Attention to meaning by algebra teachers

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ABSTRACT

Non-attendance to meaning by students is a prevalent phenomenon in school mathematics. Our goal is to investigate features of instruction that might account for this phenomenon. Drawing on a case study of two high school algebra teachers, we cite episodes from the classroom to illustrate particular teaching actions that de-emphasize meaning. We categorize these actions as pertaining to (a) purpose of new concepts, (b) distinctions in mathematics, (c) mathematical terminology, and (d) mathematical symbols. The specificity of the actions that we identify allows us to suggest several conjectures as to the impact of the teaching practices observed on student learning: that students will develop the belief that mathematics involves executing standard procedures much more than meaning and reasoning, that students will come to see mathematical definitions and results as coincidental or arbitrary, and that students' treatment of symbols will be largely non-referential.

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It is not uncommon for students to manipulate symbols without a meaningful basis that is grounded in the context in which the symbols arise; for instance, a student might write: $(\log a + \log b)/\log c = (a+b)/c$. Matz (1980) connects this error and a wide range of algebra errors to overgeneralizing the distributive property. In this example, students factor out the symbol "log" from the numerator and cancel it with the "log" in the denominator without attending to the quantitative meaning of their action. The behavior of operating on symbols as if they possess a life of their own, rather than treating them as representations of entities in a coherent reality, is referred to as the *non-referential symbolic* way of thinking (Harel, 2007). With this behavior, one does not ask questions like "What is the definition of $\log a$?", "Does $\log a$ (multiplication) have a quantitative meaning?", "Is $\log a + \log b = \log(a+b)$?", and so on, because symbols are not conceived by the person as representations of a coherent quantitative reality. Often, students' actions – even if erroneous – can be deemed meaningful to them when judged according to their current schemes and the norms of their school mathematics. However, the concern of this study is attendance to *mathematical meanings*: those judged as such by the mathematical community at large. While we fully recognize the pedagogical significance of understanding the meanings intended by students, these are not the focus of this study. This study focuses on teaching practices that may lead to students' non-attendance to mathematical meanings, creating what Brousseau (1997) calls *didactical obstacles* (as opposed to *epistemological obstacles*—those that have to do with the inherent difficulty individuals have in developing new conceptions). When teachers do not emphasize mathematical meaning, by, for example, accepting or modeling non-referential symbolic reasoning, students are likely to adopt similar reasoning practices.

Evidence of students' non-attendance to mathematical meaning has been well-documented in the literature. Work on students' mathematical behavior in elementary algebra provides a good example. Kieran's (1981) literature review reports ways that students misunderstand the meaning of the equal sign. Young children view the equal sign as an operator signal, telling them to do something like add the numbers to its left. This view persists into middle (and sometimes high) school, so

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that students have trouble understanding equalities like $4 + 8 = 2 \times 6$ (Behr, Erlwanger, & Nichols, 1976). Students also may use the equal sign to record their steps without regard to its standard meaning, as in $5 \times 3 = 15 - 7 = 8$ (Kieran, 1981). The non-referential symbolic way of thinking is only one manifestation of non-attendance to mathematical meaning by students. A different but related manifestation is students' treatment of variable. Sleeman (1984) observed high school students offering different values of x for a single linear equation: $3x + 4x = 3$ was solved by setting the first x equal to 1 and the second x equal to 0. Also, many middle and high school students fail to realize that the meaning or value of a variable is not dependent on the letter used. They thus expect equations like $7W + 22 = 109$ and $7N + 22 = 109$ to have different solutions; in fact, some believe that W should have a larger value because it occurs later in the alphabet (Wagner, 1981). More recently, Vlassis (2004, reported in Kieran, 2007) found that 8th graders mostly viewed the minus sign as a binary symbol and thus had trouble understanding that $-x$ means the opposite of x. Of course, some of these student behaviors can be explained in terms of cognitive development (Kuchemann, 1981) and the epistemology of the concepts (Lobato & Siebert, 2002; Sfard & Linchevski, 1994). More recent work has confirmed the common sense prediction that students with a better understanding of the equals sign are better able to solve tasks involving equations (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007). Referential symbolic reasoning, in which symbols are manipulated according to a coherent system of referents, is a special case of quantitative reasoning, as discussed by Thompson (e.g. Smith & Thompson, 2007). Quantitative reasoning involves reasoning with measurable properties of an object and may initially be expressed verbally. Referential symbolic reasoning develops as students appreciate the advantages of expressing their reasoning symbolically. Smith and Thompson emphasize the centrality of meaning: "If students are eventually to use algebraic notation and techniques to express their ideas and reasoning productively, then their ideas and reasoning must become sufficiently sophisticated to warrant such tools." A learner who is using referential symbolic reasoning may not always think about the meaning behind symbols she writes, but she must be able to pause at any moment and unpack the expressions, identifying the quantities referred to and the relationships between them that justify what is written. Quantitative reasoning and, in particular, referential symbolic reasoning, develop over years of practice. We believe that the way meaning is treated in the classroom is a major factor in the development of such reasoning.

In this paper, we examine teaching actions taken by two high school algebra teachers and identify specific ways in which these teachers treat mathematical meaning in the classroom. We will document specific teaching actions carried out by these teachers (labeled "A" and "B") that de-emphasize meaning, characterizing and organizing the ways this occurs. In the conclusion, we will consider various possibilities that might account for the way these teachers treat meaning. While our main focus in this paper is limited to characterizing the treatment of meaning per se, we do suggest possible impacts on student learning. More specifically, we suggest several conjectures as to the impact of the teaching practices observed on student learning: that students will develop the belief that mathematics involves executing standard procedures much more than meaning and reasoning, that students will come to see mathematical definitions and results as coincidental or arbitrary, and that students' treatment of symbols will be largely non-referential.

