Two Dual Assertions: The First on Learning and the Second on Teaching (or Vice Versa)

Guershon Harel

1. INTRODUCTION. *Ways of thinking* are students' apparatuses for filtering and interpreting what we intend to teach them. In this paper, they are distinguished from *ways of understanding*. A way of understanding is the meaning(s) students have for a specific concept. For example, students may understand the "derivative of a function" as the slope of a line tangent to the graph of a function, as the best linear approximation to a function near a point, etc., but they may understand it superficially (e.g., "derivative is nx^{n-1} for x^n") or even incorrectly (e.g., "derivative is the quotient $(f(x+h)-f(x))/h$").

I begin by showing how students' ways of thinking impact their ways of understanding mathematical concepts. Following this, I will suggest that even though ways of thinking are difficult to relinquish, they are not unalterable; how students come to understand mathematical content influences the quality of their ways of thinking.

2. UBIQUITOUS FAULTY WAYS OF THINKING

2.1 Symbolic Reasoning. In linear algebra, students learn that the solution set of a linear system is invariant under row operations on the augmented matrix of the system, as is the rowspace of a matrix under row operations on the matrix. We observed students extend these facts to conclude, and persistently hold, that the column space of a matrix is invariant under row operations on the matrix. How did these students form this misconception despite the fact they had never heard such a statement from their teacher or read it in any textbook? These students, we found, possessed a habit of mind [**5**]—a way of thinking, that is—of arriving at conclusions on the basis of association alone, without examining their meaning and truth. In some cases, this habit of mind leads not only to an incorrect statement—which, as in this case, can be refuted by a counterexample—but to nonsense reasoning, as in the following episode (taken from [**8**]): In response to the question, "Why is a homogeneous system $AX = 0$ consistent?", Hugh, a student in a linear algebra class, said:

> Take $x_1A_1 + x_2A_2 + \cdots + x_nA_n = 0$ [the A_is are the columns of A]. We want to find x_1 and we have some values $x_2, \ldots, x_n$ and $A_1, A_2, \ldots, A_n$, by moving the known values [the x_is and A_is; apparently, thinking of the former as free variables] to the other side of the equation, we can solve [for] x_1. This is the same for $x_2, \ldots, x_n$. Hence, this demonstrates that a homogeneous system has solution: $x_1 = (x_2A_2 + x_3A_3 + \cdots + x_nA_n)/A_1$.

When asked what dividing by A_1 means, Hugh replied: "It is just this is over this [pointing to his fractional expression]. It is just like one over x."

I call this way of thinking, *symbolic reasoning*. In using it, one treats symbols as if they possess a life of their own and manipulates them without examining their meaning. In linear algebra, symbolic reasoning manifests itself in the "vectors-as-numbers" misconception, for, as we have seen in Hugh's response, students treat vectors as if they were numbers. With some students this misconception is apparent, with others it is concealed, as in the following episode: In response to the problem, "M, N, K, R, V are linearly dependent. If M and N are removed, would K, R, and V be dependent?", Sam said:

> The remaining vectors are linearly dependent although we eliminate M and N, it's the same as we multiply M and N with zero. For example: $K = aR + bV + 0M + 0N$, $a, b \neq 0$. This is because linearly dependent means that any element in a group is a linear combination of the others.

Sam's reasoning behind this seemingly incomprehensible answer was this: For him (as for some of his classmates), "dependence" was not a set-wise property, but a property of a single vector. Namely, to him, "a vector is dependent" if it is a linear combination of other vectors and so "a set of vectors is dependent" if *each* vector in the set is a linear combination of the others. Based on the problem information, he therefore assumed that $K = xR + yV + zM + vN$. Removing M and N meant to him replacing them by zeros in the latter equality. But to do so, he felt he had to compensate for the "quantity" loss of $zM + vN$ by readjusting the coefficients x and y into new coefficients a and b—just as if all the symbols represented real numbers.

The key point of these observations is that Hugh's and Sam's understanding of basic concepts were governed by their ways of thinking, the most apparent of which is symbolic reasoning.

