The Linear Algebra Curriculum Study Group Recommendations: Moving Beyond Concept Definition

Guershon Harel[1]
Purdue University

Abstract

The Linear Algebra Curriculum Study Group (LACSG) have generated a set of recommendations for the first course in linear algebra [1]. Their headings are:

1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines.
2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course.
3. Faculty should consider the needs and interests of students as learners.
4. Faculty should be encouraged to utilize technology in the first linear algebra course.
5. At least one "second course" in matrix theory/linear algebra should be a high priority for every mathematics curriculum.

These recommendations have highlighted the need for a first course in linear algebra that would give students a solid understanding of this topic. This paper has three parts: Part I presents a set of indicators for such an understanding, whereas Part II points to factors essential to the building of it. The factors that will be discussed are:

- II1. The appropriateness of the time allocated to linear algebra.
- II2. Students' background and readiness for the kind of course called for by the LACSG, in regard to linear algebra ideas and language.
- II3. Students' background and readiness for the kind of course called for by the LACSG, in regard to the concept of proof.

Accordingly, in Part III, suggestions for instructional treatments will be made. In particular, these suggestions include:

[1] I am grateful to Merlyn Behr, Jane Day, Tommy Dreyfus, Richard Penny, and Jack Tedeski for the comments on an earlier draft of the paper.

III1. A proposal to make some changes in the U.S. high school mathematics curriculum that could greatly strengthen students' ability to learn linear algebra in college.

III2. A proposal to include the use of MATLAB in college calculus as a preparation to linear algebra.

III3. Specific recommendations for the teaching of proofs.

In the last section of Part III, Section III4, a general instructional principle on which these suggestions are based will be discussed.

I. Indicators for Effective Concept Image

I1. Concept Definition Versus Concept Image

Jacob, a math major at the end of his senior year, was given the problem:

> V is a 4-dimensional vector-space. U and W are subspaces of V, $\dim U = 2, \dim W = 3$. Can $\dim(W \cap W) = 0$?

Jacob responded: "No, because there must be an overlapping between the two." From a further conversation with him on why U and W should overlap, it turned out that he simply viewed U, W, and V as sets consisting of 2,3, and 4 items, respectively, from which he concluded the $U \cap W$ must be non-empty; the data concerning vector-space, subspace, and dimension was totally ignored by Jacob. When the instructor asked "Is $U \cap W$ a vector-space?", Jacob was unable to respond, nor was he able to respond to the question, "What is the definition of vector-space?" And when the instructor asked Jacob to give an example of a vector-space, he responded:

> A polynomial... The set of all polynomials of degree 10... The set of all polynomials of degree 0. No, the first is not, because when you multiply you get a higher degree so it is not closed.

Responses like Jacob's are not uncommon; they reflect the confusion students have with linear algebra concepts [2]. David Carlson [3] has observed a similar phenomenon with his linear algebra students, which he described "as if a heavy fog has rolled in over my students, and they cannot see where they are or where they are going."

Vinner [4] would say that students like Jacob have failed to build adequate "concept images" for the "concept definitions" we present them. A *concept definition*, according to Vinner, is a verbal definition, appearing in a textbook or written by the instructor on the blackboard, that accurately describes the concept in a non-circular way. A *concept image*, on the other had, is a mental scheme, a network, consisting of (a) what has been associated with the concept in the person's mind and (b) what the person can do in regard to the concept. It may include, for instance, analogies and relationships to other concepts, propositions on or relating to the concept, examples and nonexamples, ways of solving certain problems, etc. It is worth mentioning that concept images do not necessarily include spatial visualizations, as the term "image" may suggest. In fact, it has been found that some people possess effective concept images, and yet their mode of thinking is purely analytic, not spatially visual [5].

Generally speaking, to possess an effective concept image of a certain concept means to understand the concept. The most important indicator for understanding a concept is the ability to solve problems related to the concept, where by solving a problem, it is meant knowing *both* what to do and why. This indicator, however, is too general, since problems can be of different levels. For more specificity, we discuss three other, interrelated indicators for understanding a concept:

1. (a) the ability to *remember*, not just memorize
2. (b) the ability to *communicate* ideas in one's own words,
3. (c) the ability to *think in general terms*, and
4. (d) the ability to *connect* ideas.

A student with an effective concept image is one who has these abilities relative to the concept.

I2. Remembering

Unfortunately, many students place their full reliance on definitions, by attempting to memorize them verbatim, and experience, as a result, great difficulty in learning mathematics [6]. These students manage, as did Jacob, to remember the concept definition until the final exam is over. But in the absence of a concept image that sustains the concept definition, these students are unable to retain the concept definition for a long period of time. Once the concept definition is forgotten, they are unable to retrieve or rebuild it on their own. To demonstrate the role of an appropriate concept image in remembering a concept, consider the following example: In a teaching experiment in linear algebra—after several sessions in which the concept of vector-space was gradually generalized from two and three dimensional geometric vector-space models (see [2] and [7])—the students were asked the following question:

> Let U be a subspace of vector-space V, and let b be a vector in V but not in U. Is the set $U + b = \{v + b \mid v \text{ is a vector in } U\}$ a vector-space?

There were two groups of students: those who answered this question by checking the whole list of the vector-space axioms, and those whose answer was something like, "you moved the whole thing, it doesn't have the zero vector any more," or "the new thing $U + b$, is not closed under addition." Clearly, the latter group of students built a concept image of vector-space that enabled them to focus on those vector-space properties that are most relevant to the solution of the given problem, namely, the zero property or one of the closure properties. The other group of students, on the other hand, relied on the concept definition of vector-space by checking whether the individual axioms apply. That $U + b$ is a subset of the vector-space V, which guarantees the existence of most of the axioms, was not visible to these students. Moreover, many of these students failed to check some of the axioms, including those essential to the solution of the problem (e.g., the existence of zero).