Teachers A and B had participated in a professional development program as part of a larger study. During the summer before our observations, they attended an intensive 4-week workshop that addressed both content and pedagogical knowledge for high school algebra. Approximately 30 teachers from various schools and districts met for 6 h each weekday at a university setting. The teachers were chosen because they taught middle or high school mathematics and had relatively limited mathematics background. The workshop curriculum strongly emphasized meaning, and the workshop leader reinforced this at every opportunity. Asking participants to compare different solutions to nonstandard problems was one way of stressing meaning. The problems were unlike textbook exercises in that a correct solution method, and indeed the underlying mathematical content, were not initially obvious. This content included linear and quadratic functions/equations, systems of equations or inequalities, and absolute value. The leader emphasized logical distinctions such as that between "and" and "or", and brought forward for discussion the participants' understandings of the meaning of mathematical terms and processes. For instance, he asked for a definition of "average" and asked for a verbal explanation of what it means for some x to satisfy a system of inequalities. The leader also addressed the relationship between a statement and its converse and between equivalent definitions of the same concept, highlighted the meaning and purpose of "simplifying" expressions, and promoted justification of alternative solutions. Since the leader also addressed pedagogy, for example by asking how the participants might respond to some solution as a teacher or how they might alter a problem to help their students understand the importance of some concept, we expected participants' subsequent teaching to reflect some of this attention to meaning. Selected participants had one of their classes videotaped nine times at approximately equal intervals during the year following the summer workshop. Out of those videotaped, we chose to focus on the two teachers whose video data was mathematically richest (i.e. their videos contain many uninterrupted discussions of mathematics, whereas other teachers videotaped had to devote a significant part of class time to dealing with discipline issues). Teacher A had a BA in mathematics and had taught for 3 years prior to the study. He attended previous professional development that included a focus on algebra. During the study, he taught geometry in addition to the Algebra II course we observed. Teacher B's only post-secondary mathematics course sequence was in calculus, and he had a masters degree in education. He had taught for 2 years prior to the study. During the study, he taught Algebra II in addition to the lower track Algebra I course we observed. Both teachers taught at the same high-performing suburban school.

We observed many instances from both teachers in which meaning was de-emphasized (as well as some limited instances in which meaning was promoted, which will not be discussed). Our intent is not to criticize these teachers, who are dedicated professionals serving their students and schools to the best of their ability. Nor are we trying to compare the two teachers

or paint a full portrait of their teaching practice. We are aware of, and will discuss, the many constraints – institutional and others – under which they operate and the many instructional goals they must balance. Nevertheless, of crucial importance is the question: *why, despite good intentions and experiencing a professional development workshop that focused heavily on meaning, do these teachers still exhibit teaching actions that treat meaning in undesirable ways?* We discuss this question in this paper, but it is not the focus of our study; nor do we have sufficient data to answer it. Furthermore, funding constraints prevented us from obtaining baseline data on the teachers' classes prior to the professional development workshop, and we did not interview the teachers about why they made the choices they did in the classroom.

Although the role of meaning in the classroom has been studied before, we consider the identification of specific teaching actions that de-emphasize meaning to be an important contribution of our work that can facilitate future research. In particular, it can be used to define parameters for studying teacher change with respect to treatment of meaning and the impact this treatment has on student learning. A major question for future research that emerges from our analysis is: is this behavior typical? That is, do other teachers display similar treatments of meaning, and what characteristics influence this treatment? Also, future research can test the potential explanations we propose for teachers' non-attendance to meaning following intensive professional development. An analysis comparing and contrasting the sets of actions of the two teachers can be of pedagogical value, but it is outside the scope of this study.

This paper is structured in five sections. We first present episodes from our data and observations on them. Following this, we categorize these observations and remark on differences between the two teachers we observed. Next, we consider what impacts the observed treatment of meaning might have on student learning. Finally, we address several pedagogical issues related to meaning that emerged from our observations, speculate on explanations for our observations, and suggest extensions of our work.

1. Teachers' treatment of meaning[1]

In this section, we examine seven episodes from our data, which consist of videotapes of nine classes taught by each teacher over a 1-year period. Our goal is to present examples in which the teachers' treatment of mathematical meaning is, from our perspective, undesirable. Our discussion of each episode presents several observations, each of which is labeled by a particular aspect of meaning which it illuminates. Although these aspects may recur in different episodes, we choose to discuss them individually, rather than aggregately, in order to retain the different nuances of attention to mathematical meaning that are illustrated by each individual episode. After discussing the episodes, we will summarize and categorize all the observations.

We present portions of the transcript with numbered lines, where "T" denotes the teacher and "S" denotes a student. In cases involving more than one identifiable student, numerical suffixes will be used (e.g., S1, S2). When statements are made by multiple students, they are denoted by "S*".

Most of our observations are based on the episodes as quoted, which record what the teacher does over a short period of time. However, we also consider the surrounding context when it helps us answer significant questions—for instance: Is this an introduction to the topic, was a relevant idea mentioned outside the quoted episode? We also include student responses to the teachers' actions in our analysis.

1.1. Teacher A

1.1.1. Episode 1

The teacher is giving a lecture using a computer projector. He introduces the concept of determinant and later provides a formula for calculating the determinant of a two-by-two matrix. General ideas are presented first, and specific examples come later.

On screen: Every square matrix with real number entries has a real number associated with it called the determinant.

1. T: OK, so square matrix, we've looked at a lot of square matrices, we haven't really called them square matrices but if it's a 2 by 2, 3 by 3, 4 by 4, whatever, as long as it has the same number of rows, same number of columns it's a square matrix. And every square matrix with real number entries has a real number associated with it called the determinant.
2. S1: Wasn't that when we multiplied it times another number and then. . .
3. T: I'm sorry?
4. S1: Well, wasn't the determinant where you multiplied the matrix by a number and it was. . .
5. T: I think that was a scalar, was that a scalar?
   [The teacher resumes the lecture, giving the procedure for computing a two-by-two determinant and then projecting on screen the definition of an identity matrix.]
6. S2: What's that thing after identity matrix?
7. T: Oh, this is an I, this is, the identity matrix is often represented by the letter I, capital I.

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[1] When it is clear from the context that we are referring to *mathematical meaning*, we will often omit the adjective *mathematical*.