Against this devastating reasoning, I point now (to avoid puzzlement about the meaning of the term "symbolic reasoning") to a different, essential, practice of symbolic reasoning. The definition of symbolic reasoning I gave earlier may have evoked with the reader a different image from the one revealed in Hugh's and Sam's responses. For, relative to the reader's practice of mathematics, it is not uncommon that symbols *are* treated as if they possess a life of their own, and, accordingly, *are* manipulated without (necessarily) examining their meaning.

Historically, this practice of symbolic reasoning played a significant role in the development of mathematics. For example, during the nineteenth century an enormous amount of work was done in differential and difference calculus using a technique called "operational method", a method whose results are obtained by symbol manipulations without understanding their meaning, and in many cases in violation of well established mathematical rules. (See, for example, how the Euler-MacLaurin summation formula for approximating integrals by sums was derived in [**6**].) It is only with the aid of functional analysis, which emerged early in the twentieth century, that mathematicians were able to justify many of the operational method techniques. Another, better known, example is the reconstruction of calculus into real analysis at the beginning of the nineteenth century, which commenced with Fourier's "symbolic solution" to the Flow of Heat problem. Fourier reduced this problem to the problem of expressing an even function as an infinite sum of cosines, without attending the meaning of infinite summation of functions. His solution led to observations that seemed at the time inconsistent with "regular" behavior of functions. This, in turn, led to thorough investigations into the assumptions of calculus and inspection of its structure, whereby the entire calculus was reconstructed into a new mathematical field, which now is called *analysis*.

2.2 Justification By Virtue of Authority. Another common way of thinking, which in its extreme manifestation is as harmful as symbolic reasoning, is students' total acceptance of mathematical assertions on the sole basis of a statement appearing in their textbook or uttered by their teacher. Simply put, students lack the basic intellectual curiosity of wondering why a stated assertion is true. This behavior—broadly discussed in **[8]**—is hardly surprising, for current mathematics curricula emphasize *facts rather than reasons for facts*.

2.3 Lack of Multiple Ways of Understanding. That a concept *can* be understood in different ways, it *should* be understood in different ways, and it is *advantageous to change* ways of understanding of a concept while attempting to solve a problem, are ways of thinking commonly absent from most students' repertoire of reasoning. The impact of this deficiency is particularly apparent in linear algebra, which, more than any other lower-division mathematics course, requires the formation of different ways of conceptual understanding. One must understand, for example, that problems about systems of linear equations are equivalent to problems about matrices, which, in turn, are equivalent to problems about linear transformations. Students who are not equipped with these ways of thinking are doomed to encounter difficulties. A simple, yet indicative, example is students' difficulty in adopting different ways of multiplying matrices (e.g., $(AB)^{(k)} = \Sigma_j (B^{(k)})_j A^{(j)}$, $(AB)_{(k)} = \Sigma_j (A_{(k)})_j B_{(j)}$, or $AB = \Sigma_j A^{(j)} B_{(j)}$) once they have learned the standard definition, $(AB)_{i,j} = \Sigma_r (A_{(i)})_r (B^{(j)})_r$.

2.4 Lack of Effective Concept Images. A way of understanding a concept is part of one's "concept image": a mental network consisting of what the person knows about the concept (e.g., analogies to other concepts, examples and nonexamples, etc.). The notion of "concept image" (according to **[11]**) is distinguished from that of "concept definition": a verbal statement, appearing in a textbook or presented by the teacher, that accurately describes the concept in a non-circular way. Concept images, or "elaborated structures" as cognitive scientists call them, have proved to have profound effects on memory and comprehension **[3]**. A student with an effective concept image is one who can communicate its corresponding concept definition in her or his own words, can think about it in general terms, can connect it to other concepts, and can, as a result, remember its meaning for an extended period of time **[7]**.

What sort of concept images do students build in their linear algebra courses? Tables 1 and 2 present slightly edited results of a survey of 25 students, 1–3 semesters after they had completed a differential-equations-and-linear-algebra course followed by a linear algebra course. Their average grades were, respectively, 3.05 and 2.91, on the grade-scale of $A = 4$, $B = 3$, $C = 2$, and $D = 1$.

