characteristic of algebraic reasoning—was known among the teachers as the changing-the-form-without-changing-the-value habit of mind. We see here the simultaneous implementation of the duality principle, the necessity principle, and the repeated reasoning principle. In particular, the repeated application of this habit of mind helped the teachers internalize it, whereby they become autonomous and spontaneous in applying it.

6.1.4 Institutionalizing ways of thinking

Problem: Is it true that if a positive integer is divisible by 9 then the sum of its digits is divisible by 9?

Two of the solutions offered by the teachers were:

Justification 1: Yes, it is true, because I took many cases, and in each case when the number is divisible by 9, the sum of its digits is divisible by 9.
Justification 2: For 867, $867=8\times100+6\times10+7\times1$, which is $(8\times99)+(6\times9)+(8+6+7)$. Each of the first two addends, 8×99 and 6×9, is divisible by 9, so the third addend, $8+6+7$, which is the sum of number's digits, must be divisible by 9. The same can be done for any number, so it is true that if the number is divisible by 9, the sum of its digits is divisible by 9.

In our earlier research (Harel & Sowder, 1998; Martin & Harel, 1989), Justification 1 was found to be the most common among college students. A probe into the reasoning in Justification 1 has revealed that those subjects' conviction stems from the fact that the proposition is shown to be true in a few instances, each with numbers that are supposedly randomly chosen—a manifestation of the *empirical proof scheme*.

In discussing these answers with the teachers, the goal was to help them internalize a critical distinction between the two: In Justification 2, learners generalize from a pattern in the process, in contrast to the pattern in the result observed in Justification 1. In *process pattern generalization*, learners focus on regularity in the process; whereas in *result pattern generalization*, they focus on regularity in the result. Process pattern generalization is a way of thinking in which one's conviction is based on regularity in the process, though noticing regularity in the result might stimulate it. This behavior is in contrast to result pattern generalization, where proving is based solely on regularity in the result—obtained by substitution of numbers, for instance (see Harel, 2001). Through repeated discussion of this fundamental difference in the context of mathematical problems, we aimed at helping the teachers refine their ways of thinking about what constitutes justification in mathematics—their proof schemes: from proof schemes largely dominated by surface perceptions, non-referential symbol manipulation, and proof rituals, to a proof scheme that is based on intuition, internal conviction, and ultimately logical necessity.

7 Conclusions

DNR's answer to the first of the two fundamental questions presented in the opening of this paper, "What is the mathematics that we should teach in school?", is that the focus of curriculum and instruction in mathematics should be on desirable ways of understanding *and* ways of thinking, not only by the former, as is commonly the case (see Paper I). An important goal of research in mathematics education is, therefore, to identify these ways of understanding and ways of thinking; recognize, when possible, their development in the history of mathematics; and, accordingly, develop mathematics curricula and teacher education programs that aim at helping students construct them. This should not be taken to imply that mathematics curricula should mirror historical developments of mathematics. Rather, the assumption here is that historical developments can shed light on cognitive processes of learning and, in turn, help provide a perspective on teaching.

To address the second fundamental question, "How should mathematics be taught?", *DNR* offers a definition of learning in terms of change in knowledge and the stimuli that result from and instigate the change. These stimuli, in turn, are defined in terms of two types of human needs—intellectual need and psychological need—and the knowledge currently held and newly produced by the students. Of particular importance is that this knowledge must always be judged with respect to desirable mathematical knowledge (that is, with respect to desirable ways of understanding and ways of thinking). The question of paramount importance—yet to be answered—is whether this definition is operational. That is, whether it is possible to specify the nature of change in knowledge and the stimuli that result from and instigate the change. Attempts to address this question are underway.

In addition, *DNR* offers three foundational instructional principles, *duality, necessity,* and *repeated reasoning*. Each of these principles poses both methodological and pedagogical challenges:

The *duality principle,* which deals with the developmental interdependency between ways of understanding and ways of thinking, raises the question: How, methodologically, does one determine a way of thinking from a collection of ways of understanding? For example, what is the range and contextual variation of observed ways of understanding that is necessary to determine a way of thinking? Also, how do answers to these questions translate into practical teaching methods?