   8. S2: Who cares, like what is the whole point?
   9. T: I'm sorry?
  10. S2: The like, who cares, this is like, it's called identity matrix, like what. . .
  11. T: Right, the identity, what does identity mean? If we're looking at multiplication what's the identity in multiplication?

Four observations can be made from this episode. The first observation concerns *purpose for new concepts*. In this lesson, the concepts of "matrix" and "determinant" are presented out of context. That is, they are introduced without an intellectual purpose, without the context of a problematic situation viewed as such by the students. When S2 asks about the "whole point" of what has been presented (Line 8) and later for the specific purpose of the identity matrix (Line 10), the teacher does not follow up on the first question to find out what the student is asking, and he interprets the second as a request for the definition of the term "identity" (Line 11).

The second observation concerns *meaning of terms*. The teacher uses the term "associated with," presumably to denote a functional relationship between a matrix and its determinant, but there is no effort by the teacher to address the meaning of this term, even when a student raises a question about it. S1 asks (indirectly) about the meaning of "associated with" when he requests clarification of whether it refers to multiplication of a matrix by a number (Line 2). Perhaps the student interpreted the term to indicate things that appear close together, recalling a problem in which a matrix and real number were "associated" in that they had to be multiplied together.

The third observation concerns *differentiation between meanings of different terms*. In Line 4, the same student shows more confusion; he relates multiplication by a number to "determinant." The teacher's response is "I think that was a scalar," (Line 5), with no further discussion to find out the source of the student's confusion or to clarify the meaning of the different terms that emerged in the exchange with the student. We believe the present episode is the first appearance of "determinant" in this classroom, but both teacher and student seem to be referring to an earlier lesson involving multiplication of a matrix by a scalar. The teacher does not take the opportunity to remind the class of the earlier lesson and clearly distinguish that context from the present one.

The fourth, and last, observation concerns *differentiation between an object and a process*, which can be viewed as a special case of the third observation. The teacher's response in Line 5 misses an opportunity to attend to the fact that certain terms denote an object, others a process: a scalar is an object, not the act of multiplying, as students might think based on the teacher's response.

Overall, we see in this episode a manifestation of (a) lack of intellectual purpose for what is being taught, (b) lack of clarity regarding meaning of terms, (c) lack of differentiation between meanings of different terms, and (d) no distinction between an object, such as a "scalar," and a process, such as "multiplication."

1.1.2. Episode 2

In this episode, the teacher has been discussing how to solve exponential growth/decay problems. Students are working on problems from the textbook while the teacher circulates around the room. We cannot see exactly what this student is asking about, but the issue appears to be the percent increase after 1 year of some quantity whose value after t years is proportional to 1.63^t. The general form of such problems involves the expression b^t.

  1. T: So, this is 1.63. What's my growth factor? What's my growth, what's my rate, what's my percent of increase?
  2. S: I don't know.
  3. T: Look right up there [points to the equation $b = 1+\%$ on the board]. If b is 1 plus the percent, right?
  4. S: Oh, so 1 plus 1.63?
  5. T: I, no, I already did, that's already added to 1, so what's 1.63 minus 1?
     [Near the end of the class, the teacher summarizes what students should know:]
  6. T: So I did put some notes on [the website], but this is all it is right here. Growth factor is b is 1 plus the percent, decay is 1 minus the percent.

We make three observations regarding this episode, some of which are similar to those from the previous episode. The first concerns *meaning of terms*. The teacher uses four wordings for what he wants the students to find, "growth factor, growth, rate, percent of increase" (Line 1), which do not all have the same mathematical meaning. "Percent of increase" is an accurate characterization of the quantity .63 (the teacher's intended answer, see Line 5) in this problem, but "growth factor" would more likely be 1.63. "Growth" on its own usually refers to an actual amount of growth such as 1.5 million more bacteria. "Rate" in this context would refer to amount of growth per unit time. While the teacher may be trying to help the student by providing more opportunities to recognize at least one of the terms, the meanings of these terms are not clearly distinguished.

The second observation concerns *meaning of strings of symbols*. The equations "$b = 1+\%$" and "$b = 1-\%$" literally appear on the board and, presumably, the teacher's website (Line 6). However, the equation seems intended as a mnemonic to assist students in carrying out procedures rather than a meaningful statement about quantities.

The third observation has to do with *usage of mathematical symbols*. "The percent" in the teacher's statement describes the form rather than the significance or meaning of a quantity. There is ambiguity as to whether it should mean 0.63, 1.63, 63, or 163. Indeed, in Line 4 the student has mis-identified it (from the teacher's viewpoint) as 1.63. Moreover, the symbol

"%" already has a mathematical meaning: division of the preceding number by 100. The teacher's use of the "%" symbol is not consistent with this standard meaning.

Overall, we see in this episode manifestations of (a) failure to distinguish between meanings of related terms, (b) non-attendance to quantitative references in an equation, and (c) usage of mathematical symbols that conflicts with their standard meaning.

1.2. Teacher B

1.2.1. Episode 3

The teacher introduces the next chapter in the textbook, on rational expressions and functions. As he has done for earlier chapters, he distributes a handout giving the standards for the new topic, illustrated with sample problems students will be able to solve. Based on the examples on this sheet, he asks students what a rational expression might be. Following their suggestion of "fractions," he tries to connect this with rational numbers. These had been discussed about a month earlier, judging from his reference to a test question in which students had to classify numbers as rational, irrational, or undefined. Several portions of transcript are presented for this episode, separated by ellipses (. . .).