These results suggest that students do not build effective concept images; rather, they place their full reliance on concept definitions, by memorizing them verbatim. They manage to remember concept definitions until the final exam is over but are unable to retain them for an extended period of time. Once the concept definitions are forgotten, students are unable to retrieve or rebuild them on their own.

The observations I discussed in this section are consistent with a well established observation that students' background knowledge—the sum of what they

TABLE 1. Percentage Distribution of Students' Responses

Question ↓ Answers →	Correct	Incorrect	None
Independence			
Can 3 vectors in R^2 be independent?	48	52	0
Suggest 3 independent vectors in R^3.	52	36	12
Define "linear independence".	28	68	4
Span			
Suggest a set that spans R^3.	68	16	16
Define "The span of a set of vectors".	36	58	6
Vector Space			
Give an example of a subspace.	16	60	24
Fundamental Theorem			
$rank(A_{4\times 7}) = 3$. Find $nullity(A)$.	32	52	16
Matrix Transformations			
Are matrix transformations linear?	44	24	32
What relation exists between $CS(A)$ and $\operatorname{Im}(A)$?	20	16	64

TABLE 2. Sample of Students' Responses to Questions in Table 1

Concept	**Responses**
Independence	1. Dependence is when there is some sort of equation where $\alpha x_1 + \gamma x_2 + \theta x_3 = 1$ or 0. 2. Dependence is when row reducing the arbitrary numbers are $c_1 = c_2 = \cdots = c_n = 0$. 3. Independence is when there aren't zero divisors.
Vector Space and Subspace	1. Vector space: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; Subspace: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 2. The set of integers is a vector space. The set of even integers is a subspace. 3. Vector space: $x = 0, y = 0, z = 0$; Subspace: $x + y = 0$.
Fundamental Theorem	Column space is equal to the dimensionality of the space the image is projected in.
Matrix Transformation	A matrix transformation is linear if you have one to one transformation.

know—is a chief factor influencing their learning [**1**]. This general observation—labeled BKI (Background Knowledge Impact)—"assumes near axiomatic status in cognitive science" [**10**]. Unfortunately, it is usually interpreted narrowly; namely, that students' knowledge of one concept influences their learning of another concept that is *logically* dependent on it. But as teachers, we know that the BKI observation applies even when concepts are logically independent. For example, we expect students' knowledge of metric spaces to influence their learning of topology, despite the fact that the latter is logically independent of the former. We so expect, because we understand that to learn a mathematical idea students need more than factual knowledge. In this section I tried to show that they need, in particular, appropriate ways of thinking. Thus, the observed impact of the presence of faulty ways of thinking and the absence of essential ones on conceptual understanding gives a pedagogically important meaning to the BKI observation, namely: *Ways of thinking impact ways of understanding.* This is the root of the often heard, but seldom appreciated, phrase, "students construct their own knowledge".

3. THE NECESSITY PRINCIPLE: AN ORGANIZING PRINCIPLE FOR TEACHING. Students feel intellectually aimless in mathematics courses, because we usually fail to present them with a clear intellectual purpose. In this section I present a pedagogical principle, called the Necessity Principle, by which students learn that concepts are not introduced arbitrarily but with reasons—reasons they understand and appreciate—whereby they become (or at least feel as if they were) partners in knowledge development. When we implemented the Necessity Principle consistently—that is, locally, in teaching specific concepts, and globally, throughout the entire course—along with a mode of teaching that combines small group discussion, team projects, whole-class discussion, individual learning, use of technology, and lecturing, we observed positive results: Students showed adequate understanding of the material they learned and gradually abandoned faulty *ways of thinking* and acquired instead adequate ones. I suggest, therefore, that the dual assertion—*ways of understanding impact ways of thinking*—may be equally valid.