I3. Communicating

This example demonstrates another important indicator for understanding. When a student builds an adequate concept image, not just memorizing a concept definition, the student is able to read and write about the concept and communicate it with others without feeling compelled to refer to the formal definition or the concept each time the concept's name is brought up. This is a stage in which the definition of the concept and all the experiences the student associated with it become interrelated and are captured at a glance by the concept's name. The conceptual development that leads to this stage is not immediate. We only need to reflect on personal experience when certain concepts were first introduced, in order to appreciate the non-immediacy of this stage. Imagine, for example, the cognitive strain on a student who has not formed an adequate concept image for some or all of the concepts, vector-space, dimension, null space of a matrix, and rank, when he or she hears the instructor saying, "... and since the dimension of null (A) for an $m \times n$ matrix A with rank r is $n-r$..." This is what I believe happens to many students in linear algebra: because of the insufficient time allocated to linear algebra, students fail to build adequate concept images that would enable them to build up a new linear algebra knowledge. We will return to this issue of insufficiency of time for linear algebra in the next section.

I4. Thinking in General Terms

Concepts are defined in general terms. But this does not guarantee that students would be able to think about them in general terms. Take, for example, the concept of linear independence. The reason for students' difficulty with this concept—a difficulty that is recognized by almost anyone who has taught linear algebra—is not so much in comprehending its definition per se, but in applying it to solve problems that involve non-specific vectors. Most elementary textbooks introduce this concept quite clearly, so students usually have little or no difficulty understanding the meaning of linear independence. Shortly after this concept is introduced, they can even solve simple problems, such as determining whether a *specific* set of vectors is linearly independent. For example, in my experience, when students are given a specific set of vectors, say in R^5, and asked to determine whether they are linearly independent, they set up the problem correctly and understand *why* this question is equivalent to the question whether $AX = 0$, where A is the matrix whose columns are the given vectors, has only the trivial solution. Their real difficulties, however, begin when they are asked to solve problems that involve *non-specific* vectors, such as the following:

1. Are nonzero, mutually orthogonal vectors linearly independent?
2. If S is the span of linearly independent vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ in V and α is a vector in V but not in S, are the vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ linearly independent?
3. Do elementary operations on rows of matrices affect their linear independence (dependence)?

For the reader of this paper, there is no real advantage to thinking in terms of n-tuples or any other specific vectors when solving these problems. This fact is not likely to be noticed by the students for some time, and they would continue to rest their reasoning on images of concrete objects, long after the respective concept definitions were introduced to them.

The terms "specific vectors" and "concrete objects" need clarification. It has been established that students may succeed in solving a certain problem in one specific context but fail to solve the same problem in another *specific* context (see for example, [8]). For example, the reader may have observed students correctly solve problems that ask to determine whether a specific set of row vectors is linearly independent—most commonly by applying the Gauss Elimination process to the matrix whose rows are the given vectors—but have difficulties determining if another kind of specific set of vectors—a set of polynomials, for example—is linearly independent. How can we explain this phenomenon?

Recall that according to this paper's definition, understanding must imply knowing why, not just how. So if a student thinks of "linear independence" to mean "the echelon matrix which results from elimination has no rows of zeros," without being able to mathematically justify this connection, then he or she does not understand the concept of linear independence. But this conclusion is not quite accurate, for understanding is context dependent; that is, a student may understand a concept in one context but not understand it in another context. For example, it is possible that a student understands the idea of linear independence in the context of R^3 as a geometric space of directed line-segments, but not in R^5 as an algebraic space. Or, the student may understand it in R^5 but not in $P_5[x]$. Indeed, it is common phenomenon that students set up the equality

$$a(3x^2 - 1) + b(2x) + c7 = 0 \qquad (*)$$

for the problem,

$$\text{Is the set } \left\{3x^2 - 1, 2x, 7\right\} \text{ linearly independent?} \qquad (**)$$

without having a clear idea what they are doing. Their difficulty is that they cannot apply the concept of linear independence to functions because the concept of function as a vector is not *concrete* to them; namely, they have not formed the concept of function as a mathematical object, as an entity in a vector-space [7]. As a result, the idea that the zero on the right side of (*) is a function, the zero vector in the vector-space $P_5[x]$, not the number zero, is inaccessible to them, and they cannot interpret (*) as an *identity* between the two functions; rather, they view it as an *equation* in x. Once the students form the concept of function as a mathematical object, the elements of $P_5[x]$ would become concrete to them and they would be able to solve problems such as (**).

So when we say "a specific set of vectors," we mean specific to the students. For a set to be specific, its elements must be concrete (to the students). Students build their understanding of a concept in a context that is concrete to them. Such a context serves both as an anchor to building adequate concept images and a springboard to further abstraction.

I5. Connecting

No concept image is viable without mathematical connections. One of the most appealing aspects of linear algebra—yet a serious source of difficulty for students—is the "endless" number of mathematical connections one can (must) create in studying it. Relationships between systems of linear equations, matrices, and linear transformations can be built in numerous ways, and problems about systems of linear equations are equivalent to problems about matrices which, in turn, are equivalent to problems about linear transformations. In this respect, linear algebra is different from any other lower division topic in mathematics. A comparison to calculus can demonstrate this unique feature of linear algebra. For example, there are many more theorems in a one-semester linear algebra course than in one semester of calculus. The number of theorems in liner algebra that deal with equivalencies is much greater than that in calculus. Further, although calculus textbooks vary a great deal in many aspects, they seem to widely agree about the order that the content should be presented. There are major differences, however, among linear algebra textbooks in their choice of order topics. This undoubtedly is due to the unique multiperspective nature of linear algebra.

