What is Mathematics? A Pedagogical Answer to a Philosophical Question[1]

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0 Introduction

Why do we teach the long division algorithm, the quadratic formula, techniques of integration, and so on when one can perform arithmetic operations, solve many complicated equations, and integrate complex functions quickly and accurately using electronic technologies? Typical answers teachers give to these questions include “these materials appear on standardized tests,” “one should be able to solve problems independently in case a suitable calculator is not present,” “such topics are needed to solve real-world problems and to learn more advanced topics.” From a social point of view, there is nothing inadequate about these answers. Teachers must prepare students for tests mandated by their educational system, they must educate students to carry out elementary calculations independent of computer technologies, especially calculations one might encounter in daily life, and they must prepare students to take advanced courses where certain computational skills might be assumed by the instructors of these courses. These answers, however, are external to mathematics as a discipline, in that they offer justifications that are neither cognitive (about thought processes) nor epistemological (regarding the philosophical theory of knowledge) but mainly social. For example, nothing in these answers suggests the role of computational skills in one’s conceptual development of mathematics; nor do these answers reflect the role of computations in the development of mathematics. A related question is: why teach proofs? The most typical answer given by teachers to this question was, “so that students can be certain that the theorems we present to them are true.” While this is an adequate answer—both cognitively and (by inference) epistemologically—it is incomplete. The teachers who were asked this question had little to say when skeptically confronted about their answers by being asked: Do you or your students doubt the truth of theorems that appear in textbooks? Is certainty the only goal of proofs? The theorems in Euclidean geometry, for example, have been proven and re-proven for millennia. We are certain of their truth, so why do we continue to prove them again and again?

Overall, these teachers’ answers do not address the question of what intellectual tools one should acquire when learning a particular mathematical topic. Such tools, I argue, define the nature of mathematical practice. Judging from current textbooks and teaching practices, teachers at all grade levels, including college instructors, tend to view mathematics in terms of subject matter, such as definitions, theorems, proofs, problems and their solutions, and so on, not in terms of the conceptual tools that are necessary to construct such mathematical objects. While knowledge of and focus on subject matter is indispensable for quality teaching, I argue it is not sufficient. Teachers should also concentrate on conceptual tools such as problem-solving approaches, which, I argue, constitute an important category of knowledge different from the subject matter category, as I will explain shortly.

What exactly are these two categories of knowledge? And what is the basis for the argument that both categories are needed? Initially, pedagogical considerations, not philosophical ones, engendered the two questions. However, my inquiries into these questions, especially in relation to students’ conceptions of proof, have led me into historical and philosophical analyses not initially intended. These analyses have shed considerable light on my understanding of cognitive processes of learning. For example, the philosophical debate during the Renaissance as to whether mathematics conforms to the Aristotelian definition of science helped me understand certain difficulties able students have with a particular kind of proof (see Harel, 1999). The juxtaposition of such epistemological and cognitive analyses compelled me to look deeply into the nature of mathematical knowledge and its implications for curriculum development and instruction. Thus, my answers to the above two questions—the main concern of this paper—draw upon epistemological, cognitive, and pedagogical considerations. These answers are situated within a broader theoretical framework called *DNR-based instruction in mathematics* (*DNR* for short). The initials, *D*, *N*, and *R*, stand for three leading principles in the framework—*duality*, *necessity*, and *repeated-reasoning*—to be presented in Section 4. *DNR* stipulates conditions for achieving critical goals such as provoking students' intellectual need to learn mathematics, helping them acquire mathematical ideas and practices, and assuring that they internalize, organize, and retain the mathematics they learn.

The paper consists of five sections: Section 1 discusses a triad of key DNR constructs: "mental act," "way of understanding," and "way of thinking." On the basis of these constructs, a definition according to which mathematics consists of two categories of knowledge is offered in Section 2. Epistemological considerations and pedagogical consequences of this definition are discussed in Section 2 and Section 3. Section 3 focuses mainly on long term curricular and research goals, with particular attention to lessons from history. The three foundational principles of *DNR* along with examples of other *DNR constructs* are briefly presented in Section 4. However, *DNR* concepts and themes are on every page of the paper. The paper concludes with a summary in Section 5.

1 Mental Act, Way of Understanding, and Way of Thinking

1.1 Mental Act

Humans' reasoning involves numerous *mental acts* such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving.[2] These are examples of mental acts as opposed to *physical acts*. "Lifting" and "pulling" an object are examples of the latter. However, many terms may refer to either physical acts or mental acts. For example, *searching* may refer to the act of physically looking for a missing material object—such as when one searches for missing keys—or to the act of mentally looking for an abstract object—such as when one searches for the value of an equation's unknown. The distinction between "mental act" and "physical act" is not without difficulty, as one can learn from the work of Lakeoff and Johnson (2003) and Johnson (1987), who argue that meaning, imagination, and reason have a bodily basis. This debate, however, is beyond the scope and goals of this paper.

Humans perform mental acts, and they perform them in every domain of life, not just in science and mathematics. Although all the aforementioned examples of mental acts are important in the learning and creation of mathematics, they are not unique to mathematics—people interpret, conjecture, justify, abstract, solve problems, etc. in every area of their everyday and professional life. Professionals from different disciplines are

[2] The notion of "mental act" is taken as undefined in this paper.

likely to differ in the extent they carry out certain mental acts; for example, a painter is likely to abstract more often than a carpenter, a chemist to model more often than a pure mathematician, and the latter to conjecture and justify more often than a pianist. But a more interesting and critical difference among these professionals is in the nature, the characteristics, of the mental acts they perform. A biologist, chemist, physicist, and mathematician all carry out problem-solving acts in every step in their professional activities, and they may even produce similar solutions to problems their fields have in common. The four, however, are likely to differ in the nature of the problem-solving act and other related mental acts they perform while solving problems. Mental acts are basic elements of human cognition. To describe, analyze, and communicate about humans' intellectual activities, one must attend to their mental acts.

1.2 Way of Understanding Versus Way of Thinking

Mental acts can be studied by observing peoples' statements and actions. A person's statements and actions may signify *cognitive* products of a mental act carried out by the person. Such a product is the person's *way of understanding* associated with that mental act. Repeated observations of one's ways of understanding associated with a given mental act may reveal certain *cognitive* characteristics of the act. Such a characteristic is referred to as a *way of thinking* associated with that act. In the rest of this section, these definitions will be explained and illustrated.

Again, *a way of understanding is a particular cognitive product of a mental act carried out by an individual*. For example, upon seeing the symbol $3/4$ one may carry out the interpreting act to produce a meaning for this symbol. The interpretation the person produces is her or his way of understanding the symbol. Such a way of understanding may vary with context, and when judged by an observer, it can be deemed right or wrong. For example, in one context a person may produce the meaning "3 objects out of 4 objects," and in another the meaning "the sum $1/4+1/4+1/4$." One person may produce a mathematically sophisticated way of understanding, such as "the equivalence class $\left\{3n/4n \,\middle|\, n \text{ is an integer different from zero}\right\}$," and another a naive way of understanding, such as "two numbers with a bar between them." Likewise, a particular solution to a problem and a particular proof of an assertion are products of the problem-solving act and proving act, respectively; hence, each is a way of understanding.