Before I state the Necessity Principle, let us consider the following example: A current, widely used, elementary linear algebra textbook introduces the concept of "independence" with the statement:

> So far we have defined a mathematical system called a real vector space and noted some of its properties.... [In what follows], we show that each vector space V studied here has a set composed of a finite number of vectors that completely describe V. It should be noted that, in general, there is more than one such set describing V. We now turn to a formulation of these ideas.

Following this, the text defines "span" and "dependence" and gives illustrative examples for each. The entire presentation appears clear and accurate. Despite this, its effectiveness is in doubt. Specifically, one should ask: Can a lesson that is based on this presentation convince a regular student in a standard elementary linear algebra course of the need for the concept of "independence" to solve a problem? True, the text's introductory statement indicates the "problem" it intends to solve; namely, to establish that each vector space V studied in the book has a set composed of a finite number of vectors that completely describe V. But, is our student likely to view this as a problem? Can he or she in this stage of the course understand its importance? Can he or she see how "independence" contributes to its solution? In other words, what is our student's *intellectual* need—as opposed to *social* or *economic* need—in learning the concept of "independence"? There is nothing in this presentation that arouses such a need with our student.

"Intellectual need" is an expression of a natural human behavior: When we encounter a situation that is incompatible with, or presents a problem that is unsolvable by our existing knowledge, we are likely to search for a resolution or a solution and construct, as a result, new knowledge. Such knowledge is meaningful to the person who constructs it, because it is a product of personal need and connects to prior experience. This human phenomenon is the basis for what I call the Necessity Principle.

> *The Necessity Principle: Students are most likely to learn when they see a need for what we intend to teach them, where by "need" is meant intellectual need, as opposed to social or economic need.*

In concrete instructional terms, the Necessity Principle translates into three steps:

1. Recognize what constitutes an intellectual need for a particular population of students, relative to the concept to be learned.
2. Present the students with a problem that corresponds to their intellectual need, and from whose solution the concept can be elicited.
3. Help students elicit the concept from the problem solution.

Geometric demonstrations and applications in other fields should not be confused with the Necessity Principle. While the former are excellent tools to help students solidify concepts they have already learned, the latter aims at laying the ground for the concepts students are yet to learn.

The following examples demonstrate the general idea, definitely not the complete instructional plan, of the preceding three steps:

One way to introduce the pivotal cluster of concepts, "linear combination", "dependence", and "independence", in accordance with the Necessity Principle is to use one of the historical roots of linear algebra: systems of linear equations. Students entering their first course in linear algebra are familiar with system of equations and understand their importance (in solving word problems, for example). They can be brought—as our experience suggests—to appreciate the fact that in some cases we cannot or do not want to solve a given system $AX = b$, and yet we *need* to determine whether it has a solution or whether its solution is unique. We pose these Existence and Uniqueness problems early in our matrix-oriented course—before we mention any of the above concepts—to focus students' attention on a definite goal. Because we strongly emphasize—in fact define—matrix multiplication via the relation $MN^{(k)} = \sum_j (N^{(k)})_j M^{(j)}$, it is not uncommon that a few students give a correct response to the Existence problem. Namely, if b can somehow be expressed as $b = x_1 A^{(1)} + x_2 A^{(2)} + \cdots + x_n A^{(n)}$, then $AX = b$ has a solution. Of course, students seldom give such a clean answer, but with the teacher's help, the class as a whole understands why the suggested relation between b and the columns of A merits attention and, therefore, deserves a name —"linear combination".

Building on this understanding, we help students elicit the concepts of "dependence" and "independence" from the solution to the Uniqueness problem. To avoid unnecessary complications—something we learned from experience—we first pose this problem with a homogeneous system $AX = 0$. Students are now prepared to see that the relations "one of the columns of A is a linear combination of the other columns" and its negation solve the Uniqueness problem. Once this is achieved, new concepts are born, and names are assigned to them: "linear dependence" and "linear independence", respectively.

In a similar manner, we introduce the concept of "determinant" as a technique for solving characteristic equations; we elicit the concept of "orthogonal projection" from the *need* to find an approximate solution (i.e., a least-squares solution) to an inconsistent system; we develop the proof of the Jordan Theorem, from beginning to end, in the context of solving linear systems of differential equations —a *need* students well recognize.