To summarize: In order for linear algebra students to develop effective concept images, they must learn to not just memorize concept definitions but must construct concept images that will enable them to *remember* what they learn, *think in general terms*, *communicate* and *connect* mathematical ideas. I believe that the LACSG recommendations echo this need. But I suggest that these recommendations be supplemented—not altered as some people have suggested—with additional curricular actions to insure their success.

II. Factors in Building Effective Concept Image

In this section, I will discuss three factors essential to the building of effective concept images in linear algebra.

1. The appropriateness of the time allocated to linear algebra.
2. Students' background and readiness in regards to content (objects, language, and ideas that are unique to linear algebra)
3. Students' background and readiness in regards to the concept of proof.

II1. The Appropriateness of the Time Allocated to Linear Algebra

For most students, the construction of an effective concept image is a long and painstaking process. It is not always easy for us, as teachers, to realize this fact, for, as Piaget [9] pointed out, a concept is deceptively simple when it has reached its final equilibrium, but its genesis is much more complex. The building of an effective concept image in linear algebra requires a major effort and sufficient time on the part of the students as well as their teachers. Yet, we allocate only one course in the entire undergraduate mathematics curriculum to linear algebra. In comparison, as Alan Tucker [10] has pointed out, we devote an entire year and a half of the lower division core in mathematics to calculus. Even with this amount of time, calculus is still difficult for students, a fact which raises doubts on the sufficiency of the time allocated to linear algebra.

In the case of calculus, we understand that students must build solid concept images for one-variable calculus concepts and, rightly so, we devote two courses to this goal, before we introduce multivariable calculus. For example, we understand that students must gradually abstract the idea of derivative by first dealing with it extensively in the case of one-variable functions, then abstract it into higher, yet spatially imaginable, cases (i.e. real-valued functions $f : R^2 \to R$ and $R^3 \to R$), and only then move to general functions $f : R^n \to R^m$. On the other hand, we do not seem to have the same patience for the abstraction process in linear algebra. Nor do we take into consideration the obvious fact that linear algebra concepts are indispensable for understanding many multivariable calculus ideas. In most cases, ideas that require linear algebra background are shuffled under separate sections or exercises labeled "Optional."

II2. Students' Background and Readiness in Regard to Linear Algebra Content

The imbalance between the time allocated to calculus and that allocated to linear algebra is, in fact, even greater because high school mathematics is geared toward calculus needs more than linear algebra needs. This argument may not be true if examined solely from the viewpoint of *content.* High school curricula do include topics such as systems of linear equations, analytic geometry, and Euclidean space; all are part of linear algebra. But these topics are taught in high school in ways that have little to do with the basic ideas of linear algebra. High school students are not prepared for the objects, language, ideas, and ways of thinking that are unique to linear algebra.

From students' point of view, calculus is a natural continuation of the mathematics they learned in high school. After all, students deal with real numbers and functions of real numbers in high school, and continue to deal with these objects in calculus. Also, they are often impressed by the power of calculus tools to help them solve problems in familiar domains, such as finding the area of non-standard figures, or modeling projectile motion. In contrast, students make little or no connection between the ideas they learn in linear algebra and the mathematics they learn in high school. In the current situation, the only connection that potentially exists between high school mathematics and linear algebra is the study of systems of linear equations. But even this connection is superficial. High school students' involvement with systems of linear equations amounts to learning a solution procedure for 2×2 and 3×3 systems. They do not deal with matrix representations of these systems, questions about existence and uniqueness of solutions, relations to analytic geometry of lines and planes in space, geometric transformations, matrix algebra and determinants, etc. Evidence that students place a low value on the relevance of linear algebra for high school mathematics can be derived from a recent survey of mathematics education graduates. In this survey, 45% of the respondents believed that the value of linear algebra to their profession is marginal or useless, in contrast to an average of only 13% who thought so about calculus [11].

To demonstrate what the discontinuity between high school mathematics and linear algebra entails, let me focus on one aspect of this discontinuity. Students in high school deal with real numbers and continue to deal with the same type of objects in calculus. Real numbers, for all purposes of high school mathematics and elementary calculus, represent either ratio quantities, such as speed, density, price, and probability, or magnitude quantities, such as time, weight, length, and cost. Accordingly, the symbolic representations for these objects are one-dimensional. In linear algebra, on the other hand, new types of objects are added to the play; n-tuples, matrices, and functions as entities of a vector-space. These, in contrast to real numbers, represent multidimensional quantities, such as probability vectors and price vectors, directed graphs, and solutions of a differential equation that models the effect of temperature change. According to the LACSG recommendations, the first course in linear algebra should be matrix-oriented; therefore, students would be required to develop, in a relatively short period of time, a *spatial symbol manipulation* ability they never acquired before. For example, consider the statement:

> $RX = 0$, where R is a row reduced echelon matrix with r non-zero rows in which the leading entry of row i occurs in column $k(i)$ and X is a column vector. This system consists of r non-trivial equations in which the unknown $x_{k(i)}$ occurs with non-zero coefficient only in the ith equation.

To comprehend this statement, we need to carry out several mental activities, among which (a) we visualize the matrix R and the positions of the leading entries, (b) mentally carry out the product of R with a column of unknowns, (c) visualize the corresponding positions of the unknowns in the system of equations $RX = 0$, etc. Even when we express each of these steps on paper, we must first imagine and carry them out mentally; otherwise, they become entirely mechanical for us without our seeing the overall structure. In the same manner, take the useful formulas for computing a row $C_{(i)}$ and column $C^{(i)}$ in the matrix products $C = AB$.

