The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction [1, 2, 3]

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Abstract

Students' conceptions of mathematical induction in a standard teaching approach were found to be largely manifestations of deficient proof schemes, such as the authoritative and the symbolic non-quantitative proof schemes. On the other hand, when a fundamentally different instructional treatment of mathematical induction was implemented in an elementary number theory course taught to prospective secondary teachers, students' conception of mathematical induction developed as a transformational proof scheme—a mathematically mature way of thinking. The alternative instructional treatment was guided by a system of learning-teaching principles, called the DNR system, which was developed and in turn implemented in a sequence of teaching experiments on the concept of mathematical proof.

Introduction

Undergraduate programs in mathematics usually do not include a course in number theory, but students are introduced to elementary number theory concepts in other courses, most often in a discrete mathematics course, which usually includes the concepts of mathematical induction (MI). MI is a prominent proof technique in discrete mathematics and number theory, where it is used to prove theorems involving properties of the set of natural numbers. Fermat, the founder of number theory, used a form of mathematical induction to prove many of his discoveries in this field (Boyer, 1968). Beyond its significance as a proof technique in mathematics, MI can provide a context to enhance students' conception of proof, as we will see later in this paper.

"There are unresolved problems concerning the teaching of MI which should benefit from a careful analysis" asserts Ernest (1984, p. 173) and adds that "there is … no systematic account in print of the teaching of MI, of the problems that arise, of the deeper issues involved or of the treatments given by textbooks" (p. 174). With the exception of Dubinsky's (1986, 1989) work, little has been done in research on the learning and teaching of MI during the last two decades. One goal of this paper is to revive interest in this area.

The research reported here is part of the PUPA project.[4] The general questions addressed in PUPA revolved around the development of college students' proof understanding, production, and appreciation: What are students' (particularly mathematics major students') conceptions of proof? What sorts of experiences seem effective in shaping students' conception of proof? Are there promising frameworks for teaching the concept of proof so that students appreciate the value of justifying, the role of proof as a convincing argument, the need for rigor, and the possible insights gained from proof? In answering these questions PUPA produced two complementary products: (a) a conceptual framework for students' proof schemes and (b) a system of pedagogical principles, called DNR—an acronym for the three leading principles in the system: duality, necessity, and repeated-reasoning. The two products complement each other in that while the former aimed at mapping students' current conceptions of proof, the latter specifies foundational pedagogical principles for enhancing students' proof schemes.

The investigation into students' conception of MI reported in this paper is an example of a realization of these two complementary products. The first section of the paper presents an overview of the conceptual framework for students' proof schemes (a complete description of the framework is in Harel and Sowder, 1998, and Harel, in press). The second and third sections analyze students' conceptions of MI in terms of these proof schemes; the former focuses on those recorded in standard instruction, whereas the latter on those developed in an alternative, DNR-based instruction. The fourth, and last, section describes briefly the DNR system, with particular reference to the alternative instructional treatment of MI as an example of its implementation. A full description of the DNR system appears in Harel (in press).

Theoretical Perspective

The tools and language of the analyses presented in this paper are Harel and Sowder's (1998) conceptual framework for students' proof schemes. The framework has recently been revised to reflect new observations, both theoretical and empirical (Harel, in press). Since it is not essential, nor possible due to space limitations, to describe here the entire framework, only those aspects of the framework that are relevant to the concerns of this paper will, as needed, be briefly described. An overall view of the framework is worth depicting, however. Figure 1 gives a bird's-eye view of the framework structure. As can be seen, the framework consists of three classes of proof schemes: the external conviction proof scheme class, the empirical proof scheme class, and the deductive proof scheme class (to be defined below).

Insert Figure 1

Critical to this conceptual framework is the definition of proof scheme[5] (given in Harel and Sowder, 1998, pp. 241-44). Proving is defined there as "the process employed by a person to remove or create doubts about the truth of an observation" and a distinction is made between two processes of proving: ascertaining and persuading. "Ascertaining is a process an individual employs to remove her or his own doubts about the truth of an observation. … Persuading is a process an individual employs to remove others' doubts about the truth of an observation." Thus, "a person's proof scheme consists of what constitutes ascertaining and persuading for that person."

Seldom do these processes occur in separation: in ascertaining oneself, one considers how to persuade others, and vice versa. Thus, proof schemes are subjective and vary from person to person, civilization to civilization, and generation to generation within the same civilization. Yet the goal of instruction must be unambiguous; namely, to gradually refine current students' proof schemes toward the proof scheme shared and practiced by the mathematicians of today. This claim is based on the premise that such a shared scheme exists and is part of the ground for scientific advances in mathematics.

Research Questions and Method

Surprisingly, published studies on the learning and teaching of MI have not linked students' behaviors with MI to their conception of proof. Dubinsky's (1986, 1989) work includes proof as a factor in his genetic decomposition of MI, but he neither characterizes students' proof schemes nor addresses their specific role in the learning process of MI. A major goal of this paper is to fill this gap by answering two questions:

1. How are students' conceptions of and difficulties with MI in a standard instructional treatment a manifestation of their proof schemes?
2. What are students' difficulties with and conceptual development of MI in an alternative instructional treatment?