3.1 What Constitutes Intellectual Need? How would teachers find out what constitutes intellectual need for a particular population of students? Answer: They must understand their students' ways of thinking. That said, I now characterize three forms of intellectual need: need for *computation*, need for *formalization*, and need for *elegance*. It is beyond the scope of this paper to address the psychological roots of these needs. In what follows, I elaborate on the first, briefly discuss the second, and mention only that the third is what we associate with mathematical beauty, efficiency, and abstraction.

Need for Computation. The preceding examples of how to introduce a new concept in accordance with the Necessity Principle involve *obtaining* numeric values (exact or approximate) for unknowns or *determining* conditions for existence and uniqueness of solutions to linear systems. The intellectual need that such problems evoke with students I call *need for computation.* In more general terms, problems that involve computation of objects that are concrete to students or involve determination of properties of such objects are said to satisfy the need for computation. It is motivationally a powerful need in that students get highly engaged in the problems and understand and appreciate the concepts elicited from their solution. In our teaching experiments in linear algebra—a subject particularly amenable to this need—we attempted not only to elicit specific concepts but also to enhance students' reasoning skills, as the following examples demonstrate.

- Spatial Visualization

Students are not educated to use spatial visualization, and they lack rudimentary knowledge of 2- and 3-dimensional analytic geometry. Our goal was to develop our students' spatial visualization by having them witness its power in analyzing and conjecturing assertions. The problems we assign students to achieve this goal belong to the computational need category:

1. (a) $u = [u_1\ u_2\ u_3]^t$ and $v = [v_1\ v_2\ v_3]^t$ are solutions to $\begin{cases} 2x + 3y + 6z = 0 \\ -4x + 7y + 9z = 0 \end{cases}$. Are u and v dependent or independent? Justify your answer geometrically.
   (b) $u = [1\ 2\ 0]^t$ and $v = [5\ \frac{1}{2}\ 7]^t$ are solutions to $\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$. Are these equations dependent or independent? Justify your answer geometrically.
2. Lora and Tony computed the projection matrix from a certain R^n onto a specified subspace V by using the formula $P = W(W^tW)^{-1}W^t$ (W is a matrix whose columns form a basis for V). Lora chose a matrix W_1 to substitute for W, while Tony chose W_2.
   (a) Do you expect Lora and Tony's projection matrices to be the same?
   (b) Propose a 4-dimensional subspace V of R^6. Use MATLAB to compute P with five different bases of V.
   (c) Use your geometric intuition to explain your answers to (a) and (b) and to state a general conjecture.
   (d) Prove your conjecture algebraically.

- Proof Production

The pedagogical benefits of the following sample of problems are similar to those of construction problems in Euclidean geometry. By solving such problems, students apply definitions and theorems they have learned and as a result sharpen their proof skills.

1. The outputs of the vectors $a = [0\ 1\ 2\ 0]^t$, $b = [4\ 1\ -2\ 6]^t$, $c = [3\ 1\ 0\ 1]^t$, $d = [0\ 1\ -2\ 0]^t$ under a matrix transformation A are, respectively, $\alpha = [-1\ 1\ 2\ 1]^t$, $\beta = [1\ 5\ -2\ 6]^t$, $\gamma = [1\ 1\ 2\ 3]^t$, $\delta = [1\ -2\ -1\ 3]^t$.

Construct A. Is there only one such matrix transformation? Change just one of the input vectors and just one of the output vectors in the original data so one can definitely construct more than one such matrix transformation. Change just one of the input vectors and just one of the output vectors of the original data so that it is impossible to construct such a matrix transformation.

2. A, B, C, D, E are vectors in R^6. A, B, C are independent, $D = -4A + 3B$, and $E = 12B - 13C$. Let $W = [A\ B\ C\ D\ E]$ and let $R = rref(W)$. Construct R.