1. $(AB)_{(i)} = A_{(i)}B = \sum A_{(i)k}B_{(k)}$
2. $(AB)^{(i)} = AB^{(i)} = \sum A^{(k)}B^{(i)k}$

Students may be able to verify these formulas by a direct computation of the expressions involved. But to make these formulas part of their concept image so that they can apply them on their own and appreciate their usefulness, they need to develop a feel for the relations expressed by them. That feel involves, in part, spatial symbolic manipulations of the different components in these formulas.

The above statement and the latter formulas may not seem to be difficult to us. Even so, experience shows that the language of linear algebra and the new way of symbol manipulation take time to become part of the student's repertoire. But the problem is that in the midst of their struggle to adapt to this new environment, students are introduced to complex ideas, such as linear independence, spanning set, and subspace. That is when, using David Carlson's words, the "fog begins to roll in" and students lose track of what they are learning [3].

II3. Students' Background and Readiness in Regard to the Concept of Proof

The LACSG recommendations have set forth the standard for the first course in linear algebra to be an intellectually challenging course, with careful definitions and statements of theorems, and proofs that enhance understanding. From a cognitive and pedagogical viewpoint, a linear algebra course that stresses proofs is both a necessity and a challenge. It is a necessity because the emphasis on proofs is indispensable for the development of rich and effective concept images in linear algebra. Without understanding the reasoning behind the construction of concepts and the justification of arguments, students will end up memorizing algorithms and reciting definitions. It is a challenge because, as we all know, proofs are a stumbling block for many students. Research has shown that many students carry serious misconceptions about proofs. For example, students do not understand that inductive arguments are not proofs in mathematics [12]; they do not see the need for deductive verifications [12 and 13]; they are influenced by the ritualistic aspect of proof [12]; and they do not understand that a proof confers on it a universal validity, excluding the need for any further checking [14]. This situation requires, therefore, careful considerations and a special attention to the teaching and learning of mathematical proof.

In the current situation, the first course in linear algebra, if it emphasizes proofs, would be students' first experience with algebraic proofs, because calculus often is being taught proof-free and, traditionally, the idea of proof, as a deductive process, where hypotheses lead to conclusions, is stressed in the teaching of geometry but not in the teaching of algebra. Philip Davis and Reuben Hersh [15] pointed out that "as late as the 1950s one heard statements from secondary school teachers, reeling under the impact of the 'new math,' to the effect that they had always thought geometry had 'proof' while arithmetic and algebra did not." The death of the "new math" put an end to algebra proofs in school mathematics.

In the last few years, I have been working on the epistemology of the concept of mathematical proof with students at various levels. One of the conclusions coming from this work is that a major reason that students have serious difficulties producing, understanding, and even appreciating the need for proofs is that we, their teachers, take for granted what constitutes justification and evidence in their eyes [17]. Rather than gradually refine students' conception of what constitutes evidence in mathematics, we impose on them proof methods and implication rules that in many cases are extraneous to what convinces them. This begins when the notion of proof is first introduced in high school geometry. We demand, for example, that proofs be written in a two column format, with formal "justifications" whose need not always are understood by a beginning student (e.g. Statement: $\overline{AB} \cong \overline{AB}$. Reason: Reflexive property of segment congruence). Also, we present proofs of well stated, and in many cases obvious, propositions, rather than ask for explorations and conjecturing. As a consequence, students do not learn that proofs are first and foremost **CONVINCING** arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate, and that is their responsibility to read proofs and understand the motivation behind them.

No one can expect to remedy students' misconceptions and "fill in" other missing conceptions about proofs in one single course. To meet the challenge to teach a linear algebra course that emphasizes proof, we must succeed in educating our students throughout the mathematics curriculum in school and college to appreciate, understand, and produce proofs. The movement towards this important goal cannot start in the first course in linear algebra; it must begin in the high school years and continue into the calculus courses. In fact, with a careful approach and a suitable level, we should begin educating students about the value of justification (not mathematical proof, of course) in the elementary school years. Despite this, I believe that an emphasis on proof in the first course in linear algebra, as was recommended by the LACSG, is vital.

III. Instructional and Curricular Recommendations

In the previous section, I have discussed three factors essential to the building of effective concept images in linear algebra: The appropriateness of the time allocated to linear algebra; students' background and readiness in regards to objects, language, and ideas that are unique to linear algebra, and students' background and readiness in regards to the concept of proof. In the following three sections, I will make several suggestions for instructional treatments that address each of these factors. In the first two sections, I will address the need to prepare students for the unique environment of linear algebra prior to their first exposure to this topic in college. Two ideas will be discussed: one deals with the need for and feasibility of incorporating linear algebra in high school (Section III2). The other, apart from this, suggests how, prior to their first course in linear algebra, students can be acquainted with the environment of linear algebra (Section III1). In Section III3, several instructional treatments for the concept of proof will be discussed.

III1. MATLAB, Calculus, and Linear Algebra

The educational benefit of incorporating technology in the teaching of mathematics has become, through both research in mathematics education and informal observations, apparent. In this regard, the LACSG recommendation that technology be incorporated into beginning linear algebra courses is expected. I suggest to go a step further and incorporate a matrix package like MATLAB in the teaching of calculus. This, as I will explain below, would benefit students in both linear algebra and calculus.

There are many computer packages that may well serve the purpose discussed in this section—XMath, Mathematicia, Maple, Derive, MathCad, and Xplore, to mention a few. My own experience, however, is primarily with MATLAB, which I have found to be a very adequate. I will report on my experience with this package.