The alternative treatment was implemented in a teaching experiment in elementary number theory (hereafter, the NT experiment). Twenty-five junior prospective secondary teachers took part in the NT experiment. It included the concepts of divisibility, greatest common divisors, modular arithmetic, and Diophantic equations; algorithms from number theory, such as the algorithms for finding the greatest common divisor of two positive integers; applications to computer arithmetic; cryptology; and MI. The goal of the experiment was to record students' proof schemes and the progress of their development. This paper, however, focuses on a two-week period of the NT experiment during which a fundamentally different instructional treatment of MI was implemented.

The first question is answered on the basis of observations from several sources (to be referenced below), including our own, which have recently been analyzed and interpreted in terms of the proof scheme framework (Figure 1). The second question, on the other hand, was a primary research target of the NT experiment—one in a sequence of six teaching experiments on the development of college students' concept of proof.

"Teaching experiment" here is as defined by Cobb and Steffe (1983): Each teaching session is analyzed in terms of the classroom discourse and students' performance. The result of the analysis can, and usually does, adjust or amend the plan for subsequent lessons. Results accumulated from extensive analyses usually refine, and in some cases alter, the researchers' theoretical perspective.

In our teaching experiments, the data consist of classroom observations in the form of field notes and retrospective notes, clinical interviews, homework, written tests, and quizzes, video-taped classroom sessions, and clinical interviews with students.

Students' Proof Schemes of MI in Standard Instructional Treatment

A typical instructional approach to MI, as it is presented in standard undergraduate mathematics textbooks, begins with a few examples of how formulas with a single variable of a positive integer n (e.g., the formula for the sum of the first n positive odd integers) is generalized from an observed pattern. Following this, the question of how to prove propositions that state $P(n)$ is true for all positive integers n is discussed and answered by stating the principle of MI as a proof technique consisting of two steps:

1. Base step. Show $P(1)$,
2. Inductive step. Show $P(n) \Rightarrow P(n+1)$ for every positive integer n.

In discussing this technique some textbooks alert the reader to the common misconception that a proof by MI is a case of circular reasoning, and stress that the proof does not assume that $P(n)$ is true for all positive integers, a common confusion among students.

In an attempt to help students understand how the principle works, some textbooks present the domino analogy to MI: The teacher points to a figure of an infinite row of dominos, labeled $1, 2, 3, \ldots, n, \ldots$ (like in Figure 2) and asks the students to consider the proposition $P(n)$ stating domino n is knocked over. The teacher then explains that if the first domino is knocked over—i.e., if $P(1)$ is true—and if whenever the n^{th} domino is knocked over, it also knocks the $(n+1)^{\text{st}}$ domino—i.e., if $P(n) \Rightarrow P(n+1)$ is true, then all the dominos are knocked over.[6]

Insert Figure 2 About Here

The quality of a concept image (Vinner, 1985) is impacted not only by how the concept is presented but also by how the concept is applied in solving problems. It is important, therefore, to also examine the kind and order of problems textbooks assign students to practice the principle of MI. For this, the following classification is needed.

MI problems in standard textbooks can be classified into two categories, recursion problems and non-recursion problems, according to whether the mathematical solution of the problem requires the formation of a recursive representation of a function. For example, the problem, "Prove that $1+3+5+\cdots+(2n-1)=n^2$ for all positive integers n," is a recursion problem, because the left hand side of the identity must be interpreted as a recursive representation of a function.[7] In contrast, the solution to the problem, "Prove that $n<2^n$ for all positive integers $n \geq 1$" does not necessarily involve a recursive representation of a function,[8] and so it is a non-recursion problem. In some cases the recursive representation of a function is explicit in the problem statement, in others implicit. For example, in the former problem the recursive representation of a function is explicit (the rule, $f(n)=f(n-1)+(2n-1), f(1)=1$, is virtually present in the problem statement), whereas in the Tower of Hanoi problem or in the problem, "Find an upper bound to the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$" is implicit, for no recursively defined function is explicitly present in the problem statement. Accordingly, recursion problems are further classified into two categories: explicit recursion problems and implicit recursion problems (Figure 3). It should be noted, however, that the solver might not know what a recursive representation of a function is or that such a representation is involved in the problem.

Insert Figure 3 About Here

Typically in the standard instructional approach to MI, students' first exposure to MI is through three kinds of problems: (a) identity problems (e.g., "Prove that $1+3+5+\cdots+(2n-1)=n^2$ for all positive integers n), (b) inequality problems (e.g., "Prove that $n<2^n$ for all positive integers $n>3$), and (c) divisibility problems ("Prove that $3 \mid n^3-n$ for all positive integers n"). Implicit recursion problems, such as the Tower of Hanoi Problem, in which one is required to form a recursive representation of a function, usually are not included or appear in a small number at the end of the exercises list. As will be argued later in this paper, the type and sequence of MI problems is of paramount pedagogical importance.