Beyond their value of enhancing students' understanding of specific concepts (e.g., "linearity" in the first problem), these construction problems (from the computational need category) proved particularly effective in fostering students' ability to produce proofs. For the very actions students employ to construct an object involve justification. The requirement to justify is by default in that students *need* to convince themselves that they have successfully produced the expected outcome; the *need* to justify is inherent in the task.

Need for Formalization. Students may be satisfied with their intuitive explanation of why $\lim_{n\to\infty}\frac{1}{n} = 0$, which typically is something to the effect: "$\lim_{n\to\infty}\frac{1}{n} = 0$ because the larger n gets the closer $\frac{1}{n}$ is to 0 ". A teacher preparing students for the $\epsilon - N$ definition of limit might proceed, upon hearing this explanation, by writing it on the board along with the graphs of $f(n) = \frac{1}{n}$ and $g(n) = -1$. Then the teacher may point out to the students that based on their own justification one can rightly argue $\lim_{n\to\infty}\frac{1}{n} = -1$, because, by their own words, "the larger n gets the closer $\frac{1}{n}$ is to -1". This exchange usually results—as our experience suggests—in a sort of a conflict with the students, whereby they see a need to modify their conception of limit.

This is an example of a *need for formalization.* It is, I presume, less robust and effective than the need for computation, because it requires an adequate level of mathematical maturity, which beginning students usually lack. I am referring to students' lack of appreciation for the need to justify assertions with strong visual or kinesthetic interpretations, such as those appropriate to the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT). While it is not hard to show students the power of these theorems for applications, it is not always easy to convince them of the need to prove them. We applied an experimental treatment to this problem, where we assigned an additional purpose to the proofs of these theorems. Namely, not just showing the assertions are true—that was unnecessary to the students—but also testing the formalization of intuitive notions, such as those of "limit", "continuity", and "there are no gaps on the real line." So, if the class can, for example, prove the IVT, which we all believe, by using "only" the $\epsilon - \delta$ definitions of limit, the formal definition of continuity, and the completeness axiom, then we can be certain that we have been successful in formalizing the corresponding intuitive notions.

In summary, based on my own observations, I conjecture that the need for computation is the most robust and effective with beginning students. The need for formalization is, developmentally, more subtle, and even more so the need for elegance, because they require a higher level of mathematical maturity. Accordingly, a gradual refinement of students' intellectual need should proceed in this order: from the need for computation to the need for formalization to the need for elegance.

3.2 Some Observations. I conclude this section with a few observations about the effect of the teaching approach I have just described.

Two consecutive teaching experiments in linear algebra (hereafter, LAa and LAb) were taught by the same teacher and had similar populations of students, and in both we intended to adhere to the Necessity Principle. But we were less successful implementing the Necessity Principle in LAa than in LAb. For example, the idea of introducing "independence" as a concept "needed" to answer the Uniqueness Question occurred to us only in LAb; in LAa, it was introduced in the usual way. In comparing students' achievements across the two experiments, we found differences in favor of LAb. Particularly, fewer misconceptions occurred in LAb than in LAa. For example, the "naive negation" misconception (i.e., "linear independence of a set" means one vector in the set is not a linear combination of the other vectors in the set) rarely occurred in LAb but was frequent in LAa. This observation suggests that *when students have a clear purpose for a concept, they are unlikely to misunderstand its meaning.*

In the LAa experiment, we followed the "common wisdom" of commencing the course with the definition of vector space, and, as it is commonly presented, we "abstracted" its axioms from the properties of various structures, including R^n. We found that this approach is inadequate for beginning students, who have yet to witness the benefits of axiomatization. In particular, we found that no abstraction of the specific structures really takes place with these students, and they continue to think in terms of vectors in R^n—the only vectors they recognize as mathematical objects. Consequently, they equate the vector-space axioms with the properties of vector addition and scalar multiplication in R^n. But, because they view the latter as self-evident facts that merit no special attention, they fail to comprehend the meaning of the vector-space axioms and of the basic properties derived from them. For example, when asked to reproduce the meaning of the statement, "For any A in a vector space V, $(-1)A = -A$" only a few LAa students did so successfully. The decisive majority of the students viewed this statement as much-ado-about-nothing, as one of the students put it: "Of course negative one times A is negative A, what is there to prove?" This observation suggests that *when students do not have a clear purpose for a concept, even when the concept is presented clearly, they are likely to misunderstand its meaning.*