MATLAB is an interactive system and programming language for scientific and technical computation. It boasts features that make it an excellent tool for achieving the double purpose of strengthening students' understanding of concepts in calculus and acquainting students with the linear algebra environment. Of particular importance is that MATLAB's basic data element is a matrix. This requires that solutions to problems be approached vectorally. The extent to which the vectorization feature of MATLAB is utilized varies with experience. However, with direction and encouragement, students learn to gradually utilize this important feature. The pedagogical benefit is that in writing MATLAB programs, students learn to represent problems and design and interpret their solutions in the language of vectors and matrices. In the course of doing so, they must *imagine* the actions carried out by the computer in response to their commands and *anticipate* the outcomes of these actions. These mental activities of *imagination* and *anticipation* involve mental manipulations of vectors and matrices that constitute an important component of the linear algebra environment. The best way for one to appreciate this pedagogical value is to write several MATLAB programs oneself and reflect, in the course of this exercise, on one's mental activities. For this, the book *Matrices and Matlab*, by Marvin Marcus [16], is an excellent source.

Since I am suggesting that MATLAB be incorporated in calculus, I should add that there exists anecdotal evidence of the pedagogical benefit of MATLAB in the learning of calculus and linear algebra. Allen Weitsman from Purdue University is one person who has been incorporating MATLAB into his calculus classes for several years. I have taken a close look at Weitsman's use of MATLAB in a sequence of three consecutive courses: Calculus of One Variable, Multivariable Calculus, and Linear Algebra and Differential Equations. These were small-class courses taken by the same students. I have recognized through these observations the benefit of MATLAB in strengthening students' calculus and linear algebra concepts. Space does not permit to describe in detail the benefit of his approach to the conceptualization of calculus ideas. However, based on my own analysis of the projects he assigned to his students, detailed observations of one of Weitsman's students, and many conversations with Weitsman, I can point to some pedagogical values. These include the understanding of (a) the idea of Riemann sum in one- and two-variable functions, (b) three-dimensional graphing, (c) methods of approximation, including Taylor series for two-variable functions and numerical solutions to differential equations and (d) the concept of limit.

With regard to the benefit of MATLAB to the learning of linear algebra, it is interesting to note that despite the fact that the topic "block matrices" was not covered in any of these courses,Weitsman's students developed a tendency to approach, and successfully solve, problems by matrix partitioning. This may be attributed to the experience these students have had with MATLAB, which helped them to develop the spatial symbol manipulation ability I have discussed earlier (see Section II2).

Some of the features which one needs in a software package so that it is good for both calculus and linear algebra include the following, and MATLAB has these features. The following remarks are from [17].

1. MATLAB is currently used in nearly every engineering school in the United States.
2. The superb built-in Help facility in MATLAB makes it simple, and nearly instantaneous, for the reader to look up any available command or structure.
3. Programming in MATLAB should be relatively transparent to anyone who has done a month of BASIC programming in high school.
4. It is a sophisticated numerical package with dozens of special built-in commands and functions.
5. It provides the student with a powerful interactive tool to examine significant examples, to strengthen intuitive insight, and to formulate and study plausible conjectures. These

are "mathematical" rather than "computing" activities, but they cannot be carried out in any but the most trivial situations without an effective scientific computing package.

Because of these features, especially the last one, the incorporation of MATLAB into the first course in linear algebra is most natural and meets the LACSG recommendation to utilize technology.

There is a need, however, to further experiment with the idea of incorporating MATLAB, or other similar software packages, into calculus and later into linear algebra, and investigate:

1. The benefit to the learning of calculus.
2. The extent to which the use of MATLAB in calculus prepares students for their first course in linear algebra.
3. The success of MATLAB in helping build effective concept images in the first course in linear algebra.

III2. Teaching Linear Algebra in High School

I believe that there should be an introduction to linear algebra in high school. The idea may seem too ambitious to some. But, if we believe in the pedagogical importance of and the need for a continuity between high school mathematics and college mathematics and we recognize the problem of the insufficient time allocated to linear algebra in college, then it should be clear that an introductory treatment of linear algebra in high school is a necessity. Introducing linear algebra in high school does more than prepare students to do matrix algebra and compute determinants. It will lay the grounds for building rich and effective concept images for linear independence, spanning set, vector space, and linear transformation. Students who would go through a linear algebra program in high school will become motivated and more cognitively prepared to abstract these and other ideas in their first college course in linear algebra. This is a worthy long-term investment which requires a restructuring of the existing high school mathematics curricula. I suggest that this restructuring take place in two arenas: First, the traditional high school topics, such as systems of linear equations, analytic geometry, and Euclidean space, should be taught from a linear algebra viewpoint. Second, given the imbalance currently existing between the time allocated to linear algebra to that allocated to calculus, I suggest to reduce the high school program in calculus.

The following is a very brief description of a linear algebra program which I have developed for and implemented with upper-level high school students. Results from several teaching experiments showed that the program was highly successful. (For more details, see [2] and [17].) The program consists of three phases: The first phase deals with the geometric spaces of directed line segments. The objective of this phase was to begin to build effective, visually-oriented concept images for important linear algebra concepts. Further, the construction processes of these concepts were chosen to be analogous to their construction in the general case. For example, when dealing with the 3-dimensional geometric model, a basis can be defined as three non-coplanar line segments. But such a definition is restrictive and model-dependent, since it does not transfer to vector spaces in general. In this program, this process is fully explored by showing how the concept of basis in a geometric model is developed from a spanning set.

After the central concepts of dependence, independence, linear combination, basis, and dimension are thoroughly studied in the line, plane, and space, I introduced the second phase in which the algebraic spaces R^1, R^2, and R^3 along with these central concepts, are built through the idea of vector coordinates. In fact, the unit dealing with these algebraic spaces was mostly constructed by the students themselves through individual and team projects. One of the pedagogical values of this stage is that students could see in a concrete environment how one mathematical system can be transformed into another system which is more amenable to computational techniques. The resulting system is, of course, isomorphic to the original.