A way of thinking, on the other hand, is *a cognitive characteristic of a mental act.* Such a characteristic is always inferred from observations of ways of understanding—cognitive products of a mental act. For example, a teacher following her student's mathematical behavior may infer that the student's interpretation of mathematical symbols is characteristically inflexible, devoid of quantitative referents, or, alternatively, flexible and connected to other concepts. Likewise, the teacher may infer that a student's justifications of mathematical assertions are typically based on empirical evidence, or, alternatively, based on rules of deduction.

To further illustrate the distinction between ways of understanding and ways of thinking, consider the three mental acts of "interpreting," "problem solving" and "proving."

1.2.1 Interpreting

The actual interpretation one gives to a term or a string of symbols is a way of understanding because it is a particular cognitive product of her or his act of interpreting. For example, one may interpret the string of symbols $y=\sqrt{6x-5}$ in different ways: as an equation (a condition on the variables x and y), as a number-valued function (for each number x, there corresponds the number $\sqrt{6x-5}$), or as a proposition-valued function (for each ordered pair (x, y) there corresponds the value "true" or the value "false."). These ways of understanding manifest certain characteristics of the interpreting act—for example, that "symbols in mathematics represent quantities and quantitative relationships." A person who holds more than one such way of understanding is likely to posses, in addition, the way of thinking that "mathematical symbols can have multiple interpretations." And a person who is able to vary the interpretation of symbols according to the problem at hand is likely to possess the way of thinking that "it is advantageous to attribute different interpretations to a mathematical symbol in the process of solving problems." These are examples of mature ways of ways of understanding and ways of thinking, which are absent for many high school and college students. For example, when a class of calculus students was asked what $y=\sqrt{6x-5}$ meant to them, many were unable to say more than what one of their classmates said: "It is a thing where what you do on the left you do on the right." Despite the fact that current school curricula purport to introduce algebra as generalized arithmetic, for many students the act of interpreting algebraic symbols can be characterized as being free of quantitative meaning.

It is not uncommon that students manipulate symbols without meaningful basis in the context (as in $(\log a+\log b)/\log c=(a+b)/c$). Matz (1980) connects this (erroneous) way of understanding and a wide range of algebra errors to an overgeneralization of the distributive property. Students factor out the symbol $\log$ from the numerator and cancel it, without attending to the quantitative meaning of their action. The behavior of operating on symbols as if they possess a life of their own, not as representations of entities in a coherent reality, is referred to as the *non-referential symbolic* way of thinking. With this way of thinking, one does not attempt to attend to meaning. For example, one does not ask questions such as "What is the definition of $\log a$?" "Does $\log \cdot a$ (multiplication) have a quantitative meaning?" "Is $\log a+\log b=\log(a+b)$?" and so on, for symbols are not conceived as representations of a coherent mathematical reality. Of course, one may produce correct results and still operate with the non-referential symbolic way of thinking. For example, we have observed students correctly solve systems of equations without attaching meaning to the operations they apply or to the solution they obtain.

The above characterization of the non-referential symbolic way of thinking may have evoked with the reader a different image from the one portrayed here since relative to the reader's practice of mathematics it is not uncommon that symbols are treated as if they possess a life of their own, and, accordingly, are manipulated without (necessarily) examining their meaning. I will return to discuss this point in Section 3.2.

1.2.2 Problem Solving

As to the mental act of problem solving, the actual solution—correct or erroneous—one provides to a problem is a way of understanding because it is a particular cognitive product of the person's problem-solving act. A problem-solving approach, on the other hand, is a way of thinking. For example, problem-solving approaches such as "look for a simpler problem," "consider alternative possibilities while attempting to solve a problem," and "just look for key words in the problem statement" characterize, at least partially, the problem-solving act; hence, they are instances of ways of thinking.

The problem-solving act is not of the same status as the other mental acts listed above, in that any of these acts is, in essence, a problem-solving act. The acts of interpreting, generalizing, and proving, for example, are essentially acts of problem solving. Despite this, the distinction among the different mental acts is cognitively and pedagogically important, for it enables us to better understand the nature of mathematical practice by individuals and communities throughout history, and, accordingly, set explicit instructional objectives for instruction. This viewpoint will be demonstrated in Sections 2 and 3.

1.2.3 Proving

While problem-solving approaches are instances of ways of thinking associated with the problem-solving act, *proof schemes* are ways of thinking associated with the *proving* act. *Proving* is defined in Harel and Sowder (1998) as the act employed by a person to remove or instill doubts about the truth of an assertion. Any assertion can be self-conceived either as a *conjecture* or as a *fact*: A *conjecture* is an assertion made by a person who has doubts about its truth. A person ceases to consider an assertion to be a conjecture and views it to be a *fact* once the person becomes certain of its truth. In Harel and Sowder (1998), a distinction was made between two variations of the proving act: *ascertaining* and *persuading*. *Ascertaining* is the act one employs to remove one's own doubts about the truth of an assertion (or its negation), whereas *persuading* is the act one employs to remove others' doubts about the truth of an assertion (or its negation). A *proof scheme* characterizes one's collective acts of ascertaining and persuading; hence, it is a way of thinking.

A common proof scheme among students is the *authoritative proof scheme*, a scheme by which proving depends mainly on the authority of the teacher or textbook. Another common proof scheme among students is the *empirical proof scheme*—a scheme marked by its reliance on evidence from examples or visual perceptions. Against these proof schemes stands the *deductive proof scheme,* a scheme by which one proves an assertion with a finite sequence of steps, where each step consists of a conclusion which follows from premises (and previous conclusions) through the application of rules of inference.[3] Note that while a proof scheme is a way of thinking, a *proof*—a particular statement one offers to ascertain for oneself or convince others—is, by definition, a way of understanding.

Mathematical reasoning centers on the deductive proof scheme. In contrast, the authoritative proof scheme and the empirical proof scheme are examples of undesirable

[3] For an extensive taxonomy of proof schemes drawn from students' mathematical behaviors and the historical development of proof, see Harel & Sowder (1998).

ways of thinking. While undesirable, a dash of the authoritarian proof scheme is not completely harmful and is unavoidable; people may use this scheme to some extent when they are sampling an area outside their specialties. In two of its worst forms, however, either the student is helpless without an authority at hand, or the student regards a justification of a result as valueless and unnecessary. As with the authoritarian proof scheme, the empirical proof scheme does have value. Examples and nonexamples can help to generate ideas or to give insights. The problem arises in contexts in which a deductive proof is expected, and yet all that is necessary or desirable in the eyes of the student is a verification by one or more examples.

1.2.4 Terminology

Two remarks on terminology are in order: The first remark concerns the adjective "cognitive" in the definitions of "way of understanding" and "way of thinking:" a way of understanding is a *cognitive* product of a mental act, and a way of thinking is a *cognitive* characteristic of a mental act. This is to indicate that the focus here is on cognition rather than affect or physiology. For example, the product of feeling confusion or frustration as one attempts to interpret a statement, prove an assertion, or solve a problem, is not dealt with in the conceptual framework offered here. Nor does this framework deal with physiological characteristics of mental acts—those that include, for example, certain neurological activities in the brain. Thus, the adjective "cognitive" in the above definitions intends to single out one type of products and characteristics—that which signifies *cognition.* The focus on cognition rather than affect and physiology is also evident in the examples discussed to illustrate the definitions.