The *necessity principle* is linked to the duality principle in that the T_U Part—students develop ways of thinking through the production of ways of understanding—can only be implemented by devising problematic situations that intellectually necessitate particular ways of understanding from which targeted way of thinking may be elicited. This raises the question: How does one determine students' intellectual need? *DNR* provides a framework for addressing this question, but detailed methodologies, together with suitable pedagogical strategies, for dealing with this question are yet to be devised. The framework consists of a classification of intellectual needs into five interrelated categories:

- The *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines—by whatever means he or she deems appropriate—that an assertion is true. Truth alone, however, may not be the only need of an individual, and he or she may also strive to explain *why* the assertion is true.
- The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is.
- The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them. It also includes the need to optimize calculations.
- The *need for communication* includes the need to persuade others than an assertion is true.
- The *need for connection and structure* includes the need to organize knowledge learned into a structure, to identify similarities and analogies, and to determine unifying principles.

Back to the notion of "instructional principle," the use of the term "principle" here is not without difficulty. The difficulty is twofold. First, the *DNR* instructional principles might be viewed as crude generalizations: that certain relations between teaching actions and student learning observed in a limited number of teaching experiments are not just local observations but principles, relations that are valid for all mathematics instruction. Second, and coupled with the first, is the question: are these principles merely empirical inferences from local observations or do they account for them? It is important to highlight these difficulties because it is with these difficulties in mind that the *DNR* instructional principles must be understood. One must reserve a distance between the notion of "instructional principle" in education and that of "physical principle" in science. The latter clearly means that a physical principle, more than being consistent with empirical observations, actually accounts for them. The choice of the term "conception"—rather than "law," for example—in the definition of "instructional principle" (in Sect. 5) is intended to reserve this distance. The hope, of course, is that the validity of the *DNR* instructional principles will be confirmed with additional and more systematic investigations, both empirical and theoretical, whereby their status would change from mere empirical inferences to statements with an explanatory power.

The repeated reasoning principle aims at helping students internalize and organize the knowledge they learn. It, too, raises methodological and pedagogical challenges. For example: How should mathematical activities be sequenced so that they repeatedly call for thinking through problematic situations, on the one hand, and respond to students' changing intellectual needs, on the other?

DNR defines TKB in terms of three components of knowledge: knowledge of mathematics (the desirable ways of understanding and ways of thinking possessed by the teacher), knowledge of student learning (a teacher's view of "mathematics learning" and her or his understanding of the cognitive and epistemological issues involved in learning particular mathematical ways of understanding and ways of thinking), and knowledge of pedagogy (a teacher's teaching practices and instructional principles). What is crucial here is the content of the three components comprising the TKB, not their label, which may resemble those offered by others (e.g. Ball & Bass, 2000). In Harel (1993), the content of these components was described in general, less precise terms, though undoubtedly was influenced by the *DNR* ideas, which at the time were neither explicit nor well-formed. Here, in contrast, the content of TKB is explicitly implied from the *DNR* framework. Of particular importance is that according to *DNR,* a teacher's knowledge of pedagogy and student learning rests on that teacher's knowledge of mathematics. That is to say, although each of the three components of knowledge is indispensable for quality teaching, they are not symmetric: the development of a teachers' knowledge of student learning and of pedagogy depends on and is conditioned by their knowledge of mathematics. This position is implied from the definition of the TKB and *DNR* premises: on the one hand, instructional objectives, by the Mathematics Premise, must be formulated in terms of mathematical knowledge, i.e., desirable ways of understanding and ways of thinking. On the other hand, "student learning" in the definition of the TKB is the sense of *DNR*'s definition of "learning," which, as we have discussed earlier, involves the knowledge utilized and newly constructed by the student during the learning phases. Hence, as a teacher attends to the learning process of her or his students, he or she must judge their current knowledge and the knowledge they produce relative to the targeted mathematical knowledge. Clearly, a teacher with insufficient mathematical knowledge will not be able or develop the necessary understanding to make such judgments. This raises a critical question: What mathematics should a teacher know? As we have argued in Paper I, thinking of teachers' mathematical knowledge in terms of ways of thinking (not only in terms of ways of understanding) helps address this question more decisively.