Finally, in the third phase, these central concepts are defined a third time in R^n, with the introduction of linear transformations and their matrix representations in R^2 and R^3. More specifically, this phase dealt with: (a) analytic geometry of the plane and space; (b) systems of linear equations (algebraic and geometric investigations); (c) linear transformations in the plane and their matrix representations, and (d) matrix algebra.

III3. Instructional Approach to Proof

The following are suggestions with illustrative examples of how we can make proofs a tangible experience for students.

1. *Students should take an active part in the construction of relations between ideas and in the production of their justifications.* Instructors should set the tone for this approach right at the beginning of the course and continue throughout. At first, students may resist this approach because it does not meet their expectation and definition of a class-session. For this reason, it is advised to begin with instructional activities that are elementary enough that all or most students can participate. For example. students could be assigned the homework problem:

Let

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -6 & 1 & 0 \end{pmatrix}$$

Find for which (y_1, y_2, y_3, y_4) the system of equations $AX = Y$ has a solution. You are encouraged to use MATLAB.

There will be students who would use a direct approach to solve this problem by row-reducing the matrix $[A \quad Y]$ into $[A' \quad Y']$ and solving for Y'. (In this approach, there is, obviously, no gain from using MATLAB.) The answer to the problem (if it is derived from a row-reduced-echelon matrix) is:

$$Y \text{ is a linear combination of } Y_1 = (-1, 3, 1, 0) \text{ and } Y_2 = (3, -2, 0, 1).$$

There will be other students who would use a different method. Using MATLAB, these students may find a basis for the column space by applying the rref command to A^T. Their answer would be:

$$Y \text{ is a linear combination of } Z_1 = (1, 0, 0.2857, 0.4286) \text{ and } Z_2 = (0, 1, 0.4286, 0.1429)$$

In the class session. these solution approaches would be presented by the students (if the students had not chosen the second approach, it would be presented by the instructor), and a class discussion would follow. In addition, the relation

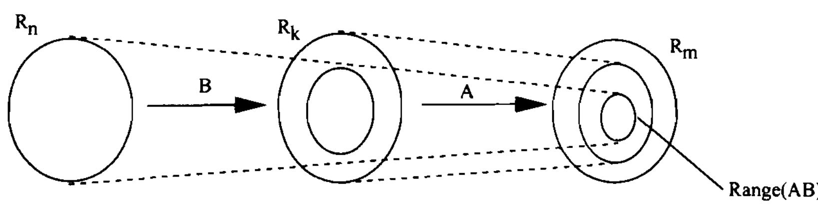
$$\text{rref}([Y_1 \;\; Y_2]) = [Z_1 \;\; Z_2]$$

may be brought up either by one of the students and, if not, by the instructor. (If a computer display system is available, this can be displayed instantly.) The learning opportunities that this activity can offer to students are apparent and valuable, I believe. They are valuable not only because students would learn a specific mathematical connection (the connection between a system $AX = Y$ and the column-space of A) or a specific mathematical property (the property that a subspace has more than one spanning set), but also because the student took an active part in discovering these ideas.

2. *Students should be helped to build proofs on their intuitions.* Consider, the following example: After running several examples in MATLAB, students can easily conjecture the relation:

For two matrices A and B, $rank(AB) \leq \min[rank(A), rank(B)]$.

This relation can be quite intuitive to the students if they are encouraged to think of a matrix as a linear transformation and consider the geometrical meaning of rank, as shown in the following figure:



Occasionally, textbooks and instructors present a geometrical explanation of this kind, and succeed, as a result, to convince the student in the truth of the relation. Rather than building on this understanding by expanding it into a more formal proof, they often abandon it and start from scratch an analytic proof. Even when students understand all the steps of the analytic proof, it often happens that they do not see a connection between this proof and the geometric explanation. Once the student is convinced of the truth of a relation through an intuitive explanation, with the help of the instructor if necessary, the challenge is to refine this explanation into an analytic proof.

3. *Students should be encouraged to read proofs.* I don't know of any study about undergraduate students' ability to read mathematical texts, but based on my own and some of my colleagues' observations, it does not seem that we are giving enough attention to this important goal. Carl Cowen [18] has suggested an approach of how to encourage students to read mathematics by including on the exam new theorems, together with their proofs, followed by questions that can be answered only by those who read and understood the theorems and proofs. This idea, I believe, should be extended to homework as well as classroom activities. One of Cowen's example is from linear algebra. I will present it here in its entirety [18].

> This example comes from a linear algebra course for engineers. At the time of the test, the students had studied the spectral mapping theorem for polynomials which implies that the positive integer powers of a diagonalizable matrix are similar to the powers of the diagonal matrix, but had not seen the theorem on square roots of positive definite matrices.
>
> **Theorem.** If A is a diagonalizable matrix all of whose eigenvalues are non-negative, then there is a matrix B with non-negative eigenvalues such that $B^2 = A$.
>
> **Proof.** Since A is diagonalizable, there is an invertible matrix S such that $L = S^{-1}AS$ is diagonal. The diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ of L (which are the eigenvalues of A) are non-negative by hypothesis. Let $\mu_j = \sqrt{\lambda_j}$ be the non-negative square roots of the eigenvalues and let M be the diagonal matrix with diagonal entries $\mu_1, \mu_2, \ldots, \mu_n$. Clearly, $M^2 = L$. Since similar matrices have the same eigenvalues, the matrix $B = SMS^{-1}$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, which are non-negative. Moreover,
>
> $$B^2 = (SMS^{-1})^2 = SMS^{-1}SMS^{-1} = SM^2S^{-1} = SLS^{-1} = A$$
>
> **Problem**: The matrix
>
> $$A = \begin{pmatrix} 10 & -9 \\ 6 & -5 \end{pmatrix}$$
>
> has eigenvalues 1 and 4. Find a matrix S as above and use it to find a matrix B with positive eigenvalues such that $B^2 = A$.
>
> In this question, I expect the students to understand from the proof of the theorem that the problem is to be solved by diagonalizing A and solving the corresponding problem for the diagonal matrix. In particular, I expect them to use the given eigenvalues of A to find a basis of eigenvectors, and thereby to construct S, L, M, and B.