1. T: Ok, we're going on to the next chapter, chapter eleven, which is rational expressions and functions, and like every other chapter I'm going to start out by giving you what we're going for. . . What do you suppose rational expressions are, looking at all those things? What's common about all of them? . . .Number six, solve rational equations, there's that word, equations, equality, equals. . . We're talking about rational expressions and functions and the first lesson we're going to do is we're going to take care of the first one, rational expressions. What is a rational expression? Do we know what a rational number is?
2. S*: A whole number?
3. T: Give me an example, well rather than rational, give me an example of an irrational number.
4. S1: Something that, I don't know.
5. S*: Go on and go on and go on.
6. T: Goes on and on forever? So give me an example of one.
7. S2: Three.
8. T: Is three an irrational number?
9. S*: Yes. No. (. . .)
10. T: So give me an example of an irrational number, one that goes on and on forever.
11. S*: Pi.
12. T: Pi. Pi is irrational. Why is it irrational? Because if I write it out it's equal to 3.141628 [sic] da da da da da da and we're going to need work in the middle of next period, of the next century, still writing numbers out, it keeps on going. (. . .)
13. S3: The square root of 99 is irrational.
14. T: The square root of 99 is irrational. Why is it irrational?
15. S4: Because two numbers (inaud). (. . .)
16. S5: An irrational number are all numbers that can't be expressed as a ratio of two integers.
17. T: OK. She just said a mouthful. Did you hear what she said? (. . .)
18. T: If I can't express it as the ratio of two integers, it's irrational. If I can, can I express this number [3] as a ratio of two integers? (. . .)
19. T: How about this guy here [$\sqrt{99}$]? No, I can't. It just so happens the only square roots that work out to be numbers are the numbers we dealt with, the perfect squares in here. (. . .)
20. T: So this is between 9 and 10 but if I was to write it out, it's not equal to 9, it's not equal to 10, it's equal to nine point and the numbers are going to go on forever. Can this be written. . .and it can't be written as a fraction because it's going on forever. It just so happens every time that it goes on forever, you can't express it as a fraction.

In this episode, we observe inattention to meaning in several ways. As in episode 1, there is no *purpose for new concepts*; in this case, no context or purpose is presented for the classification of numbers as rational versus irrational.

Our second observation concerns *correctness of definitions*. Due to the lack of purpose, even correct definitions are unlikely to be truly meaningful. However, the definition of irrational initially accepted by the teacher is incorrect (Lines 5 and 6). Taken literally, "goes on and on forever" describes a non-terminating decimal. That is, students are explicitly given an incorrect meaning for irrational numbers. The possibility of distinguishing between non-terminating and non-repeating decimals is never raised. From the data, we cannot tell to what extent the teacher or the students may be aware of such a distinction, although it is crucial.

The third observation concerns the *role of meaning in justifications*. Once the above definition is accepted, there is no use of it to justify claims about irrationality, with the exception of explaining why 3 is not irrational. The teacher asserts that π is irrational because it is non-terminating, but that claim rests only on his authority (Line 12). Without a way to apply the definition in practice, students have no alternative to justification based on authority. Of course, a proof that pi is irrational is not accessible to precalculus students, and we would not expect the teacher to present one. However, his claim raises

the issue of how in principle one could know that the decimal expansion of some number is non-terminating (actually, non-repeating), and the applicability of the definition depends on resolving this issue.

The fourth observation concerns *meaning of symbolic representations*. The specific numbers used as illustrations, π and $\sqrt{99}$, are treated non-referentially, without attention to their meaning or how to move between alternative representations. The teacher quotes an incorrect series of digits for π (Line 12), and there is no reference to its definition in terms of a circle or discussion of how its decimal expansion might be computed from its meaning. Similarly, there is no discussion of how the meaning of $\sqrt{99}$ determines its decimal expansion. The teacher does not pay explicit attention to the mental images students may have for either of these numbers. This is relevant to the above point about whether and how the decimal expansion of some number could be known to be non-terminating.

The fifth observation concerns *consistency of terms*. The distinction between rational and irrational numbers is critical for understanding how and why the rational number system is enlarged to the reals, which in turn expands students' conception of what a "number" is. Throughout most of this episode, "number" is indeed used in the expanded sense of "real number." However, in Line 19 the teacher says "the only square roots that work out to be numbers are the numbers we dealt with, the perfect squares in here." Both uses of the word "numbers" seem to connote "familiar numbers." The second "numbers" is clarified as "perfect squares," but it is not clear whether the first "numbers" refers to rational numbers or natural numbers. The word "number" is unavoidably ambiguous, but the teacher does not distinguish different definitions of "number" that may be implicit in different contexts.

The sixth observation concerns *treatment of different definitions of the same concept.* In Line 16, a student presents the textbook's definition of irrational number, "all numbers that can't be expressed as a ratio of two integers." The teacher immediately accepts this new definition and uses it to revisit the examples 3 and $\sqrt{99}$. However, there is still no purpose or context for the definition and no nontrivial use of it to justify claims about irrationality. The presence of two definitions of the same concept raises a critical issue of meaning: the need to establish their equivalence. It appears (Line 20) that the teacher recognizes this need at some level, since he asserts (incorrectly) the equivalence on his own authority. However, we do not see in this classroom any expectation that such issues of equivalence must be justified by reasoning, or even tested against examples such as 1/3 (which would refute the claimed equivalence).

The seventh, and final, observation concerns *treatment of mathematical definitions and claims.* We have seen that when justifications of claims do not rest on the meaning of concepts, only the teacher's authority remains to justify them. However, not only are the claims justified by authority, but also they appear to be arbitrary. No reason is provided or presumed to exist for these claims, as emphasized by the teacher's "It just so happens" (Lines 19 and 20). This undermines the function of justification in providing insight—explaining *why* claims hold rather than simply *that* they do. Likewise, in the absence of purpose or context, the definitions appear essentially arbitrary.

To summarize, this episode illustrates (a) lack of context or purpose for what is presented, (b) acceptance of incorrect definitions, (c) claims asserted by authority rather than established by meaning-based reasoning or tested against examples, (d) no attention to the meaning of symbolic representations, (e) inconsistent use of a term, (f) acceptance of multiple definitions without demonstrating their equivalence, and (g) expectation that definitions and facts are arbitrary rather than based on purpose and meaning. Although the lesson appears to focus on the meaning of irrationality, students learn only to classify a limited set of numbers as rational or irrational based on the teacher's authority.