The second remark concerns ease of terminology. It may not be easy to get accustomed to the technical distinction between the terms "way of understanding" and "way of thinking" as is made here. This is partly because in communication among educators and in the literature on learning and teaching the two terms are often used interchangeably (and without exact definitions). Also, the phrase "way of" seems to connote a sort of a process and, hence, a dynamic image, whereas the definition of "way of understanding" as a *product* of a mental act may connote an outcome, a static image. My intention in using the phrase "way of" is to insinuate "one of several possible ways," which suggests that a mental act in mathematics can, and should, have multiple products and characteristics—an implied view in the *DNR* perspective, as we will see. The verbs "to understand" and "to think" are used in this paper in accordance with the definitions of the corresponding terms: "to understand" means to "have a way of understanding," and "to think" means to "apply a way of thinking." In DNR, and throughout this paper, "ways of understanding" and "ways of thinking" are distinguished from their values. For example, one's way of understanding can be judged as correct or wrong, useful or impractical in a given context, etc.

2 A Definition of Mathematics: Epistemological Considerations and Pedagogical Implications

The notions of "ways of understanding" and "ways of thinking" as defined here are key constructs in the definition of mathematics I will now state. Mathematicians, the practitioners of the discipline of mathematics, practice mathematics by carrying out mental acts with particular characteristics (ways of thinking) to produce particular constructs (ways of understanding). Accordingly, mathematics consists of these two categories of knowledge. Specifically:

Definition: Mathematics consists of two complementary subsets:

- **The first subset is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the institutionalized[4] ways of understanding in mathematics throughout history. It is denoted by WoU.**
- **The second subset consists of all the ways of thinking, which are characteristics of the mental acts whose products comprise the first set. It is denoted by WoT.**

By this definition, mathematics is like a living organism. It grows continually as mathematicians carry out mental acts and their mathematical communities assimilate the ways of understanding and ways of thinking associated with the mathematicians' mental acts. The assimilation is attained when new ways of understandings are integrated into an existing mathematical edifice and ways of thinking are adopted in subsequent mathematical practices. As one can learn from the history of mathematics, the assimilation process is gradual and often not without conceptual struggle. Some ways of understanding and ways of thinking are regarded as inaccurate or faulty—sometimes long after they have been institutionalized. They, two, are part of mathematics according to this definition, as I explain later in this paper. In the rest of this section, I shall discuss several epistemological issues concerning this definition and examine their pedagogical consequences.

2.1 Listability

Mathematics as a union of WoU and WoT is not listable—capable of being completely listed. WoU contains more than the collection of all the statements appearing in mathematical publications, and the members of WoT are largely unidentified. I explain:

Consider a statement—say, a new theorem—that has appeared in a mathematical publication, such as a book or research paper. Its publication indicates recognition by a community that a new way of understanding has been accepted. Individual mathematicians might believe and act as if the published theorem represents a way of understanding *shared* by the community at large, whereas, in fact, each individual mathematician possesses an idiosyncratic way of understanding the theorem. Of all the latter "private" ways of understanding, consider only the subset of those that are consistent with the former "public" ways of understanding. These, too, are considered institutionalized, since it is assumed by the mathematics community that any way of understanding that is consistent with a "public" way of understanding is acceptable. Thus, WoU contains all the statements that have appeared in mathematical publications—which the mathematics community views as representations of shared ways of

[4] *Institutionalized* ways of understanding are those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems.

understanding—together with those possessed by individual mathematicians that are consistent with them. While the former are listable, the latter are not. The reason they are not listable is this: Let S be a statement that has been published, and let S' be a particular person's way of understanding S. Once this person has expressed S' to the community, S' moves to the domain of those ways of understanding assumed-to-be shared by the community. But the members of the community, including this person, possess idiosyncratic ways of understanding S'.

A pedagogical implication of this analysis is that a way of understanding, such as a definition, theorem, proof, or solution to a problem, cannot and should not be treated by teachers as an absolute universal entity shared by all students. Any statement a teacher (or a classmate) utters or puts on the board will be translated by each individual student into a way of understanding that depends on her or his experience and background. The goal of the teacher is then that these necessarily different individual mental constructs are compatible with each other. A classroom environment that promotes discussion and debate among students is both necessary for and instrumental in achieving this goal.

As to the WoT subset of mathematics, its members are not formally recognized by the mathematics community. They are neither explicitly targeted as instructional objectives by mathematicians nor investigated and reported in formal publications. Occasionally, however, they are informal parts of communications between collaborators. Polya's (1957) book "How to Solve It" is a rare attempt by a professional mathematician to explicate desirable problem-solving approaches, which, as was explained earlier, are ways of thinking. [For a discussion on Polya's pedagogical and epistemological assumptions for his on mathematical heuristics, see Schoenfeld, 1992]. It is much more difficult to reflect on and express in precise words ways of thinking than ways of understanding. In DNR-based instruction, considerations of ways of thinking are central; they are an essential part of curriculum development and instruction, as we will see.

2.2 Boundaries

A consequence of my definition of mathematics is that mathematics must include ways of understanding and ways of thinking that from the vantage point of contemporary mathematicians are imperfect or erroneous; Euclid's *Elements* is an example. This leads to the following question: Should ways of understanding and ways of thinking used or produced by individuals (students, for example) while they are engaged in a mathematical activity be considered mathematical even if they are narrow or faulty? My answer to this question is affirmative *in so far as* the individual has utilized—with or without the help of an expert—such ways of understanding and ways of thinking for the construction of institutionalized knowledge—knowledge accepted by the mathematics community at large.

This position is consistent with the definitions of "way of understanding" and "way of thinking." As can be seen from the examples discussed in the previous section, these terms do not imply correct knowledge. The terms only indicate the knowledge currently held by a person, which may be correct or erroneous, useful or impractical. Having said this, it must be emphasized that *the ultimate goal of instruction must be unambiguous: to help students develop ways of understanding and ways of thinking that are compatible with those that are currently accepted by the mathematics community at large*. From a pedagogical point of view this goal is meaningless without realizing that the process of learning necessarily involves the construction of imperfect and even erroneous ways of understanding and deficient, or even faulty, ways of thinking. Teachers must be aware of this phenomenon when working toward an instructional goal, and their teaching actions must be consonant with this awareness. In particular, they must attempt to identify students' current ways of understanding and ways of thinking, regardless of their quality, and help students gradually refine and modify them toward those that have been institutionalized—those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems.

The repeated use of the term "institutionalized" here raises the question: what about creativity—the discovery of *new*, not necessarily institutionalized, ways of understanding and ways of thinking? Are such discoveries mathematical? By my definition of mathematics they are not. This position is based on the premise that mathematics is a human endeavor, not a predetermined reality. As such, it is the community of the creators of mathematics who makes decisions about the inclusion of new discoveries in the existing edifice of mathematics. Such decisions may never be made by the community at large, and the new discoveries may be forever lost as a result. The work of Ramanujan would have likely been lost hadn't G. H. Hardy recognized the precious mathematical discoveries in the letter Ramanujan sent to him around 1913. Other decisions may be delayed; the work of Grassman (19^{th} century) and the work of Cantor (20^{th} century) are examples. Grassman's work was ignored for many years but became later the basis for vector and tensor analysis and associative algebras. Cantor's set theory, too, was ignored or boycotted for some time, but was later recognized as one of the most important discoveries of the twentieth century mathematics. At the time of their discoveries, prior to their institutionalization, these works did not belong to mathematics, according to the definition of mathematics I propose in this paper.