References

Anderson, J. R. (1980). *Cognitive psychology and its implications*. San Francisco: Freeman.

Anderson, J. R. (1982). Acquisition of cognitive skill. *Psychological Review, 89*, 369–406. doi:10.1037/0033-295X.89.4.369.

Anderson, J. R., Reder, L. M., & Simon, H. A. (1996). Situated learning and education. *Educational Researcher, 25*(4), 5–11.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning*, 83–104.

Behr, M., Erlwanger, S., & Nichols, E. (1976). How children view equality sentences: PMDC technical report, no. 3. Tallahassee, Fla.: Florida State University, 1975. ERIC No. ED, 144, 802.

Behr, M. J., Wachsmuth, I., & Post, T. R. (1984). Tasks to assess children's perception of the size of a fraction. In A. Bell, B. Low, & J. Kilpatrick (Eds.), *Theory, research and practice in mathematical education. Working group reports and collected papers from the 5th International Congress on Mathematical Education, Adelaide* (pp. 179–181). Nottingham, UK: Shell Centre for Mathematical Education, Nottingham University.

Bransford, J. R., Brown, A. L., & Cocking, R. R. (Eds.). (1999). *How people learn: Brain, experience and school*. Washington, DC: National Academy Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Boston: Kluwer Academic Publishers.

Brownell, W. A. (1946). Introduction: Purpose and scope of the yearbook. The forty-fifth yearbook of the National Society for the Study of Education, Part I, the measurement of understanding, (pp. 1–6). University of Chicago, Chicago.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359–387. doi: 10.1007/BF01273371.

Chiesi, H., Spilich, G., & Voss, J. (1979). Acquisition of domain-related information in relation to high and low domain information. *Journal of Verbal Learning and Verbal Behavior, 18*, 257–273. doi:10.1016/S0022-5371(79)90146-4.

Cohen, D. K., & Ball, D. L. (1999). Instruction, capacity, and improvement (CPRE research report no. RR-043). Philadelphia: University of Pennsylvania, Consortium for Policy Research in Education.

Cohen, D. K., & Ball, D. L. (2000). Instructional innovation: reconsidering the story. Paper presented at the American Educational Research Association.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist views of the teaching and learning of mathematics (Journal for Research in Mathematics Education, Monograph No. 4, pp. 107–122). Reston: National Council of Teachers of Mathematics.

Cooper, R. (1991). The role of mathematical transformations and practice in mathematical development. In L. Steffe (Ed.), *Epistemological foundations of mathematical experience*. New York: Springer-Verlag.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. New Jersey: Ablex Publishing.

Davis, R. B. (1992). Understanding" Understanding. *Journal of Mathematical Behavior, 11*(3), 225–242.

DeGroot, A. D. (1965). *Thought and choice in chess*. The Hage: Mouton.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In Tall D. (Ed.), Advanced mathematical thinking (pp. 95–123). Boston: Kluwer.

Dubinsky, E. D., & McDonald, M. A. (2001). APOS: A constructive theory of learning in undergraduate mathematics education research. The teaching and learning of mathematics at university level*:* An ICMI study.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3–17.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), Proceedings of the Sixth International Conference for the Psychology of Mathematics Education (128–131). Antwerp, Belgium: Universitaire Instelling.

Goetting, M. M. (1995). The college student's understanding of mathematical proof (doctoral dissertation). University of Maryland.