In the standard approach, students are asked to practice the application of theorems' results, not theorems' proofs. In this respect, Cowen's idea is refreshing.

Another crucial contributor to students' ability (or inability) to read mathematical texts is the way textbooks are written and structured. Many textbooks are written in a style that is more appropriate for a research paper than a text that is intended for sophomore or junior mathematics majors. The textbooks' role is not just to present their material logically and succinctly, but also to share with the students the thought processes behind the choice of definitions, the intellectual need for the theorems, and the motivations for the proofs.

Other textbooks have the tendency to redundantly multiply theorems. By formulating many results as theorems, they believe students would have an easy access to the results that are needed to solve problems. I believe that this approach can handicap students' learning. For, in so doing, these textbooks can end up training students to become theorem-searchers and theorem-users rather than theorem analyzers. In addition, it is difficult for a beginning student who uses such a text to see the trees from the forest. Textbooks should help students to develop the sense that elementary linear algebra can be structured around a few central and unifying ideas. Such a sense can be built only by a strong emphasis on proofs where students would come to realize the centrality of ideas such as linear independence, spanning, elementary operations, vector-space, and linear transformation.

4. *Students should learn that understanding a proof is more than understanding each of the proof's steps.* Following Cowen's suggestion, we can give problems that require a deep understanding of the proof they are asked to read. For example, to solve the following problem, one must understand the core idea behind the proof presented in (3):

> **Problem**: A is similar to D, and f is a polynomial for which the equation $f(X) = A$ has a solution. Prove that $f(X) = D$ has a solution.

This does not mean that the promotion of this goal should be done only with complex problems. On the contrary, the insistence on understanding the ideas behind the proof should begin with simple proofs, such as in the following example: Students are presented with the definitions of a left inverse and a right inverse of a matrix, followed by the following theorem and its proof.

> **Theorem**. If A has a left inverse B and a right inverse C, then $B = C$.
>
> *Proof.* Suppose $BA = I$ and $AC = I$. Then $B = BI = B(AC) = (BA)C = IC = C$.

The statement of this theorem is simple but somehow surprising, especially to one for whom matrix multiplication is still a peculiar operation. The proof too is simple and its steps can be understood easily by most students, but would it demystify the theorem for the students? The challenge, I must emphasize, is not to look for a "better" proof, which in some cases is advisable. The challenge is to bring students to (a) question the expectedness of a theorem and (b) insist to be convinced by its proof, not just understand each of its steps. As teachers, we would be delighted to see students complaining that they don't understand how the use of associativity of matrix multiplication is what is necessary to prove the theorem and then see them turning to a general 2×2 case to examine why the proof proves. Unfortunately, because of the reasons I mentioned earlier, students are usually passive toward theorems and proofs, so it is unlikely that they would raise any questions at all.

A reader who is a teacher may be skeptical about the applicability of these four suggestions because of the amount of class time they require. I agree with the reader's observation. But is there really any alternative?

III4. A General Instructional Principle: The Necessity Principle

The instructional treatments I have suggested in this paper are based on the needs and interest of students as learners—an approach that was highlighted by the LACSG in Recommendation 3. The main principle behind these instructional treatments is a principle called the *necessity principle*, which states that for students to learn, they must see a need for what we intend to teach them [2,19]. By "need" I mean an intellectual need, as opposed to a social or economic need.

The necessity principle pertains to any act of teaching and learning. My approach to the teaching of proof, for example, rests on this principle. In dealing with this concept, I begin with whatever the current understanding of proof for the student is; that is, what he or she conceives as a means for conviction. Often, students' certainty and conviction of the truth of an argument is not based on logical and deductive implications, but on empirical evidence, intuitions, or even authority (e.g. an argument is true because the teacher said so). My goal as an instructor is to bring the students to doubt their beliefs about what constitutes a truth, whereby they would appreciate a need to justify an argument. The development of the concept of proof is, therefore, a process of continuing refinement of students' doubts; its culminating point occurs when rigor and an axiomatic approach to justification are fully appreciated by the student.

In a different context, an example of a violation of the necessity principle would be "deriving" the definition of vector-space from a *presentation* of the properties of R^n that correspond to the vector-space axioms [2]. This statement does not hold for an advanced student, who understand the role of the postulational approach in mathematics. It does not hold for a beginning student, one who has yet to witness the economy of thought in thinking in terms of vector-space axioms. For this student, properties in R^n are self evident: thus, they do not deserve the attention they get. A more adequate approach that can lead students to better appreciate the properties of vector-space is involving students in an investigation of the solutions of homogeneous system of linear equations. Students do appreciate the need to solve equations and the need to find all their possible solutions. In this investigation, they would realize that the set of these solutions is infinite, but it forms a structure that enables us to generate the entire set from a "small" number of solutions. This can be a revelation for the students.