1.2.2. Episode 4

The issue of irrational numbers reappears in the context of removing perfect squares from radical expressions. The specific question is whether the factor 2 can be removed from the radical $\sqrt{2 \cdot 81}$.

1. T: What times itself will give me 2? Is 1 times itself, 1 times 1, equal 2?
2. S*: NO.
3. S*: 1.4 (inaud).
4. T: Say it again.
5. S*: 1.4.
6. T: Yes, go back and look at that table. And it turns out that 1.414 and it keeps going on forever, times 1.414 going on forever will give me 2. So, the only way I can take 2 outside is, what would come outside is a square root of that, 1.414.

We see again several of the issues described in the previous episode: the ambiguity of "going on forever" (non-terminating or non-repeating?), the arbitrariness of the claims ("it turns out"), and the absence of attention to how a decimal approximation for $\sqrt{2}$ might be computed. Students have a table giving such decimal approximations of square roots, but it functions only as an additional authoritative source of information. A new observation concerns *the meaning of operations*. The teacher claims that the product of two infinite decimals is defined and equal to 2 even though students know no algorithm for doing such a computation. Thus, multiplication here refers to a procedure which cannot be carried out by any means known to the students. Students' familiarity with multiplication conceals its lack of a referent in this case. Overall, the instructional treatment assumes that no important conceptual issues arise from extending arithmetic operations to infinite decimals.

1.2.3. Episode 5

During this episode, the teacher is addressing the entire class as they simplify radical expressions. He has posed the practice problem of simplifying $\sqrt{75/25}$.

On screen: A radical expression is in simplest form when all three statements are true.

(1) The expression under the radical sign has no perfect square factors other than 1.
(2) The denominator does not contain a radical expression.
(3) The expression under the radical sign does not contain a fraction.

 1. T: OK, Amanda, tell me what you did.
 2. S1: I put square root of 75 over square root of 25.
 3. T: Now we see a definite violation of rule number 2. It's busted. What's next?
 4. S1: Square root of 25.
 5. T: Are we done?
 6. S*: No.
 7. T: What now? Square root of 75 is 5 times 5 times 3 all over 5. 5 times 5 is 25 times 3 is 75. (. . .)
 8. S2: Can I just divide 25 into 75?
 9. T: 25 divided by 75. Three. So I could have gone straight that way. Very good. See that? But it's the same thing. You're using different routes to get to the same end. This is a good way of checking it.
    Later on, the teacher helps an individual student simplify $\sqrt{13}\sqrt{13}$:
10. T: OK, keep going. Did you have to write that line in between here? Look at what you did. Square root of 13 times the square root of 13 equals the square root of 169. The square root of 169 is 13. OK, so as long as you realize you know how to do this you can take a shortcut.

We make the following closely related observations regarding this episode. The first concerns *focus on symbolic form*. The simplification rules as stated do not refer to the quantitative value or meaning of an expression, only to its form. Moreover, we did not observe any discussion of what it means to "simplify," so this term may not be meaningful for students. That is, students may not realize that the simplified form of an expression must still have the same value as the original expression but that it should be easier to use or evaluate—a realization that could have led the students to see intellectual purpose in simplification tasks. The teacher does not state how the rules he presents are related to simplification, nor does he clarify whether they are intended to define, or merely describe, the simplest form of a radical expression. Also, no attention is drawn to the implicit claim that such a form is unique.

The second observation concerns *treatment of procedures*. In this episode, procedures are presented as primary, meaning as secondary. No specific procedure is discussed in the class that was observed, but it appears that the teacher had previously demonstrated procedures for simplifying radical expressions. We see evidence here that these procedures encourage students to work problems without considering the meaning of "simplest form" or the meaning of the expression to be simplified, and we classify such a procedure as non-referential. In the first problem above, one student follows the non-referential procedure that has been taught and another student considers it necessary to ask whether it is acceptable to evaluate the expression directly according to its meaning (Line 8). The teacher acknowledges that this meaning-based approach is valid, but he suggests that it is a good way to check the non-referential procedure rather than a reasonable starting point (Line 9). In the last portion of the episode, a student has *computed* that $\sqrt{13}\sqrt{13} = 13$ without realizing that this is true by the definition of a square root. The student's paper is not visible, so we do not know whether the teacher's proposed "shortcut" is an approach based on meaning. Either way, the teacher's statement "as long as you realize you know how to do this you can take a shortcut" (Line 10) seems to once again uphold the non-referential procedure as primary.

In sum, we see in this episode (a) emphasis on the form of expressions rather than their quantitative value, and (b) procedures treated as primary, meaning as secondary.

1.2.4. Episode 6

The teacher leads a discussion of a warm-up problem he has assigned the class:

James scored 87%, 79%, 76%, and 72% on five tests. He needs an 80% average to get a B in the class. What score on the last test does he need to achieve this goal?

1. T: First off, to get an 80% average, what does that mean? Mathematically what do we need to do to make it 80%? How do you figure an average?
2. S: Don't you add everything out and divide by. . .no wait (inaud) is that right? Yeah, you divide it by how many there are?
3. T: OK, so we want to figure out the average here. We're going to add 87 plus 79 plus, read them off to me please.

We make two observations regarding this episode. The first concerns *meaning of terms*. After asking, "what does that mean?", the teacher immediately associates the *meaning* of "average" with the *procedure* for computing the average (Line 1). The second observation concerns *usage of mathematical symbols.* Although the teacher uses the word "percent" in his

explanation of the solution, the "%" symbol is non-referential in that the problem data are treated as if they were simply integers, not percentages. In other classroom contexts, a percentage such as 87% would be converted to the decimal 0.87; why not here?

We see in this episode (a) conflation of meaning with a procedure and (b) a mathematical symbol that is ignored.

1.2.5. Episode 7

The teacher addresses the entire class, going over the homework problem of solving the equation $x/3 = 10/4$. We write [Problem] in the dialogue below when a speaker refers to this equation on the board.