Greeno, J. (1980). Conceptual entities. In D. Gentner, & A. Stevens (Eds.), Mental models (pp. 227–252). Hillsdale: Lawrence Erlbaum Associates

Harel, G. (1993). On teacher education programs in mathematics. *International Journal of Mathematical Education in Science and Technology, 25*, 113–119. doi:10.1080/0020739940250114.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-Based instruction. In S. Campbell & R. Zaskis (Eds.), Learning and teaching number theory. In C. Maher (Ed.), Journal of Mathematical Behavior. New Jersey, Ablex Publishing Corporation, 185–212.

Harel, G. (2008a). DNR perspective on mathematics curriculum and instruction: Focus on proving, part I. *ZDM. Zentralblatt für Didaktik der Mathematik, 40*, 487–500. doi:10.1007/s11858-008-0104-1.

Harel, G. (2008b). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold & R. Simons (Eds.), Proof and other dilemmas: Mathematics and philosophy. Washington: Mathematical American Association.

Harel, G., & Sowder, L. (1998). Student's proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (Vol. 3, pp. 234–283). Providence: American Mathematical Society.

Harel, G., & Sowder, L (2008). Toward a comprehensive perspective on proof, In F. Lester (Ed.), *Handbook of research on teaching and learning mathematics* (2nd ed.). Greenwich, CT: Information Age Publishing. (in press).

Hiebert, J. (1997). Aiming research toward understanding: lessons we can learn from children. In A. Sierpinsda & J. Kilpatrick (Eds.), Mathematics education as a research domain: a search for identity: An ICMI study (pp. 141–152). Dordrecht: Kluwer Academic Publishers.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. New York: Cambridge University Press.

Lawson-Tancred, H. T. (1998). Aristotle: The Metaphysics. Harmondsworth: Penguin.

LaBerge, D., & Samuels, S. J. (1974). Toward a theory of automatic information processing in reading. *Cognitive Psychology, 6*, 293–323. doi:10.1016/0010-0285(74)90015-2.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), Multiple Perspectives on Mathematics Teaching and Learning (19–44). Westport, CT: Ablex.

Lesgold, A. M., Rubison, H., Peltovich, P., Glaser, R., Klopfer, D., & Wang, Y. (1988). Expertise in a complex skill: Diagnosing X-ray pictures. In M. T. H. Chi, R. Glaser & M. Farr (Eds.), *The nature of expertise* (pp. 311–342). Hillsdale: Erlbaum.

Martin, G., & Harel, G. (1989). Proof frame of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*, 41–51. doi:10.2307/749097.

Miller, G. A. (1956). The magical number seven, plus or minus two: some limits on our capacity to process information. *Psychological Review, 63*, 81–87. doi:10.1037/h0043158.

Piaget, J. (1952). The origins of intelligence in children. New York: International Universities Press.

Piaget, J. (1969). Early growth of logic in the child. London: W. W. Norton & Company.

Piaget, J. (1973). The child and reality: problems of genetic psychology. New York: Grossman.

Piaget, J. (1978). Success and understanding. Cambridge, Mass.: Harvard University Press.

Piaget, J. (1985). *The equilibration of cognitive structures: the central problem of intellectual development*. Chicago: University of Chicago Press.

Schneider, W., & Shiffrin, R. M. (1977). Controlled and automatic human information processing: Detection, search and attention. *Psychological Review, 84*, 1–66. doi:10.1037/0033-295X.84.1.1.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36. doi:10.1007/BF00302715.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22.

Steffe, L. P., Cobb, P., & Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Steffe, L. P., & Thompson, P. W. (2000). *Teaching experiment methodology: Underlying principles and essential elements*. Hillsdale: Lawrence Erlbaum Associates.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: philosophy, theory, and application*. New York: Praeger.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: considerations in developing. In E. A. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives. Hillsdale: Lawrence Erlbaum Associates.

Tournaire, F. (1986). Proportions in elementary school. *Educational Studies in Mathematics, 17*(4), 401–412. doi:10.1007/BF00311327.

von Glasersfeld, E. (1983). Learning as constructive activity. Proceedings of the 5th Annual Meeting of the North American Group of Psychology in Mathematics Education, 41–101.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds. & Trans.): Cambridge: Harvard University Press.(Original work published 1934).