I began looking closely into the learning and teaching of MI about ten years ago. Until then my instructional treatment followed the standard approach—both in content presentation and in the kind and order of problems assigned. My students' understanding of MI was unsatisfactory. Their difficulties were consistent with those reported in the literature, and in retrospect, when analyzed in terms of students' conceptions of proof, they were found to be mostly manifestations of the external and empirical proof schemes (Figure 1), as we will now see.

MI as an Empirical Proof Scheme: The Case of Result Pattern Generalization

One of the difficulties students have with MI, also observed by Show (1978) and reviewed in Ernest (1984), is their view of induction as a technique of drawing a general conclusion from a number of individual cases. This observation, however, clomps two distinct ways of thinking into one: in one students generalize from a process pattern, in the other from result pattern. In process pattern generalization students focus on regularity in the process, whereas in result pattern generalization on regularity in the result. For example, some students responded to the log-identity problem (Prove that for all positive integers n, $\log(a_1 \cdot a_2 \cdot \cdots \cdot a_n) = \log a_1 + \log a_2 + \cdots + \log a_n$) by relying on the regularity of the results of their calculations, as in the following response by a student:

Response 1:

$\log(4 \cdot 3 \cdot 7) = \log 84 = 1.924$ $\qquad$ $\log(4 \cdot 3 \cdot 6) = \log 72 = 1.857$

$\log 4 + \log 3 + \log 7 = 1.924$ $\qquad$ $\log 4 + \log 3 + \log 6 = 1.857$

Since these work, then $\log(a_1 \cdot a_2 \cdot \cdots \cdot a_n) = \log a_1 + \log a_2 + \cdots + \log a_n$.

A probe into these students' reasoning has revealed that their conviction stems from the fact that the proposition is shown to be true in a few instances, each with numbers that are randomly chosen. This behavior is a manifestation of the empirical proof scheme, a scheme in which students rely on evidence from perception or examples of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, etc. (Harel and Sowder, 1998).

MI as a Deductive Proof scheme: The Case of Process Pattern Generalization

Process pattern generalization is a way of thinking in which one's conviction is based on regularity in the process, though it might be initiated by regularity in the result. This behavior is in contrast to the result pattern generalization, where students' proving is based solely on regularity in the result—obtained by substitution of numbers for instance.

To demonstrate the process pattern generalization, consider the following response by one of the students to the log-identity problem we have just discussed:

<u>Response 2</u>:

(1) $\log(a_1a_2) = \log a_1 + \log a_2$ by definition

(2) $\log(a_1a_2a_3) = \log a_1 + \log a_2a_3$. Similar to $\log(ax)$ as in step (1), where this time $x = a_2a_3$.
Then
$\log(a_1a_2a_3) = \log a_1 + \log a_2 + \log a_3$

(3) We can see from step (2) any $\log(a_1a_2a_3 \cdots a_n)$ can be repeatedly broken down to $\log a_1 + \log a_2 + \cdots + \log a_n$

In this excerpt the student recognizes that the process he employs in the first and second cases constitutes a pattern that recursively applies to the entire sequence of propositions, $\left\{\log(a_1a_2a_3 \cdots a_n) = \log a_1 + \log a_2 + \cdots + \log a_n\right\}_n$.

Both the result pattern generalization and the process pattern generalization are based on a limited number of specific cases—in both Responses 1 and 2 above, for example, the generalizations are made from two cases. This may suggest, therefore, that both kinds of generalizations are empirical. This is not so, however. The process pattern generalization is <u>not</u> a manifestation of the empirical proof scheme but an expression of the transformational proof scheme. To see why, one only needs to examine the two generalizations against the definitions of the two schemes. The transformational proof scheme is characterized by (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory—an attempt to predict outcomes on the basis of general principles—and (c) transformations of images that govern the deduction in the evidencing process (Harel and Sowder, 1998).[9] While both responses share the first characteristic—i.e., in both the students respond to the "for all" condition in the log-identity problem statement—they differ in the latter two: Whereas the mental operations in Response 1 are incapable of anticipating possible subsequent outcomes in the sequence and are devoid of general principles in the evidencing process, the mental operations in Response 2 correctly predict, on the basis of the general rule, $\log(ax) = \log a + \log x$, that the same outcome will be obtained in each step of the sequence. Further, in Response 1 the reference rule that governs the evidencing process is empirical; namely, $(\exists r \in R)(P(r)) \Rightarrow (\forall r \in R)(P(r))$. In Response 2, on the other hand, it is deductive; namely, it is based on the inference rule $(\forall r \in R)(P(r)) \wedge (w \in R) \Rightarrow P(w)$. (Here r is any pair of real numbers a and x; R is the set of all pairs of real numbers; $P(r)$ is the statement "$\log(ax) = \log a + \log x$" and w in step n is a pair of real numbers $a_1 a_2 \cdots a_{n-1}$ and a_n.)

<u>MI as an External Proof Scheme</u>

Dubinsky (1986) has indicated "if you question students—even those who have had several mathematics courses—although almost all of them will have heard of induction, not many of them will be able to say anything intelligent about what it is, much less actually use it to solve a problem" (p. 305). Our experience corroborates Dubinsky's observation. "It is a proof with steps" was the only response to the instructor's question "What is MI?" despite the fact that the entire class confirmed that they were familiar with the term, MI.