As students progress into their study of linear algebra, they come to appreciate this idea and other uses and roles of the concept of vector space, especially its role as a unifying concept. But they can arrive at this appreciation only by solving problems; for example, problems such as those presented in the section "Thinking in general terms," where they learn that their solutions are independent of any specific model, or problems through which they learn that in some cases it is more advantageous to think in terms of the general vector-space than in terms of a specific model. As examples of the latter, consider the problems: Given a linear operator T on V, when is T diagonalizable? When is T orthogonal? In searching for solutions to these problems, there is no advantage in thinking of V as R^n. On the contrary, thinking in terms of R^n may add concerns that are irrelevant to the problem in hand. For example, if the student thinks in terms of R^n, he or she may be tempted to involve the standard basis—because of its "special" feature in representing vectors—in their search for solutions.

The key factor in the application of the necessity principle is students' involvement in solving problems that are theoretical and challenge their understanding and creativity, as well as the active involvement in the construction of definitions,theorems, and proofs. The mode of teaching that fits this approach is, I believe, a *combination* of small-group discussion and team projects, whole-class discussion, individual learning, use of technology, and lecturing (yes, lecturing). The focus here is on the balance among these modes of teaching—a balance that each individual teacher has to develop for herself or himself. I emphasize this combination of modes of teaching against the fanaticism toward one single mode, nowadays either the lecturing mode or the mode of cooperative learning.

Summary

This paper touches upon certain aspects of each one of the LACSG recommendations (see the Abstract):

The discussion in [1] of Recommendation 1 highlights the need for an intellectually challenging course, with emphasis on proofs that enhance understanding. I have suggested that the focus on proofs should not begin in the first course in linear algebra; rather, with a careful approach and appropriate level of rigor that matches students' ability, it must be throughout the mathematics curriculum. I have also made some suggestions of how we can help make proofs a tangible experience for the students.

Recommendation 5 can be viewed as a recognition by the LACSG that the time allocated to linear algebra in the current undergraduate mathematics curriculum is insufficient. In my opinion we should adopt this recommendation and go beyond it by incorporating basic, geometrically-oriented, linear algebra ideas in high school mathematics. This suggestion, if implemented, would make the first course in linear algebra a natural continuation of what students have learned in high school. Accordingly, it could build on geometrically-based concept images of linear algebra already possessed by students.

I have also suggested incorporating MATLAB (or any other similar software package) in the teaching of calculus for the double purpose of strengthening students' understanding of concepts in calculus and acquainting students with the linear algebra environment. In particular, because MATLAB's basic data element is a matrix, programming in MATLAB can help students make n-tuples and matrices concrete. This would prepare students for a matrix-oriented course as was suggested in Recommendation 1. Also, I have indicated, as other people have already done, that the use of MATLAB in the teaching of linear algebra is most natural. These suggestions are in line with Recommendation 4, which calls for a utilization of technology in linear algebra course.

References

1. David Carlson, Charles Johnson, David Lay, and A. Duane Porter, The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra, *College Mathematics Journal* **24**, (1993) 41–46. Also in this volume.
2. Guershon Harel, *Teaching Linear Algebra in High School*, Unpublished doctoral dissertation, Ben-Gurion University of the Negev, Beer-Sheva, Israel, 1985.
3. David Carlson, Teaching linear algebra: must the fog always roll in? *College Mathematics Journal* **24**, (1993) 29–40. Also in this volume.
4. Shiomo Vinner, Concept definition, concept image, and the notion of function, *International Journal of Mathematics Education, Science and Technology* **14**, (1985) 293–305. Also in this volume.
5. Theodore Eisenburg and Tommy Dreyfus, On visual versus analytical thinking in mathematics, *The Proceeding of the Tenth International Conference on the Psychology of Mathematics Education*, University of London, (1986) 153–158.
6. Robert Davis, *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Ablex Publishing Corporation: New Jersey, 1984.
7. Guershon Harel and James Kaput, The role of conceptual entities in building advanced mathematical concepts and their symbols. In D. Tall(Ed), *Advanced Mathematical Thinking*, Ridel, Dordrecht, Holland, (1991) 82–94.
8. Ed Dubinsky and Guershon Harel, The nature of the process conception of function, *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA Notes, Vol. 25.
9. Jean Piaget, Barbel Inhelder, and Alina Szeminska, *The Child's Conception of Geometry*, Basic Books, New York, (1960).
10. Alan Tucker, The growing importance of linear algebra in undergraduate mathematics, *College Mathematics Journal* **24**, (1993), 3–9.
11. Office of Academic Services, School of Education, Purdue University.
12. Gary Martin and Guershon Harel, Proof frames of preservice elementary teachers, *Journal for Research in Mathematics Education* **20**, 41–51.
13. Shlomo Vinner, The concept of exponentiation at the undergraduate level and the definitional approach. *Educational Studies in Mathematics* **8**, (1977) 17–26.
14. Efraim Fischbein and Ehood Kedam, Proof and certitude in the development of mathematical thinking, *Proceedings of the Sixth International Conference of the Psychology of Mathematics Education*, Antwerp, Belgium, Universitaire Instelling Antwerpen, (1982) 128–131.
15. Philip Davis and Reuben Hersh, *The Mathematical Experience*, Houghton Mifflin, (1982).
16. Marvin Marcus, *Matrices and Matlab*, Prentice Hall, (1992).
17. Guershon Harel, Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes, *Focus on Learning Problems in Mathematics* **11**, (1989) 139–148.

18. Carl Cowen, Teaching and testing mathematics reading, *Amer. Math. Monthly* **98**, (1991), 50–53.
19. Guershon Harel, Applying the principle of multiple embodiments in teaching linear algebra; aspects of familiarity and mode of representation, *School Science and Mathematics* **89**, (1989), 49–57.
20. Ed Dubinsky, Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed), *Advanced Mathematical Thinking*, Ridel, Dordrecht, Holland, (1991).