2.3 Relation to Ontology

There is a danger of confounding the above definition of mathematics with a particular philosophical stance with which I vehemently disagree. Mathematics, according to this definition, consists of ways of understanding and ways of thinking that have evolved throughout history. Inevitably, some of these constructs are narrow and even faulty if judged from a contemporary perspective. *This does not entail that particular mathematical statements could be true for some people and false for others*—a view that is implied by an extreme form of post-modernism, which asserts that mathematical truth depends on the culture or bias of the mathematician (Buss, 2005). Such disputed statements cannot be part of mathematics according to my definition, for they have never been institutionalized by any mathematics community in the history of mathematics. That is, no mathematics community, as far as I know, has ever accepted that a statement A *and* its negation, ~A, can both be true within the same system of rules of inference. A statement such as "Every function on the real numbers is continuous" is true for intuitionists but false for the rest of us because the two communities are considering the statement within different systems of rules of inference. Also the term "function" has different meanings for the two communities. Thus, a statement must not be considered in isolation but within a context that constitutes its meaning.

What is disputed among philosophers, and to a lesser extent among mathematicians, is the answer to an ontological question: What is the nature of the *being* *and existence* of mathematics? For example, is mathematical practice an act of discovery of eternal objects and ideas that are independent of human existence, an intuition-free game in which symbols are manipulated according to a fixed set of rules, or a product of constructions from primitive intuitive objects, most notably the integers? The three positions expressed in this question correspond, respectively, to the three major schools of thought, Platonism, Formalism, and Constructivism. Since Constructivism insists that mathematical objects must be computable in a finite number of steps, it does not admit many results accepted by the other schools as true. The basis for this rejection is not "cultural difference" or "personal bias," as the extreme forms of post-modernism imply; rather, the basis for the rejection is philosophical: nothing can be asserted unless there is a proof—a constructivist proof—for it.

It is an open, empirical question whether mathematicians' ontological stances on the nature of mathematical practice have any bearing on their views of how mathematics is learned and, consequently, how it should be taught. I conjecture that teachers' approach to the learning and teaching of mathematics is not determined by their ontological stance on the being and existence of mathematics. Dieudonné, a prominent member of the Bourbaki group, calls in the following statement for an uncompromising Formalist view:

> Hence the absolute necessity from now on for every mathematician concerned with intellectual probity to present his reasoning in *axiomatic* form, i.e., in a form where propositions are limited by *virtue of rules of logic only*, all intuitive "evidence" which may suggest expressions to the mind being deliberately disregarded. (Dieudonné, 1971, p. 253).

Yet, he cautions his reader:

> *We are saying that this is a form imposed on the presentation of the results; but this does not lessen in any way the role of intuition in their discovery*. Among the majority of researchers the role of intuition is considerable, and no matter how confused it may be, an intuition about the mathematical phenomena being studied often puts them on the track leading to their goal. (Emphases added; Dieudonné, 1971, p. 253).

One can reasonably infer from these statements that Dieudonné's approach to teaching is to emphasize intuition despite his adherence to the Formalist school.

What does determine then one's approach to learning and teaching of mathematics?

2.4 Quality of Teacher's Knowledge Base

Quality of instruction is determined largely by what teachers know. Building on Shulman's (1986, 1987) work and consistent with current views (Brousseau, 1997; Cohen & Ball, 1999, 2000), *teacher's knowledge base* was defined in Harel (1993) in terms of three components: *knowledge of mathematics, knowledge of student learning*, and *knowledge of pedagogy*. Here I present a refined definition of these components that is aligned with the definition of mathematics I have just discussed:

- *Knowledge of mathematics* refers to a teacher's *ways of understanding* and *ways of thinking*. It is the quality of this knowledge that is the cornerstone of teaching for it affects both what the teachers teach and how they teach it.
- *Knowledge of student learning* refers to the teacher's understanding of fundamental psychological principles of learning, such as how students learn and the impact of their previous and existing knowledge on the acquisition of new knowledge.
- *Knowledge of pedagogy* refers to teachers' understanding of how to teach in accordance with these principles. This includes an understanding of how to assess students' knowledge, how to utilize assessment to pose problems that stimulate students' intellectual curiosity, and how to help students solidify and retain knowledge they have acquired.

Thus, while mathematical knowledge is indispensable for quality teaching, it is not sufficient. Teachers must also know how to address students as learners. In DNR, however, teacher's knowledge of student learning and pedagogy rests on the teacher's knowledge of mathematics. That is to say, although each of the three components of knowledge is indispensable for quality teaching, they are not symmetric: the development of teachers' knowledge of student learning and of pedagogy depends on and is conditioned by their knowledge of mathematics. A brief example to illustrate this claim follows: The example is from an on-site professional development study, currently underway, aimed at investigating the evolution of teachers' knowledge base. One of the findings of this study is that teachers' appreciation for students' struggle with a particular concept is a function of the quality of the teachers' way of understanding that concept. For example, Lisa, one of the teacher participants in this study, developed and enthusiastically implemented an instructional activity where her tenth-grade class gradually discerned the formula for the sum of the interior angles in a convex polygon along with a mathematically acceptable justification for it. In one of the interviews with Lisa, she pointed out, with great satisfaction and a sense of accomplishment, that the class understood well the proof of the formula and some students even developed it on their own. On the other hand, Lisa, who had insufficient understanding of graphical representation of solutions to systems of linear inequalities, struggled to see the benefit of a multi-stage instructional activity that was designed to involve students in developing a solid understanding of how to solve and graph the solution of such systems. She inclined, instead, to provide the students with a prescribed procedure of how to solve these systems. Thus, Lisa's lack of a deep understanding of systems of linear inequality prevented her from pursuing good teaching of this topic. Overall, Lisa's knowledge of pedagogy and of student learning seems to evolve hand in hand with the growth of and self reflection on her knowledge of mathematics, not out of institutional demand to improve her students' mathematical performance.

3 Long-Term Curricular and Research Goals

In the opening of this paper, it was argued that the instructional objectives teachers set for their classes correspond merely to subject matter in terms of products of mental acts—ways of understanding, such as particular definitions, procedures, techniques, theorems, and proofs. Neither the actions of the teachers nor the justifications they provide for their objectives indicate attention to the characteristics of mental acts—to the ways of thinking that students are to develop by learning particular subject matter. Objectives formulated in terms of ways of understanding are essential, as it is asserted in one of the *DNR* principles, to be presented in Section 4, but without targeting ways of thinking, students are unlikely to become independent thinkers when doing mathematics. This brings up the question, when should we start targeting ways of thinking with students?