When pressed, some students admit that they view the step of verifying $\mathrm{P}(1)$ as non-essential, and one is required to perform it just to satisfy the rule stated by the teacher. For, according to these students, the reason that $P(n)$ is true for all positive integers n is that "We proved it for $n+1$, so we proved it for n." Seldom do students express doubts about the process of MI; they accept it, as they have been trained to accept without question any other mathematical statement. This behavior is a manifestation of the authoritative proof scheme (Figure 1)—a scheme in which proving depends mainly on the authority of the teacher or textbook.

In the authoritative proof scheme, one follows the rule prescribed by the principle of MI without understanding its meaning: first verify $\mathrm{P}(1)$, then follow the procedure of assuming $\mathrm{P}(n)$, go through a certain manipulation, and then derive $\mathrm{P}(n+1)$ (Baxandall, Brown, Rose, and Watson, 1978); thus reinforcing another external proof scheme*,* the <u>symbolic non-quantitative</u> proof scheme—a scheme by which one thinks of symbols as if they possess a life of their own without reference to their functional or quantitative meaning (Harel and Sowder, 1998). As was observed by Woodall (1981, p. 100), for many students, induction means, "Take an equation involving n and add something to both sides so as to produce a similar equation with $n+1$ in place of n" (quoted in Ernest, 1984). Many students perform this procedure correctly without understanding what they are doing. For example, some students did not know what is the "something" they are adding to both sides of the equality, as the following case illustrates:

A student was given the problem: "Prove that for any positive integer n, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$." He first performed the <u>base step</u>, for $n=1$, $\frac{1}{1\cdot 2}=\frac{1}{2}$, and then added $\frac{1}{(n+1)(n+2)}$ to both sides of the equality, but did not know whether he was adding the $(n+1)^{\text{st}}$ term of the sequence or the sum of the first $n+1$ terms. Although he performed the algebraic computation correctly and obtained the expression $\frac{n+1}{n+2}$ as he desired, his understanding of the process was merely procedural, not relational (in the sense of Skemp, 1976).

In summary, in the standard instructional treatment of MI the three most prevalent proof schemes among students are the authoritative proof scheme, the symbolic, non-quantitative proof scheme, and the result pattern generalization, which is a manifestation of the empirical proof scheme. On the other hand, process pattern generalization—an expression of the transformational proof scheme—is uncommon among students. Its mere occurrence, however, offers an important hint as to an alternative, more effective instructional approach to MI, which is the topic of the next section.

Students' Proof Schemes of MI in an Alternative Instructional Treatment

The standard instructional treatment of MI has two major deficiencies. The first deficiency is that the principle of MI is introduced abruptly: students do not see how the principle is born out of a need to solve problems, nor do they see how it is derived from previous more elementary experience. Rather, it is handed to them as a prescription to follow; thus reinforcing the authoritative and symbolic, non-quantitative proof schemes.

The second deficiency is in the type and order of problems used. The significance of implicit recursion problems, as it will be demonstrated later in this section, is that they induce students to focus on process pattern generalization rather than on result pattern generalization, whereby they build and reinforce the transformational proof scheme. As it has been indicated earlier in this paper (see the paragraph following Figure 3), standard textbooks focus almost exclusively on identity problems, inequality problems, and divisibility problems. These problems require little understanding of MI, for students learn to solve them by blindly following the two MI steps: verify $\mathrm{P}(1)$ and then go through a certain manipulation to obtain $\mathrm{P}(n+1)$—not necessarily understanding that the latter is derived from $\mathrm{P}(n)$.

Some textbooks authors, in an attempt to sequence the problems assigned to students in what they perceive as an increasing order of difficulty, begin with "easy" problems, such as "Prove by MI the statement, for any positive integer $n \geq 4$, $n! > 2^n$," and cause as a result an unexpected hindrance (see below the subsection, Essentiality of MI). Most of the students in our teaching experiments viewed such statements as "trivial," as ones that do not require MI but can be solved in an "easier" way. The demand to use MI for these statements was, therefore, non-intrinsic, alien, to the students. This is not to say, however, that such problems should not be assigned. As we will see below, they can play an important role in the interiorization process of MI, provided they are introduced in a suitable stage in the students' conceptual development.

The new instructional treatment of MI takes into considerations these deficiencies. It is structured to help students develop the principle of MI gradually through intrinsic problems—problems they can understand and appreciate. The treatment consists of three phases corresponding to the three levels of conceptual development:

1. Quasi-induction as an internalized process pattern generalization
2. Quasi-induction as an interiorized process pattern generalization
3. MI as an abstraction of quasi-induction.

These phases, and the terms used in their labels, will be discussed in the next three subsections.