One of the most positive results of our teaching experiments is that students' use of symbolic reasoning, justification by virtue of authority, and the like gradually diminished as they developed alternative, mathematically adequate modes of justification [**8**]. We attribute this change to the fact that students acquired conceptual tools to analyze situations and solve problems. In linear algebra teaching experiments, for example, row reduction was such a conceptual tool. Our students were assigned numerous problems on the structure of the reduced row echelon form and the meanings and implications of row reduction in questions of existence and uniqueness of solutions and in questions of independence and span. We let students work their way through the problems, justifying and rejustifying assertions they use. It was only at the end of an instructional unit that the teacher summarized the relevant key theorems and provided a coherent structure for what was learned. To solidify students' image of these theorems, we concluded each instructional unit with a set of review problems in which the students applied the newly formulated theorems. It is our judgment that this effort yielded fruitful outcomes: Because row reduction is a simple *computational* tool that students could easily understand, and, more importantly, because they *repeatedly* witnessed its power in solving problems, they successfully internalized it as a problem-solving heuristic. It became for them a dominant method of thinking about linear algebra problems, and, most importantly, it replaced the faulty ways of justification they previously possessed.

3.3 A Concluding Remark. A well documented observation in the cognitive science literature is that once a way of thinking is established as a behavior, it becomes tenacious and extremely difficult to relinquish [**3**]. The implication of this finding is that the seeds for good ways of thinking must be laid early on in students' mathematical experience—in elementary and secondary education. Students' mathematical education in these levels accounts for the presence (absence) of the undesirable (desirable) ways of thinking I have discussed here. Beginning in elementary school, students learn to add and subtract multidigit numbers without understanding the concept of place value [**4**], and they perform operations on fractions meaninglessly [**10**]. They continue to use symbolic reasoning in their secondary school years and fail, as result, to learn rudimentary ideas, such as what it means to solve an equation [**12**].

Our experience suggests, however, that—with effort—ways of thinking can be altered. Our teaching approach was guided by the Necessity Principle. To implement this principle successfully, we learned that one must set high, yet realistic, expectations from students, and must make students accept responsibility for learning. The mode of teaching that fits the Necessity Principle approach is—as I have said in [**7**]—a combination of small group discussion, team projects, whole-class discussion, individual learning, use of technology, and lecturing (yes, lecturing). There must be balance among these modes of teaching—a balance that each teacher has to develop individually. I recommend this combination of modes of teaching rather than dogmatic devotion to one single mode, nowadays either the mode of lecturing or the mode of cooperative learning.

We often hear the phrase, "students must not be passive receivers of information but active participants in knowledge construction". The Necessity Principle pours hard content into this otherwise trivial declaration. It suggests in concrete terms how to make students active learners, and how—speaking metaphorically—to transfer definitions and theorems from the ownership of the text and teacher to the ownership of the students.

The Necessity Principle has its roots in the Piagetian theory of learning and is consistent with the current theory of Problématique put forth by French mathematics educators [**2**]. According to the latter, for example,

> ... pupils' learning depends on their recognition and re-construction of problems as being their own.... A problem is a problem for a student only if she or he takes the responsibility for the validity of its solution. This transfer of the responsibility for truth from teacher to pupils' must occur in order to allow the construction of meaning. [**2**, p. 259]

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GUERSHON HAREL, born in Southern Tunisia, received his Ph.D. (1985) in mathematics from Ben-Gurion University, Israel. His research centers on the epistemology of mathematics and its application in mathematics curricula and teacher education. He is a professor at Purdue University, associate editor of the Monthly, site director of the Park City Mathematics Institute, and principal investigator of an NSF-funded research project on the concept of mathematical proof.
Purdue University, West Lafayette, IN 47907
harel@math.purdue.edu