3.1 Elementary Mathematics

The formation of ways of thinking is extremely difficult and those that have been established are hard to alter. This is one of the main findings of our research (see for example, Harel & Sowder, 1998). Hence, the development of desirable ways of thinking should not wait until students take advanced mathematics courses; rather, students must begin to construct them in elementary mathematics, which is rich in opportunities to help students begin acquiring crucial ways of thinking. Consider, for example, the concept of fraction. In current mathematics teaching, even when students learn mathematics symbolism in context, the context is usually limited. For example, the most common way of understanding the concept of fraction among elementary school students is what is known in the literature as the *part-whole* interpretation: m/n (where m and n are positive integers) means "m out of n objects." Many students never move beyond this limited way of understanding fraction and encounter, as a result, difficulties in developing meaningful knowledge of fraction arithmetic (Lamon, 2001) and beyond (Pustejovsky, 1999). Seldom do students get accustomed to other alternative ways of understanding such as m/n means "the sum $1/n+\cdots+1/n$, m times" or "the quantity that results from m units being divided into n equal parts" or "the measure of a segment m-inches long in terms of a ruler whose unit is n inches" or "the solution to the equation $nx=m$" or "the ratio $m:n$; namely, m objects for each n objects." This range of ways of understanding a fraction makes the area of fractions a powerful elementary mathematics topic—one that can offer young students a concrete context to construct desirable—indeed, crucial—ways of thinking, such as: mathematical concepts *can* be understood in different ways, mathematical concepts *should* be understood in different ways, and *it is* advantageous to change ways of understanding of a mathematical concept in the process of solving problems. These ways of thinking will be needed in the development of future ways of understanding. Indeed, without the above cluster of ways of thinking students are bound to encounter difficulties in other parts of mathematics. In calculus, for example, depending upon the problem at hand, one would need to interpret the phrase "derivative of a function at a," or the symbol $f'(a)$, as "the slope of a line tangent to the graph of a function at a" or "the $\lim_{h\to 0}\big(f(a+h)-f(a)\big)/h$" or "the instantaneous rate of change at a" or "the slope of the best linear approximation to a function near a." Likewise, in solving linear algebra problems it is often necessary—or at least advantageous—to convert one way of understanding into another way of understanding by using the equivalence among problems on systems of linear equations, matrices, and linear transformations.

The history of mathematics can provide a guide to ways of thinking worth pursuing—in the classroom and in mathematics education research. In the rest of this section, I will illustrate this claim with examples from the history of algebra and proof.

3.2 Algebra

According to Klein (1968) the revival and assimilation of Greek mathematics during the 16th century resulted in a conceptual transformation that culminated in Vieta's development of symbolic algebra. Until then, mathematics had evolved for at least three millennia with hardly any symbols. The following is an example to illustrate the colossal role symbolic algebra played in defining modern mathematics. The work of Vieta that led to the creation of algebra and that of Descartes that led to the creation of analytic geometry constituted the conceptual foundation for the critical shift from "results of operations" as the object of study to the operations themselves as the object of study. While the Greeks restricted their attention to attributes of spatial configurations and paid no attention to the operations underlying them, 19th century mathematics investigated the operations, their algebraic representations, and their structures. In particular, Euclidean constructions using only a compass and straightedge were translated into statements about the constructability of real numbers, which, in turn, led to observations about the structure of constructible numbers. A deeper investigation into the theory of fields led to the understanding of why certain constructions are possible whereas others are not. The Greeks had no means to build such an understanding, since they did not attend to the nature of the operations underlying Euclidean construction. Thus, by means of analytic geometry, mathematicians realized that all Euclidean geometry problems can be solved by a single approach, that of reducing the problems into equations and applying algebraic techniques to solve them. Euclidean straightedge-and-compass constructions were understood to be equivalent to equations, and hence the solvability of a Euclidean problem became equivalent to the solvability of the corresponding equation(s) in the constructible field.

The monumental role that symbolic algebra played in defining modern mathematics might be obvious to many, but it is worth pointing out in debates on the future direction of school mathematics, particularly when attempts are made to deemphasize symbolic manipulation skills. Often the rationale behind these attempts is the availability of electronic technologies equipped with computer algebra programs that can carry out complex computations of all kinds and in all areas of mathematics. While these technologies can have a positive role in the teaching of mathematics (see, Kaput & Hegedus, 2003) they can, if not used wisely, deprive the students of the experience necessary for developing critical mathematical ways of thinking. In particular, they can deprive students of the opportunity to develop one of the most crucial mathematical ways of thinking, that of *algebraic invariance.*

Algebraic invariance refers to the way of thinking by which one recognizes that algebraic expressions are manipulated not haphazardly but with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant. If this way of thinking were set as an instructional objective, elementary algebra—especially symbol manipulation skills—would be taught differently and more meaningfully. The method of completing the square, for example, would have an added value, not just as a method for solving quadratic equations but as an activity to advance students toward acquiring the algebraic invariance way of thinking. Assuming the students have already learned how to solve equations of the form $(x+T)^2 = L$, the teacher's action would be geared toward helping them manipulate the quadratic equation $ax^2 + bx + c = 0$ with a goal in mind—that of transforming the latter equation form into the former known equation form but maintaining the solution set unchanged. The intellectual gain is that students learn that algebraic expressions are re-formed for a reason and would, accordingly, develop a sense of the actions needed in order to reach a desired algebraic form. Without this ability, symbol manipulation is largely a mysterious activity for students—an activity they carry out according to prescribed rules but without a goal in sight. With this ability, on the other hand, symbol manipulation is not a matter of magic tricks performed by the teacher but goal-directed operations learnable by all students. Of course, one reason symbolic manipulation is being deemphasized is that this is *not* how it's being taught!

With the algebraic invariance way of thinking as an instructional objective, teaching techniques of integration, for example, will have an added value: would one teach such techniques not only so that students know how to determine antiderivatives of functions and values of integrals, but also to help students develop a critical way of thinking in mathematics—that of utilizing the power of mathematical symbolism to solve problems and make and prove conjectures. Techniques of integration provide an excellent context to advance students toward this goal, which is why I believe this topic should be maintained as part of the calculus curriculum. Take, for example, the simplest technique of Reduction to Standard Formulas. In solving an integral such as $\int \tan\theta d\theta$, students in freshman calculus learn to set a goal of transforming this unknown integral into an equivalent form that is familiar. Even if the students do not note that the symbolic representation $\tan\theta = \sin\theta / \cos\theta$ suggests the substitution $u = \cos\theta$, they would learn to appreciate such a representation when they see how it is utilized to *change the form of the integral without changing its value* through a sequence of symbolic transformations (e.g., $\int \tan\theta d\theta = -\int du/u = -\ln|u| + C = \ln|u^{-1}| + C = \ln|\sec\theta| + C$). Likewise, the algebraic invariance way of thinking is the basis for the concept of "equivalent systems;" that is, for manipulating a system of equations but maintaining its solution set.

The algebraic invariance way of thinking is not learned at once—one constructs it gradually by applying it in different contexts, such as techniques of integration, systems of linear equations, matrix factorization, etc. Students can start acquiring it in elementary mathematics, for example when transforming fractions into decimals, and vice versa. It is crucial, however, that such transformations are carried out meaningfully. Often students are taught to carry out symbolic transformations without adequate emphasis on their justification. For example, we have seen students learn to solve division problems involving decimal numbers (e.g., $0.14\overline{)12.91}$) by transforming them into division problems involving whole numbers (e.g., $14\overline{)1291}$) without ever attending to the mathematical basis for the transformation. Such exercises—devoid of meaning—have no value in advancing the algebraic invariance way of thinking among students, and they deprive the students of the opportunity to develop other critical ways of thinking. When justifying the equivalence of $0.14\overline{)12.91}$ and $14\overline{)1291}$, for example, students reason proportionally (e.g., when justifying that $\frac{0.14}{12.91} = \frac{0.14 \times 100}{12.91 \times 100}$), attend to the nature of the number system (e.g., when justifying that $0.14 \times 100 = 14$ and $12.91 \times 100 = 1291$), and begin to develop algorithmic way of thinking (when dividing 1291 into 14 by using the long-division algorithm). Obviously, such opportunities will not occur if the non-referential way of thinking dominates students' actions or if the students obtain the answer to $0.14\overline{)12.91}$ by using a calculator.