1. T: How do I simplify, or how do I solve this [Problem]?
2. S1: Flip it.
3. T: Flip it? What's different from this, the difference between this and what we had for warmups [division of rational expressions]? In the warmups, were the warmups an equality? Were they equalities? Is this [Problem] an equality? It is, why? (...)
4. T: What makes an equality? What makes something an equality?
5. S1: Are you asking (inaud)
6. T: Yeah, I don't know.
7. S2: Why is that equal to that one [unclear references]?
8. T: Is this [Problem] an equality?
9. S2: Yeah.
10. T: Why?
11. S2: No, I said if it's an equality? (...)
12. T: What is an equality?
13. S3: An inequality is [pause]
14. T: An inequality is what? What is this [writes "EQUALITY" on the board]? (...)
15. T: Is that [points to a previously worked problem off-camera, which we see later does not involve an "=" sign] an equality, no. Is that [Problem] an equality, yes.
16. S*: Why?
17. S1: Nobody knows what it means!

Our first observation concerns *purpose for new concepts*. The first student seems to confuse the equal sign with the division symbol that appeared in a warmup problem, so she wants to invert and multiply. The teacher tries to distinguish that the current problem involves an equality, but the ensuing confusion shows that students do not understand this term. Indeed, since they already know the term "equation," there seems to be no reason for the additional term "equality." The teacher does not explain the relation between these terms or what is new about an "equality." Instead, he demonstrates the way to solve "equalities." The episode continues:

18. T: This is an equality, different from what we had on the warmup. This is to solve it, we can cross-multiply. [S], tell me what to write?
    [with students' help, he writes $3(10) = 4x$ and solves for x] (...)
19. T: If I want to get x all alone, I want to get x all alone, this is an equality, I want to get x all alone on one side of the equals sign. How am I going to get x all alone?
20. S*: (inaud)
21. S1: No, do the thing that you had before, the 3 times 10 equals $4x$.
22. S2: Yeah, that's what I said.
23. T: Multiply both sides by 3? Oh, I like her, we got to keep her around for awhile. What happens if I multiply both sides by 3?
24. S1: Then you just (inaud)
25. S2: The 3 goes away.
26. T: Three, this is the same thing as that. The threes cancel out. I get x all alone on this side. What's over here? I get 3 times 10, or, over 4, or 30 over 4. Does that make sense? And to do the long math, what's 30/4, which is equal to?
27. S1: 7.5.
28. T: Seven and one half or 7.5, however you want to write it. OK, that's one way of solving it. Actually, that is, that is the way of solving it. But what most of us see is that there's a shortcut here. How about, instead of doing all this, what we did was we multiplied 3 times 10 is equal to x times 4 and then what's the next thing you do?
29. S1: You do 10 times 3 which equals 30 and then equals $4x$ and then (inaud).
30. T: And what did we end up with here?
31. S1: 7.5.
32. T: It's the same thing as up here. We did the same steps but you just take a different approach to it. Does that make sense?

Our second observation concerns *meaning of terms*. Although the teacher correctly shows that his method of cross-multiplication is equivalent to the procedure of isolating *x* by multiplying both sides by 3, he uses the term "shortcut" to describe this cross-multiplication (Line 28). Because the problem can be solved more quickly and easily by simply multiplying by 3, we see no objective reason to call cross-multiplication a shortcut. There is no discussion of what a "shortcut" would mean, other than a method recommended by the teacher. Our third observation concerns *meaning of symbolic representations*. The string of symbols "10/4" is treated as a pair of numbers without attention to the overall quantity being represented. The teacher treats "10/4" as containing two inputs to the cross-multiplication procedure without calling attention to the fact that it represents a single quantity. In particular, neither the teacher nor any student suggests that "10/4" can be replaced by the quantity "2.5."

We see in this episode (a) introduction of a new term with no purpose, (b) incorrect usage of the term "shortcut," and (c) symbols treated as inputs to a procedure rather than representations of a quantity.

2. Summary and categorization of observations

Our observations fall roughly into four categories, based on what they primarily concern: purpose, distinctions, terminology, and symbols.

2.1. Observations concerning purpose

1. There is no intellectual purpose for what is being taught.
2. Terms are introduced with no purpose.
3. Claims are asserted by authority rather than established by meaning-based reasoning or tested against examples, thus removing one of the logical functions of meaning.
4. There is expectation that definitions and facts are arbitrary rather than based on purpose and meaning.

2.2. Observations concerning distinctions in mathematics

1. There is a lack of differentiation between meanings of different terms.
2. No distinction is made between an object, such as a "scalar," and a process, such as "multiplication."
3. Multiple definitions are accepted without discussing their differences or demonstrating their equivalence.

2.3. Observations concerning mathematical terminology

1. There is a lack of attendance to meaning of terms.
2. Terms are used inconsistently.
3. There is acceptance of incorrect definitions.
4. Operations are assumed to be meaningful in the absence of an algorithm for carrying them out [which might be appropriate at a higher level of mathematical sophistication].
5. There is conflation of meaning with a procedure.
6. The term "shortcut" is used arbitrarily.
7. Procedures are treated as primary, meaning of terms as secondary.

2.4. Observations concerning mathematical symbols

1. Strings of symbols are presented without attention to their meaning.
2. Mathematical symbols are used in ways that conflict with their standard meaning.
3. There is no attention to the meaning of symbolic representations.
4. There is emphasis on the form of expressions rather than their quantitative value.
5. Mathematical symbols are ignored.
6. Symbols are treated as inputs to a procedure rather than representations of a quantity.

In the above characterizations, we did not connect our observations to the particular teacher in any episode. However, we did observe some differences between the teachers, in addition to the happenstance that we present more data involving teacher B. We emphasize that a teacher's classroom presentation of a mathematical topic may not be reliable evidence of his own understanding of that topic, but may reflect pedagogical choices or constraints. In our study, teachers A and B both made statements that were incorrect from a literal mathematical perspective. However, teacher A appeared to understand most of the terms and symbols he used; they were referential for him. Despite this, he did not clearly communicate meanings to the students, as evidenced by their behavior and questions. He also used certain terms and expressions in ambiguous and confusing ways, which probably increased students' tendency to pay attention to keywords and triggers rather than overall meanings.