Phase 1: Quasi-induction as an Internalized Process Pattern Generalization

Students' first exposure to MI was through engagement with implicit recursion problems typified by:

1. Find an upper bound to the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$
2. You are given 3^n coins, all identical except for one which is heavier. Using a balance, prove that you can find the heavy coin in n weighings.
3. Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces, where these pieces cover three squares at a time.
4. The Tower of Hanoi Problem: Three pegs are stuck in a board. On one of these pegs is a pile of disks graduated in size, the smallest being on top. The object of this puzzle is to transfer the pile to one of the other two pegs by moving the disks one at a time from one peg to another in such a way that a disk is never placed on top of smaller disk. How many moves are needed to transfer a pile of n disks?
5. Prove that every third Fibonacci number[10] is even.
6. Into how many regions is the plane cut by n lines, assuming no two lines are parallel and no three intersect at a point.

In the standard instructional approach these problems are usually kept to the end of the exercise list or not introduced at all. The students in the NT teaching experiment, on the other hand, were engaged for a relatively long period of time—before the principle of MI was mentioned—in working on problems of this kind. These problems proved to be effective in shifting students' attention from result pattern generalization to process pattern generalization. For example, in working on Problem 1 in small groups, students first noticed that 2 is an upper bound—typically by finding on their calculators the values of several items in the sequence. Some students saw this as sufficient evidence—a typical result pattern generalization phenomenon. Other students established a process pattern generalization, as did one of the small working groups, who argued via their representative that:

> The third item is less than 2 because it is the square root of a number that is smaller than 4, this number being the sum of 2 and a number that is smaller than 2.

The representative proceeded by applying this form of argument to the next few terms in the sequence and concluded that all the terms in the sequence must be less than 2 because, according to her, the same relationship exists between any two consecutive terms in the sequence. While the result pattern generalization argument persuaded only some of the students, this process pattern generalization argument convinced all the students; hence, the class as a whole seemed to view the latter as superior to the former.

The language and notation used by the students within the class discussion developed gradually. For example, during the discussion of Problem 1 the students hardly used any mathematical symbols in referring to the terms in the sequence. In the discussion of Problem 4, on the other hand, they used mathematical symbols to convey their solution, as the following response by one of the small working groups illustrates:

> [The] formula is derived for moving the disks on 3 pegs. The formula is $2(S_n)+1$. [Let's say] we moved n disks, and it took S_n moves. In order to move $n+1$ disks, first we move n disks. This takes say S_n moves. Now, we move the $(n+1)^{st}$ disk—that is

> the $+1$ of the formula. Last we move the n disks on top of the $(n+1)^{\text{st}}$ disk, [which take another S_n moves]. We have now made a total of $2(S_n)+1$ moves.

These students seem to understand that with the recursive formula, $S_{n+1} = 2S_n + 1$, one can compute the number of moves needed for any number of disks by simply starting from the number of moves needed for 1 disk and substitute it in the formula to find the number of moves needed for 2 disks, and so on.

Not all students pursued a process pattern generalization for these problems. Some students obtained a formula for the number of moves by result pattern generalization. For example, the representative of another working group presented the table

Disk #	# of moves
1	$1 = 2^1 - 1$
2	$3 = 2^2 - 1$
3	$7 = 2^3 - 1$
4	$15 = 2^4 - 1$
5	$31 = 2^5 - 1$

and then said:

> Therefore we can infer that $2^{31} - 1 = 2{,}147{,}483{,}647$,[11] where $2^n - 1 = \#$ of moves and $n =$ number of disks moved.

This result intrigued some of the students. They saw that the two formulas constructed by the two working groups, $S_{n+1} = 2S_n + 1$ and $S_n = 2^n - 1$, produced the same outcome for any specific natural number n they choose. But this form of equivalency was foreign to them, because they did not know how to transform one formula into another—in the same way they knew how to transform trigonometric identities, for example. With direction from the instructor, the students easily established the implication $S_n = 2^n - 1 \Rightarrow S_{n+1} = 2S_n + 1$, but the converse was postponed to the third phase.

The current phase continued for about a week, during which the students repeatedly applied process pattern generalization in solving about twenty problems like Problems 1-6. The general solution approach that emerged from this <u>repeated reasoning</u> I refer to as <u>quasi-induction</u>. Students' expressions of quasi-induction are either in the form of the <u>method of ascent</u> or in the form of the <u>method of descent</u>.

The two methods will be exemplified now even though the latter appeared only in the second phase.

<u>The method of ascent</u>.

Problem: "Find an upper bound to the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$"

Student 1: Since $\sqrt{2}$ is less than 2, $2+\sqrt{2}$ is less than 4, and so $\sqrt{2+\sqrt{2}}$ is less than 2. Since $\sqrt{2+\sqrt{2}}$ is less than 2, $2+\sqrt{2+\sqrt{2}}$ is less than 4. Hence, $\sqrt{2+\sqrt{2+\sqrt{2}}}$ is less than 2. And so on.

Student 2: Each item of the sequence can be obtained from the item on [its] left in the same way [pointing to how she derived $P(2)$ from $P(1)$ and $P(3)$ from $P(2)$], and this way is adding to the one on the left 2 and taking the square root [i.e., $a_n = \sqrt{2+a_{n-1}}$]

<u>The method of descent</u>.

Problem: Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all positive integers n.