It should be clear that in applying the algebraic invariance way of thinking, it is never the case that every single symbol is referential. It is only in critical stages—viewed as such by the person who carries the symbol manipulations—that one forms, or attempts to form, referential meanings. One does not usually attend to interpretation in the middle of symbol manipulations unless one encounters a barrier or recognizes a symbolic form that is of interest to the problem at hand; thus, for most of the process the symbols are treated as if they have a life of their own. It is in this sense that symbol manipulation skills should be understood and, accordingly, be taught.

One might ask, what is then the difference between the algebraic invariance way of thinking and the non-referential symbolic way of thinking? The answer is that the former includes the ability to pause at will to probe into a referential meaning for the symbols involved, whereas the latter does not. In applying the algebraic invariance way of thinking, the attempt to form a referential meaning does not have to occur, and even if it occurs it does not have to succeed. It is only that the person who carries out the manipulation has the ability to investigate the referential meaning of any symbol and transformation involved. In the non-referential symbolic way of thinking this ability is largely absent.

It is worth pointing out that the practice of manipulating symbols without *necessarily* examining their meaning played a significant role in the development of mathematics. For example, during the nineteenth century a significant work was done in differential and difference calculus using a technique called "operational method," a method whose results are obtained by symbol manipulations without understanding their meaning, and in many cases in violation of well-established mathematical rules. (See, for example, the derivation of the Euler-MacLaurin summation formula for approximating integrals by sums, in Friedman, 1991.) Mathematicians sought meaning for the operational method, and with the aid of functional analysis, which emerged early in the twentieth century, they were able to justify many of its techniques. Hence, the operational method technique is a manifestation of the algebraic invariance way of thinking, not the non-referential symbolic way of thinking.

In sum, with the algebraic invariance way of thinking, teachers would recognize that the goals of teaching manipulation skills include both learning how to compute solutions to particular problems and constructing conceptual tools that are an essential part of mathematical practice. The goal of teaching techniques of integration, for example, is not just to obtain an antiderivative for a given function, but also to help students acquire an important way of thinking—that of manipulating symbols with a goal of changing the form of an entity without changing a certain property of the entity, a way of thinking that is ubiquitous and essential in mathematical practice. The role of symbolic algebra in the reconceptualization of mathematics raises a critical question about the role of symbolic manipulation skills in students' conceptual development of mathematics. In response to increasing use of electronic technologies in schools, particularly computer algebra systems, educators should ask: Might these tools deprive students of the opportunity to develop algebraic manipulation skills which are needed for the development of the algebraic invariance way of thinking?

3.3 Proof

Certain obstacles students encounter with the concept of proof seem to parallel obstacles in the development of mathematics. I discuss here two related observations. The first involves the transition from Greek mathematics to modern mathematics and the second the notion of Aristotelian causality.

3.3.1 Transition from Greek mathematics to Modern Mathematics

The deductive mode of thought was conceived by the Greeks more than 20 centuries ago and is still dominant in the mathematics of our day. The mathematicians of the civilizations that preceded the Greeks established their observations on the basis of empirical measurements; hence, they mainly possessed and employed empirical proof schemes—schemes marked by their reliance on evidence from examples or visual perceptions. In Greek mathematics, logical deduction is central in the reasoning process, and it necessitated the geometric edifice they created. This edifice, however, represents a single model—that of idealized physical reality. This ultimate bond to a real-world context had an impact on the Greeks' deductive proof scheme, in that Euclid often uses arguments that are not logical consequences of his initial assumptions but are rooted in humans' intuitive physical experience. While Euclid's *Elements* is restricted to a single interpretation—namely that its content is a presumed description of human spatial realization—Hilbert's *Grundlagen* is open to different possible realizations, such as Euclidean space, the surface of a half-sphere, ordered pairs and triples of real numbers, etc., including the interpretation that the axioms are meaningless formulas. In other words, the *Grundlagen* characterizes a structure that fits different models. To reflect this fundamental conceptual difference, I refer to the Greeks' method of proving as the *Greek axiomatic proof scheme* and to the modern mathematics' method of proving as the *modern axiomatic proof scheme*. The transition between these two proof schemes is revolutionary: It marks a monumental conceptual change in humans' mathematical ways of thinking. Understanding this transition may shed light on epistemological obstacles students encounter upon moving from concrete models of their quantitative or spatial reality—such as the ones held by the Greeks—to a more abstract setting—such as that offered by Hilbert. As a historian might ask what events—social, cultural, and intellectual—necessitated the transition from one way of thinking to another (e.g. from pre-Greek mathematics to Greek mathematics to the mathematics of the Renaissance and to modern mathematics), a mathematics educator should ask what is the nature of the instructional interventions that can bring students to refine and alter an existing way of thinking to a more desirable one?

3.3.2 Causality

According to the definition of "proof scheme" presented in Section 1, certainty is achieved when an individual determines—by whatever means he or she deems appropriate—that an assertion is true. Truth alone, however, may not be the only aim of an individual, and he or she may also desire to know *why* the observation is true—the cause that makes it true. An individual may be certain of the truth of an observation and still strive to understand what in that truth liberates her or him from doubt. "Proofs really aren't there to convince you that something is true—they're there to show you why it is true," said Gleason, one of the solver of Hilbert's Fifth Problem (Yandell, 2002, p. 150).

Two millennia before him, Aristotle, in his Posterior Analytic, asserted: "To grasp the why of a thing is to grasp its primary cause."

The 16-18th century conception of mathematics reflects global epistemological positions that can be traced back to this position of Aristotle, according to which explanations in science must be causal. According to the philosophers of the time, this position entails the rejection of proof by contradiction, for when a theorem "A implies B" is proved by showing how not B (under the assumption of A) leads logically to an absurdity, a person does not learn anything about the causality relationship between A and B, nor—one might add—does one gain insight into how the theorem was—or might have been—conjectured. Some students' behavior with proof can be explained in terms of this epistemological position, in that many able students search for causal relationships in proofs and dislike indirect proofs (see Harel, 1999). Likewise, for the decisive majority of mathematicians the purpose of a proof is not only demonstrating that the assertion is true, but also explaining why it is true. Proofs by contradiction, while accepted in modern mathematics, usually lack the explanatory power direct proofs can have. As an example, it is worth mentioning the controversy that Hilbert's proof of Gordan's Conjecture[5] raised. Hilbert didn't find a basis that everyone had searched for but merely proved that if we accept Aristotle's law of the excluded middle ("Any statement is either true or its negation is true") then such a basis had to exist, whether we could produce it or not. Why was Hilbert's use of proofs-by-contradiction so controversial—after all, he was not the first to use this method of argument? According to Yandell (2002), previous uses had not dealt with a subject of such obvious calculational complexity. A pure existence proof does not produce a specific object that can be checked—one had to trust the logical consistency of the growing body of mathematics to trust the proof. The presence of an actual object that can be evaluated provides more than mere certainty; it can constitute a cause—in the Aristotelian sense—for the observed phenomenon. The philosophers of the Renaissance rejected proof by contradiction, and the practice of many mathematicians of that period, such as Cavalieri, Guldin, Descartes, and Wallis, reflected this position by explicitly avoiding proofs by contradiction in order to conform to the Aristotelian position on what constitutes perfect science (Mancosu, 1996).