Teacher B made many statements that seemed to be non-referential or improperly referential for him. He did not always view expressions as representing quantity, and he misused certain important terms (such as "irrational"). It appears that his goals included explicit discussions of meaning, for instance to help his students understand the meaning of "rational expression" and "equality." However, he spent a significant amount of time trying to elicit particular terms from students or focusing on a term itself rather than the concept behind it. Both teachers fall back on authority at times, so that the teacher's assertions outweigh sense-making. A prominent example is the discussion of irrationality in teacher B's classroom: students accept the teacher's claim that "goes on forever" is equivalent to "can't be written as a fraction" without attempting to check examples or consider how the equivalence might be shown.

3. Potential impact on student learning

We conjecture that the types of behavior we observed are likely to result in students learning the following undesirable lessons. We emphasize that our study does not provide direct evidence for these conjectures.

1. Mathematics involves executing standard procedures much more than meaning and reasoning.

   When students are encouraged to treat symbols simply as inputs for a procedure, they are unlikely to attend to the meaning of those symbols. Students' conception of mathematics is defined by their experience with it. Thus, students who are exposed to standard procedures and rules, but rarely see useful explorations of meaning or justifications based on meaning, are likely to view mathematics as a set of rules and procedures with little meaning attached. For instance, teacher A's equation $b = 1 + \%$ might lead students to view exponential growth as a collection of exercises in which they just find "the percentage" and plug it into formulas, not needing to understand the idea of a growth factor. They are encouraged to attend to superficial aspects of a problem which could be altered simply by rewording it, such as which quantity is given as a percentage. In this conception, mathematics is identified with the actions the student is expected to take in a given list of problem situations.

2. Mathematical definitions and results are coincidental or arbitrary.

   When definitions are not motivated by anything but curricular requirements, they can seem arbitrary. For instance, teacher A introduces determinants as scalars associated with a matrix by a particular formula, without a meaningful grounding in the information they can provide about that matrix. Neither teacher made it clear to students that mathematical definitions are carefully chosen to describe objects of interest in a precise manner. It is thus unlikely that students will pay careful attention to mathematical definitions, beyond memorizing what they need for testing. The teachers' use of authority rather than meaning to justify results may make them seem coincidental rather than due to definitions and previously established facts. Teacher B's use of the phrase "it just so happens" while discussing irrational numbers exemplifies this. Moreover, such behavior may lead students to see authority as the only conceivable justification for procedures and facts. If these are arbitrary, then students cannot verify them by reasoning based on meaning. They are thus forced to trust the teacher rather than checking for themselves.

3. Non-referential symbolic reasoning.

   The lack of attention to the meanings of symbols used to solve problems may encourage the undesirable non-referential symbolic way of thinking, wherein students view symbols as having a life of their own and manipulate them based on arbitrary rules. Students might also learn that neither a problem nor the procedure used to solve it need to have clear meaning, as long as the necessary steps can be carried out. Episodes such as teacher A's treatment of exponential growth may lead students to ignore the literal meaning of problems and the teacher's statements, attending only to what procedure to use. When the meaning of symbols and operations is not considered, practice problems become procedure drills rather than opportunities to reason repeatedly, a critical step for students to internalize their knowledge.

4. The form of an expression is more important than its value.

   At times, both teachers concentrated more on the form of expressions than their values. Indeed, it may not always have been clear to students that expressions *had* values. Sometimes, the expressions or numbers in a problem were treated merely as inputs to some procedure. Teacher A's equation $b = 1 + \%$ specified only the form of the input: that it should be a percentage. When solving $x/3 = 10/4$, we did not observe teacher B treating the right hand side as a quantity, which could be replaced by the decimal value 2.5. Instead, he treated 10 and 4 as separate inputs to the cross-multiplication procedure. Treatment of expressions in certain forms as inputs to a standard procedure can lead students to view certain expressions as calling out for a particular manipulation, independent of the context in which that expression appears or the goal of the problem.

4. Remarks on possible causes for observed phenomena

In this section we offer several remarks, in the form of conjectures, as to what might account for the teaching practices observed in our study. As mentioned earlier, these two teachers had participated in an intensive professional development institute that emphasized meaning. However, we have documented cases of inattention to meaning by the teachers that we suggest could have a negative impact on student learning regardless of the teachers' actual intent or knowledge. A question that should be answered by further research is: what major factors account for their inattention to meaning, even after this professional development? Although we do not have enough data to answer this question, we suggest possible contributing

factors. These factors overlap with each other and are not intended to be exhaustive, but they provide a basis for further study. Of course, it may be that different factors are dominant for the two teachers.

We believe that institutional constraints were a factor in these teachers' treatment of meaning. That is, the curriculum and the assessment of both students and teachers are largely determined by factors external to the classroom. In conversations, both teachers refer to standardized testing, e.g. ". . .we want them to pass those tests. . .we want to make sure they're exposed to everything that could possibly be on the test and the only way to do that is to maintain a schedule" (teacher A); "their goal is to pass the high school exit exam, that's one of our primary goals out of this course" (teacher B). Although they say they want students to understand what they learn, both teachers feel pressure to cover the prescribed topics in a limited amount of time, which may leave little time for emphasis on meaning. The criterion of success is procedural: whether students can rapidly solve the problems likely to appear on standardized tests. Teacher A in particular believes that his teaching practice is already successful in this sense, and that both parents and school administrators would object to changes from this successful focus. Thus, it may be that the teachers appreciated emphasis on meaning at the professional development workshop, but they felt too constrained to pay more attention to meaning in their own classrooms.