Student 3:

$n=2 \quad \frac{1}{2}+\frac{1}{6}=\frac{2}{3}$

$n=3 \quad \frac{2}{3}+\frac{1}{2}=\frac{3}{4}$

$n=4 \quad \frac{3}{4}+\frac{1}{20}=\frac{4}{5}$

$n=5 \quad \frac{4}{5}+\frac{1}{30}=\frac{5}{6}$

If $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$

Then $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}-\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}-\frac{1}{n\cdot(n+1)}$

$\Rightarrow$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}&=\frac{n^2-1}{n(n+1)}\\ &=\frac{(n+1)(n-1)}{n(n+1)}\\ &=\frac{n-1}{n}\\ &=\frac{n-1}{\left[(n+1)-1\right]}\end{aligned}$$

which is of the form $\frac{n'}{(n'+1)}$

where $n'=n-1$.
These steps could be continued down to eventually get n to a small enough number so it can be evaluated easily. A few of these are shown above.

Problem: Prove that $\log(a_1\cdot a_2\cdot\cdots\cdot a_n)=\log a_1+\log a_2+\cdots+\log a_n$, for any positive integer n.

Student 4: Since $\log a_1a_2=\log a_1+\log a_2$, if we look at $\log\left((a_1\cdot a_2\cdot\cdots\cdot a_{n-1})a_n\right)$, we need to show $\log(a_1\cdot a_2\cdot\cdots\cdot a_{n-1})=\log a_1+\log a_2+\cdots+\log a_{n-1}$. Then this equation can be reduced by the same method, until we get $\log a_1a_2=\log a_1+\log a_2$.

The NT students invented quasi-induction while attempting to solve implicit recursion problems. They practiced the method by repeatedly applying it in solving problems similar to

Problems 1-7 above. A total of 20 implicit recursion problems were assigned in two sets of homeworks—11 problems in the first set, 9 problems in the second. After completing the first set of homework, students' application of quasi-induction, as was expressed in their individual work and the class discussions, seemed autonomous and spontaneous.

Quasi-induction, as the NT students conceived it in this instructional phase, may be viewed as a theorem-in-action, a term coined by Vergnaud (1988) to refer to "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (p. 144). A theorem-in-action is usually implicit knowledge, and so was quasi-induction in this instructional phase. More important, at this phase quasi-induction was an internalized theorem-in-action; namely, the students were able to apply it autonomously, and often spontaneously. The instructional goal, however, was that quasi-induction become an interiorized theorem-in-action; namely that the students be able not only to apply quasi-induction autonomously, but also represent it conceptually as a method of proof—in the same way they might evoke proof-by-contradiction, for example.

Phase 2: Quasi-induction as an Interiorized Process Pattern Generalization

Implicit recursion problems used in the first phase necessitated for the students the formation of recursive relations, which in turn led them to construct, and with repeated application internalize, quasi-induction. Explicit recursion problems, such Problem 7-9 below, were kept to the second phase, whose ultimate goal was to advance students to the state of conceiving quasi-induction as a method of proof.

7. Prove that for any positive integer $n \geq 1$ $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$.
8. Find a formula for the sum $a + a^2 + a^3 + \cdots + a^n$

9. Compute the sum $1+3+\cdots+2n-1$

The rationale for reserving the explicit recursion problems to the second phase was this: The essential common feature to explicit recursion problems and implicit recursion problems is that the problem statement is about a proposition-valued function whose domain is the set of positive integers. Based on pilot experiments, we expected NT students not to recognize this fact spontaneously. But, we hypothesized, when they do, they would apply, or attempt to apply, quasi-induction to the new set of problems. And through this activity, the students would ascertain themselves and persuade others of the quasi-induction's applicability range expansion, whereby they would reflect on quasi-induction and recognize it as a method of proof.

This predicted scenario was in the main realized. At first, when explicit recursion problems were introduced, the students did not see any relation between them and the implicit recursion problems from the previous phase. Their approach to problem 7, for example, was to find a closed form to the summation on the left-hand side of the equality—mostly by trying to find a common denominator to the fractional expressions. For problem 8, some stated they knew the formula from their Statistics class, but admitted they had never understood why it works. Interestingly. Only a few students attempted to verify the equality in problem 7 with specific positive integer values—an indication that most of the students did not see the problem statement as representing an infinite sequence of statements on the positive integers. Pointing out this fact to the students did not lead them to realize the common structure to the two sets of problems. The students had to be explicitly asked to see if such a common structure existed. Even when they recognized the common structure, they did not spontaneously attempt to apply quasi-induction to the new problems. Some of students even fell back to mere empirical reasoning, especially in problems in which a result pattern generalization was apparent, such as in problem 9. Here many students observed easily that

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$

and from this sequence, they concluded that $1 + 3 + \cdots + 2n - 1 = n^2$ for all positive integers n.