The implication of this history is not to avoid proofs by contradiction in mathematics curricula. On the contrary, proofs by contradiction represent an important, institutionalized way of thinking, which students should acquire. The point of this history is that modern proof schemes were born out of an intellectual struggle—a struggle in which Aristotelian causality seems to have played a significant role. It is an open question whether the development of students' proof schemes necessarily includes some of these epistemological obstacles. The fact that even able students encounter these obstacles makes this question even more relevant to the matter at hand. An answer to this question may shed light on some of the roots of the obstacles students encounter with certain kinds of proof, such as proof by contradiction. Accordingly, appropriate

[5] The conjecture states: There is a finite basis from which all algebraic invariants of a given polynomial form could be constructed by applying a specified set of additions and multiplications.

instructional interventions can be devised to help students develop desirable proof schemes as they encounter these obstacles, which, perhaps, are unavoidable.

4 DNR Based Instruction in Mathematics

DNR-based instruction in mathematics (*DNR*, for short) is a theoretical framework for the learning and teaching of mathematics—a framework that provides a language and tools to formulate and address critical curricular and instructional concerns. In this framework the *mathematical integrity* of the content taught and the *intellectual need* of the student are at the center of the instructional effort. *DNR* has been developed from a long series of teaching experiments in elementary, secondary, and undergraduate mathematics courses, as well as teaching experiments in professional development courses for teachers at each of these levels. Briefly, *DNR* can be thought of as a system consisting of three categories of constructs:

1. *Premises*—explicit assumptions underlying the *DNR* concepts and claims.
2. *Concepts*—referred to as *DNR determinants.*
3. *Instructional principles*—claims about the potential effect of teaching actions on student learning.

It goes beyond the scope of this paper to do more than present a brief outline of these constructs. For more about *DNR*, see Harel (1989, 2001, in press, in preparation).

Premises. One of the *DNR* premises is the *conceptualization premise:*

> Humans—all humans—possess the ability to develop a desire to be puzzled and to learn to carry out physical and mental acts to fulfill their desire to be puzzled and to solve the puzzles they create.

This premise, which follows from Aristotle, is one of eight *DNR* premises. Note that it assumes not only humans' desire *to solve puzzles* but also humans' desire *to be puzzled*. It serves as a basis for many themes in *DNR*—the *necessity principle*, to be stated shortly, is one of them. It is also the basis of *DNR*'s interpretation of *equity*: *All* students are capable of learning if they are given the opportunity to be puzzled, create puzzles, and solve puzzles.

Concepts. "Mental act," "way of understanding," and "way of thinking" are examples of *DNR determinants*.

Instructional Principles. Not every *DNR* instructional principle is explicitly labeled as such. The system states only three foundational principles: the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle;* hence, the acronym *DNR.* The other principles in the system are derivable from and organized around these three principles.

Recall, according to my definition, mathematical knowledge consists of ways of understanding and ways of thinking. The *duality principle* concerns the developmental interdependency between these two constructs:

> **The *Duality Principle*:** Students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess.

The reciprocity between ways of understanding and ways of thinking claimed in the duality principle is of mutual effect: a change in ways of thinking brings about a change in ways of understanding, and vice versa. The claim intended is, in fact, stronger: Not only do these two categories of knowledge affect each other but a change in one cannot occur without a corresponding change in the other.

Implied from the duality principle is that preaching ways of thinking to students would have no effect on the quality of the ways of understanding they would produce. For example, talking to them about the nature of proof in mathematics or advising them to use particular heuristics would have minimal or no effect on the quality of the proofs and solutions they would produce. Only by producing desirable ways of understanding—by way of carrying out mental acts of, for example, solving mathematical problems and proving mathematical assertions—can students construct desirable ways of thinking. This seems obvious until one observes, for example, teachers teaching problem-solving heuristics explicitly and students following them as if they were general rules rather than rules of thumb for solving problems.

Attention to ways of thinking, on the other hand, is necessary—according to the duality principle—for they direct teachers as to which teaching actions to avoid and which to pursue. As we have discussed earlier, attention to desirable ways of thinking—such as algebraic invariance, proportional reasoning, and algorithmic reasoning—highlights the need to focus on particular ways of understanding certain concepts and processes (e.g., the solution process of quadratic functions, techniques of integration, and division of decimal numbers; see Section 3.2). In particular, teachers must take into consideration students' current ways of thinking. For example, a college instructor may start a course in geometry with finite geometries as a preparation for non-Euclidean geometries. We found (Harel & Sowder, 1998) that most undergraduate students taking college geometry are not prepared for such an instructional treatment because they do not possess the modern axiomatic proof scheme—which includes the way of thinking that geometric properties are not limited to spatial imageries.[6] As was discussed earlier, this way of thinking was born at the turn of the 20th century with the publication of Hilbert's *Grundlagen* and is considered revolutionary in the development of mathematics.

Of critical pedagogical importance is the question: What is the nature of instructional treatments that can help students construct desirable ways of understanding and ways of thinking? This is addressed by the other two *DNR* principles: the *necessity principle* and the *repeated reasoning principle.*

> **The *Necessity Principle*:** For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to *intellectual need*, not social or economic need.

Most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics classes because we—teachers—fail to help them realize an *intellectual need* for what we intend to teach them. The term *intellectual need* refers to a

[6] For example, students in our study encountered insurmountable difficulty interpreting the statement "Given a line and a point not on the line, there is a line which contains the given point and is parallel to the given line" in a finite geometry.

behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate. For example, students might encounter a situation that is incompatible with, or presents a problem that is unsolvable by, their existing knowledge. Such an encounter is intrinsic to the learners because it stimulates a desire within them to search for a resolution or a solution, whereby they might construct new knowledge. There is no guarantee that the learners construct the knowledge sought or any knowledge at all, but whatever knowledge they construct is meaningful to them since it is integrated within their existing cognitive schemes as a product of effort that stems from and is driven by their personal, intellectual need. While one should not underestimate the importance of students' social need (e.g., mathematical knowledge can endow me with a respectable social status in my society) and economic need (e.g., mathematical knowledge can help me obtain comfortable means of living) as learning factors, teachers should not and cannot be expected to stimulate (let alone fulfill) these needs. Intellectual need, on the other hand, is a prime responsibility of teachers and curriculum developers.

Even if ways of understanding and ways of thinking are necessitated through students' *intellectual need* there is still the task of ensuring that students internalize, organize, and retain this knowledge. This concern is addressed by the *repeated-reasoning principle:*

The *Repeated Reasoning Principle:* Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.

Research has shown that repeated experience, or practice, is a critical factor in these cognitive processes (see, for example, Cooper, 1991). *DNR-based instruction* emphasizes repeated reasoning that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization—a conceptual state where one is able to apply knowledge autonomously and spontaneously. The sequence of problems must continually call for reasoning through the situations and solutions, and they must respond to the students' changing intellectual needs.