A related factor that may have contributed is the teachers' goals for their classrooms. In part because of their constraints, the teachers may have had goals for the classroom that they did not feel would be advanced by emphasis on meaning. Our data do not include statements of their goals, but we can infer that many of these goals are procedural. For instance, in episode 3, teacher B brings up the curriculum standard, "solve rational equations" without mentioning any standards concerned with meaning; indeed, it is relatively rare for standards to address issues of meaning. It may be that the ensuing discussion was only intended to set the scene for solving rational equations, without intending to enable students to apply definitions of rational and irrational. Similarly, the teachers may want to give their students a better understanding of mathematical meanings, but they do not have the curricular materials to support them. Without supporting materials, the teachers may feel that it would be a disservice to their students to try to engage in discussions of meaning. Moreover, the teachers may appreciate the importance of meaning but feel that their students are incapable of appreciating it. Thus, their expectations of what students can do limit discussions of meaning.

Finally, the *teachers' knowledge base* may be limiting their actions in the classroom. Building on Shulman's (1986, 1987) work and consistent with other views (Ball & Cohen, 1999; Brousseau, 1997), Harel (1993) defines *teacher's knowledge base* in terms of three components: *mathematics content*, *epistemology*, and *pedagogy*. *Mathematics content* refers to the depth and breadth of a teacher's mathematics knowledge. This knowledge is the cornerstone of teaching; it affects both what the teachers teach and how they teach it. *Epistemology* refers to the teacher's understanding of fundamental psychological principles of learning in general and mathematical learning in particular, such as how students learn and the impact of their existing knowledge on the acquisition of new knowledge. *Pedagogy* refers to teachers' understanding of how to teach in accordance with these principles. This includes an understanding of how to assess students' knowledge, how to pose problems that stimulate students' intellectual curiosity, and how to help students solidify and retain knowledge they have acquired. While mathematical knowledge is indispensable for quality teaching, it is not sufficient. Teachers must also know how to address students as learners. Harel (2007) argues that a teacher's knowledge of epistemology and pedagogy rests on that teacher's knowledge of mathematics. In other words, although each of the three components of knowledge is indispensable for quality teaching, they are not symmetric: the development of teachers' knowledge of epistemology and of pedagogy depends on and is conditioned by their knowledge of mathematics. From this perspective, some of the teachers' actions can be explained by a lack of awareness of other options: the teacher does not know enough to provide an alternative treatment. We observed weak evidence that this is a contributing factor: some of the teachers' actions appear to be attempts to attend to meaning, but (from our perspective) these attempts were not effective in emphasizing meaning.

5. Pedagogical issues and conclusions

Teachers set the tone for how meaning is treated in several ways. Teachers explicitly introduce and review terminology while discussing concepts, but they also implicitly set the norms for how terminology is used by using it themselves. When the teacher does not demonstrate precise and referential definitions, it is unlikely that students will value referential usage of terminology. Teachers also help students execute procedures in many ways. They might demonstrate a solution procedure, describe what must be done in a class of situations, or help individual students with particular steps. However, the goal should be not only for the students to learn the procedure itself, but also for them to understand the meanings and reasoning involved.

The distinction between mathematical meaning, which is part of the concept image (Tall & Vinner, 1981) taken to be shared by the mathematical community at large, and concept definition is an important one. The determinant of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$ by definition, but this alone does not provide an idea of what a determinant means. The meaning of a determinant might be that it is a convenient device with which to test invertibility or uniqueness of solutions, find eigenvalues, etc. Both teachers seemed to equate definitions and meanings in many cases: they answered questions about the meaning of some concept by providing a definition or formula for the concept. As evidenced by the confusion students exhibited in these situations, the definition alone may not be helpful.

Students can benefit from seeing the usefulness of terminology and procedures. To help them understand the purpose of particular ideas, teachers can motivate the ideas before introducing them. For instance, they can present several problems that require a procedure *before* explaining the procedure, making students more likely to appreciate it. Correspondingly, teachers can wait to define an entity until students have already encountered that entity through solving problems. Thus, students will already have a mental image of the entity when they encounter its technical definition. Providing context can also help to motivate new ideas. For instance, in his discussion of rational and irrational numbers, teacher B could have mentioned the historical significance of this distinction, the computational distinction between how arithmetic operations are performed on rational numbers versus irrational ones, the significance of incommensurability in geometry, or the reasons for enlarging the rational number system to the reals. We have no evidence as to whether such topics were mentioned when the rational/irrational distinction first appeared in this classroom, but we note that in our data teacher B does not appear to be aware of these significances. This is hardly surprising, as studies (e.g. Fischbein, Jehiam, & Cohen, 1995) have shown that many pre-service teachers have trouble correctly defining irrational numbers and distinguishing them from rational ones. When teachers understand and convey to students the purpose of the ideas they introduce, they increase the likelihood that students pay attention to meaning. The professional development leader had modeled and pointed out the aforementioned principles (that is, making procedures and entities explicit *after* students have seen the need for them, and providing context for this need) during the summer institute, but the participants had little opportunity to discuss how they might apply these principles to their own classrooms. Two questions for future research can be: why did the modeling of these pedagogical principles have minimal effect on the participants' teaching, and what professional development activities would enable teachers to apply these principles in their own classrooms? The former question will be addressed in a paper in preparation (Rabin, Fuller, & Harel, in preparation).

We have presented actions by two teachers that are likely to have negative effects on students' mathematical development and beliefs. Based on personal knowledge of the teachers, we believe that their actions are well-intentioned. Nevertheless, such actions seem to pervade their classrooms, inhibiting considerations of meaning. We conjectured that students in these classrooms are likely to learn several undesirable lessons as a result of this treatment of meaning. That is, they will believe that mathematics involves executing standard procedures more than meaning and reasoning and that mathematical definitions and results are coincidental or arbitrary; also, they will engage in non-referential symbolic reasoning and treat the form of an expression as more important than its value. This research contributes to the understanding of mathematical meaning in the classroom by critically examining specific actions taken by two teachers that seem to de-emphasize meaning. This level of specificity also allows us to make conjectures about potential effects such teaching might have on student learning. Further investigation is required in several directions. First, to collect evidence linking teachers' treatment of meaning with their students' learning and beliefs. Second, to identify other common teaching actions that de-emphasize meaning and quantify the prevalence of these. Most importantly in our view, there is a need for further research to examine the role of different factors that contribute to teachers' de-emphasis of meaning, as well as how these factors can be overcome.

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