Only one of the small working groups discovered how to apply quasi-induction to solve the new problems. In expressing their solution to problem 9, they explicitly referred to their solution to problem 1 from phase 1. The group's representative first represented the expression $1 + 3 + \cdots + 2n - 1$ as a sequence 1, $1+3$, $1+3+5$, $\ldots$ and then said:

> Like in Problem 1, where the relationship between two consecutive elements a_k and a_{k+1} is $a_{k+1} = \sqrt{2 + a_k}$, here the relationship is $a_{k+1} = a_k + 2k + 1$. And like in Problem 1, where we used the fact that a_1 is smaller than 2 to derive that a_2 is smaller than 2, then used the fact that a_2 is smaller than 2 to derive that a_3 is smaller than 2, and so on, here we use the fact that $a_1 = 1^2$ to derive that $a_2 = 2^2$, because $a_2 = a_1 + 2 \cdot 1 + 1 = 1^2 + 2 \cdot 1 + 1 = 1^2 + 2 \cdot 1 + 1^2 = (1+1)^2 = 2^2$, and in a similar way to use that $a_2 = 2^2$ to derive that $a_3 = 3^2$, and so on.

This discovery was novel but not alien to the rest of the class—it was in the their zone of proximal knowledge. From this point on, the rest of the working groups began to adopt this pattern of reasoning in solving explicit recursion problems. The students continued for some time, especially in addressing their classmates, to point to parallels between their solutions to explicit recursion problems and specific solutions they used to solve implicit recursion problems—just as in the last excerpt.

Students' repeated attempts to attend to the underlying common structure of the two categories of problems—the implicit recursion problems from phase 1 and the explicit recursion problems from phase 2—and to apply a process pattern generalization approach in the form of quasi-induction to problems in the latter category led the students to reflect on the proof method they had acquired and successfully used in the previous phase. This together with the didactical contract established in class—that persuasion, not only ascertainment, is an essential part of proving—induced students to be explicit about their use of quasi-induction.

It is both interesting and important to note that historically quasi-induction was an antecedent to the concept of MI—a topic to be addressed in a sequel to this paper (Harel, McClure, & Sowder, in preparation). The institutionalization of quasi-induction as a means for persuasion and ascertainment was not achieved by mere democratic consensus, nor was it independent of the instructor's endorsement. It was from the very start an explicit instructional objective, for it was judged to correspond to and be the cognitive root (in the sense of Tall, 1992) for the principle of MI—a judgment supported by an analysis of the historical epistemology of MI. However, the process by which the students developed this method of proof was largely adidactical (Brousseau, 1997)—free from considerations of pleasing the teacher or conforming to his will.

Phase 3: MI as an Abstraction of Quasi-induction

From a mathematical point of view, the formal principle of MI might be seen as a precise formulation of quasi-induction. From a cognitive point of view, on the other hand, the gap between the two is considerable: while the latter deals with local inference step, the former with global inference form, as I will now explain.

Inference step versus inference form. In quasi-induction the conviction that $P(n)$ is true for any given natural number n stems from one's ability to imagine starting from $P(1)$ and going through the inference steps, $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$, … $P(n-1) \Rightarrow P(n)$. This does not mean that one actually runs through many steps, but that he or she realizes that in principle this can be done for any given natural number n. In particular, in quasi-induction one views the inference, $P(n-1) \Rightarrow P(n)$, just as one of the inference steps—the last step—in a sequence of inferences that leads to $P(n)$. In MI on the other hand, one views the inference, $P(n) \Rightarrow P(n+1)$, as a variable inference form, a placeholder for the entire sequence of inferences. Consequently, he or she has no need to run through the specific inference steps, $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$, …, $P(n-1) \Rightarrow P(n)$, for the instantiation of these steps into the inference form, $P(n) \Rightarrow P(n+1)$, are viewed as a logical necessity. Thus, while quasi-induction and MI are both instantiations of the transformational proof scheme, the latter is an abstraction of the former.

The introduction of the formal principle MI in the NT experiment. Unfortunately, at the time we conducted the NT experiment, we were unaware of the cognitive gap between quasi-induction and the formal principle of MI. We viewed the latter merely as a concise formulation of the former, and so was it presented to the students.

The students were accustom to gradual reformulation of mathematical ideas—a practice we established in each of our teaching experiments. With a few exceptions, our students' ability to express mathematical ideas properly was initially very limited. Specific interventions were needed to bring about improvement in the quality of their mathematical exposition. The following are two of the interventions:

Individual students are asked to see the instructor to explain or articulate some of their homework or exam works. In these individual meetings the student is helped to recognize that although her or his idea is correct or partially correct, it is expressed deficiently or insufficiently. The recognition often occurs when the instructor presents an interpretation of the student's answer that the student deems legitimate, yet initially unintended by her or him. Once the student arrives at the recognition of the need to modify her or his work, he or she is offered the option to rewrite the respective assignment, upon whose completion the initial grade is adjusted according to the quality of the change.

The other intervention is that each instructional unit in the teaching experiment ends with an assignment in which students are asked to summarize what they perceive as the most important ideas of the unit. With these summaries in mind, and with further inputs from the students, the instructor then gives a presentation in which the main ideas learned are expressed in standard mathematical language. Following this, the students are assigned a set of review problems aimed at helping them solidify the ideas learned in terms of the final formulations.

It was in the context of the latter classroom practice that the formal principle of MI was presented as a reformulation of quasi-induction. The formal statement presented was the following:

$P(n)$ is true for all positive integers n if

1. $P(1)$ is true
2. For any positive integers k, whenever $P(k)$ is assumed true, $P(k+1)$ is also true.