These instructional principles are the basis for many of the pedagogical positions expressed in this paper, and they have been used to organize my instruction, in general, and teaching experiments, in particular. Consider the following unit taken from a recent teaching experiment with secondary mathematics teachers with limited mathematics background. The teachers worked on justifying the quadratic formula. Prior to this problem, they had repeatedly worked with many quadratic functions, finding their roots by essentially completing the square. They abstracted this process to develop the quadratic formula. In doing so they repeatedly transformed particular equations of the form $ax^2 + bx + c = 0$ into an equivalent equation of the form $(x+T)^2 = L$ for some terms T and L, in order for them to solve for x (as $-T+\sqrt{L}$ and $-T-\sqrt{L}$). To get to the desired equivalent form, they understood the reason and need for dividing through by a, bringing c/a to the other side of the equation, and completing the square. For these teachers, the symbolic manipulation process stems from an intellectual need—the need to arrive at a particular form in order to determine the equation's unknown—and conditioned by quantitative considerations—to transform the algebraic expressions without altering their quantitative value. In these activities, the teachers practiced the algebraic invariance way of thinking, whose importance I have discussed in Section 3.2. We see here the simultaneous implementation of the duality principle, the necessity principle, and the repeated reasoning principle. In particular, the repeated application of the invariance way of thinking helped the participant teachers internalize it, whereby they become autonomous and spontaneous in applying it.

5 Summary

Current teaching practices tend to view mathematics in terms of subject matter, such as definitions, theorems, proofs, problems and their solutions, not in terms of the conceptual tools that are necessary to construct such mathematical objects. The tenet of this paper is that instruction should focus on both categories of knowledge: subject matter and conceptual tools. The paper defines these two categories and explains why both categories are needed. The definitions and explanations are oriented within the *DNR* framework. Central to *DNR* is the distinction between "way of understanding" and "way of thinking." "Way of understanding" refers to a cognitive product of a person's mental act, whereas "way of thinking" refers to its cognitive characteristic. Accordingly, mathematics is defined as the union of two sets: the set WoU, which consists of all the institutionalized ways of understanding in mathematics throughout history, and the set WoT, which consists of all the ways of thinking that characterize the mental acts whose products comprise the first set.

The members of WoT are largely unidentified in the literature, though some significant work was done on the problem-solving act (e.g., Schoenfeld, 1985; Silver, 1985) and the proving act (see an extensive literature review in Harel & Sowder, in press). The members of WoU include all the statements appearing in mathematical publications, such as books and research papers, but it is not listable because individuals (e.g., mathematicians) have their idiosyncratic ways of understanding. A pedagogical consequence of this fact is that a way of understanding should not be treated by teachers as an absolute universal entity shared by all students, for it is inevitable that each individual student is likely to possess an idiosyncratic way of understanding that depends on her or his experience and background. Together with helping students develop desirable ways of understanding, the goal of the teacher should be to promote interactions among students so that their necessarily different ways of understanding are compatible with each other and with that of the mathematical community.

Since mathematics, according to the definition offered in this paper, includes historical ways of understanding and ways of thinking, it must include ones that might be judged as imperfect or even erroneous by contemporary mathematicians. Also included in mathematics are imperfect ways of understanding and ways of thinking used or produced by individuals in the process of constructing institutionalized knowledge. The boundaries as to what is included in mathematics are in harmony with the nature of the process of learning, which necessarily involves the construction of imperfect and erroneous ways of understanding and deficient and faulty ways of thinking. These boundaries, however, are not to imply acceptance of the radical view that particular mathematical statements could be true for some people and false for others.

My definition of mathematics implies that an important goal of research in mathematics education is to identify desirable ways of understanding and ways of thinking, recognize their development in the history of mathematics, and, accordingly, develop and implement mathematics curricula that aim at helping students construct them. This claim was illustrated in the contexts of algebra and proof. The discussion on algebra highlights the need to promote the algebraic invariance way of thinking among students. With it, students learn to manipulate symbols with a goal in mind—that of changing the form of an entity without changing a certain property of the entity. It also points to the risk that the use of electronic technologies in schools, particularly computer algebra systems, can deprive students of the opportunity to develop this crucial way of thinking. The discussion on proof focuses on the transition from Greek mathematics to modern mathematics and the role of Aristotelian causality in the development of mathematics during the Renaissance. It raises the question of whether the development of students' proof schemes parallels those of the mathematicians and philosophers of these periods. An answer to this question would likely have important curricular and instructional implications.

Since the formation of desirable ways of thinking is difficult and those that have been formed are hard to relinquish, an effort must be made in early grades to help students acquire desirable ways of thinking. The concept of fraction, for example, can be taught with multiple ways of understanding, and in a context where students can develop ways of thinking necessary for the acquisition of advanced mathematics. Similarly, arithmetic problems such as division of decimals can provide invaluable opportunities to engage in proportional reasoning and algorithmic reasoning and revisit the nature of the decimal-number system.

Pedagogically, the most critical question is how to achieve such a vital goal as helping students construct desirable ways of understanding and ways of thinking. DNR has been developed to achieve this very goal. As such, it is rooted in a perspective that positions the mathematical integrity of the content taught and the intellectual need of the student at the center of the instructional effort. The mathematical integrity of a curricular content is determined by the ways of understanding and ways of thinking that have evolved in many centuries of mathematical practice and continue to be the ground for scientific advances. To address the need of the student as a learner, a subjective approach to knowledge is necessary. For example, the definitions of the process of "proving" and "proof scheme" are deliberately student-centered (see Section 1). It is so because the construction of new knowledge does not take place in a vacuum but is shaped by one's current knowledge. What a learner knows now constitutes a basis for what he or she will know in the future. This fundamental, well-documented fact has far-reaching instructional implications. When applied to the concept of proof, for example, this fact requires that instruction takes into account students' current proof schemes, independent of their quality. Again, despite this subjective definition *the goal of instruction must be unambiguous—namely, to gradually refine current students' proof schemes toward the proof scheme shared and practiced by contemporary mathematicians.* This claim is based on the premise that such a shared scheme exists and is part of the ground for advances in mathematics.

Instruction concerns *what* mathematics to teach as well as *how* to teach it. While the definition of mathematics offered in this paper dictates the kind of knowledge to teach—ways of understanding *and* ways of thinking—the three DNR principles stipulate how to teach that knowledge:

The duality principle concerns the developmental interdependency between ways of understanding and ways of thinking: Students would be able to construct a way of thinking associated with a certain mental act or refine or modify an existing one *only* if they are helped to construct suitable ways of understanding associated with that mental act. Conversely, students would be able to construct a way of understanding associated with a certain mental act or refine or modify an existing one *only* if they are helped to construct suitable ways of thinking associated with that mental act in the form of problem-solving approaches or proof schemes.

According to the necessity principle, problem solving is not just a goal but also the means—the only means—for learning mathematics. Learning grows only out of problems intrinsic to the students, those that pose an intellectual need for them. In general, an intellectually-based activity is one where students' actions are driven by a desire to solve intrinsic problems. In a socially-based activity, on the other hand, students' actions are carried out merely to satisfy a teacher's will.[7] In an intellectually-based teaching environment, students are continually challenged with new problems from which they elicit new concepts and ideas. Such an environment is necessary for learning, and is conducive to creativity.

The repeated reasoning principle is complementary to the duality principle and the necessity principle, in that its aim is for students to internalize what they have learned through the application of these two principles. Through repeated reasoning in solving intrinsic problems, the application of ways of understanding and ways of thinking become autonomous and spontaneous.

[7] The notion of intellectually, but not socially, based activity is similar to what Brousseau (1997) calls an *adidactical situation.*

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