This was the first time in the course that the term mathematical induction was used; none of the above terms, including the term quasi-induction, had ever been used in class. Following this, the students were assigned a set of review problems consisting of combinations of implicit and explicit recursion problems in a variety of contexts: algebra, geometry, and calculus. The set also included non-recursion MI problems (to be discussed in the next subsection).

Unlike the previous two stages, no significant classroom discussion ensued from this presentation. The students seemed to have easily assimilated the principle of MI into their scheme of quasi-induction: About 75% of the problems assigned were solved correctly and in terms of the principle of MI; the rest were either solved by quasi-induction or by other means (see the next subsection below). The following is an example of one students' solution by MI to the problem, "Prove that a set of n elements has $2^n - 1$ nonempty subsets."

> Consider the case for $n = 2$, set= $\{a_1, a_2\}$
> Non empty subsets= $\{a_1\}, \{a_2\}, \{a_1, a_2\}$
> Now we can form the subsets for $n = 3$ by adding
> (i) the subset of $n = 2$, $= \{a_1\}, \{a_2\}, \{a_1, a_2\}$
> (ii) the subsets of $n = 3$, where each now includes element a_3:
> $\{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}$ and the subset $\{a_3\}$
> Examining this process, we see that if a set of n elements has S_n subsets, a set of $n+1$ elements has $2S_n + 1$ subsets.
> Now assume $S_n = 2^n - 1$
> Then $S_{n+1} = 2S_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$
> Also, for $n = 1$, $S_1 = 1 = 2^1 - 1$
> By induction, $S_n = 2^n - 1$ for all n.

The nearly effortless reconceptualization of quasi-induction into MI by the students is attributed to the conceptual foundations cemented in phase 1 and 2. The kind and order of the problems the students dealt with in these phases constituted two critical intellectual needs for them: The first is the intellectual need to compute a solution to problems that were intrinsic, not alien, to them. By "compute," it is meant developing a process pattern generalization to explain the reason 2 is an upper bound for the sequence. The second is the intellectual need to formulate this process—making it explicit to themselves—in order to persuade others of the validity of their solutions. Quasi-induction, therefore, marks a vital developmental step toward the concept of MI.

Essentiality of MI

Beyond the level of abstracting the formal principle of MI, there is yet another level the NT teaching experiment did not set as a goal. It is the understanding that from a formal, axiomatic perspective any proof of an assertion on the set of positive integers must use MI. This level of understanding can be reached only when the students possess the axiomatic proof scheme—the understanding that a mathematical justification must, in principle, have started originally from undefined terms and axioms. Although the axiomatic proof scheme should be an explicit goal of undergraduate mathematics curricula, it was a non-realistic goal in the NT teaching experiment, given the state of knowledge and mathematical maturity of the participants.

That the NT students did not reach this level of understanding can be seen in their responses to problems such as:

10. Prove that for any positive integer n, $\log(a_1 \cdot a_2 \cdot \dots \cdot a_n) = \log a_1 + \log a_2 + \dots + \log a_n$

11. Prove that for any positive integer $n \geq 4$, $2^n \leq n!$

12. Prove that the sum of the measure of the interior angles in a convex n-gon is

$180^0(n-2)$.

Such problems introduced to many students no need to search for consecutive relations, because, they argued, "the problems can be solved in an easier way." Most of these students offered solutions to these problems that were not based on MI. For example, problem 10 appeared in succession to problem 9. Most of the students solved problem 9 by MI but argued that problem 10 can be solved in an easier way, as did one of the students:

> Let $y = \log a$
> We can say $a = 10^y$ by definition of $\log$
> Now we can express $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ as $10^{y_1} \cdot 10^{y_2} \cdot \ldots \cdot 10^{y_n} = 10^{y_1+y_2+\ldots+y_n}$.
> Therefore,
> $\log(a_1 \cdot a_2 \cdot \ldots \cdot a_n) = \log(10^{y_1+y_2+\ldots+y_n}) = (y_1 + y_2 + \ldots + y_n)\log 10 =$
> $= (y_1 + y_2 + \ldots + y_n)1 == y_1 + y_2 + \ldots + y_n = \log a_1 + \log a_2 + \ldots + \log a_n$

Problem 11 in particular was viewed as trivial, as one that needs no MI. For example, one of the students responded to this problem by presenting 2^n and $n!$ on the top of each other as follows:

$$2^n = \underbrace{\underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{16} \cdot 2 \cdot \ldots \cdot 2}_{n \text{ times}}$$

$$n! = \underbrace{1 \cdot 2 \cdot 3 \cdot 4}_{24} \cdot 5 \cdot \ldots \cdot n$$

explaining that:

> The first factor in $n!$ [i.e., 24] is greater than the first [corresponding] factor in 2^n [i.e., 16], and the rest of the factors, 5, 6, 7, …, are [correspondingly] greater than the factors, 2, 2, 2, …

For problem 12, one of the students presented his solution to the class by drawing on the board a figure similar to Figure 4, explaining how the formula is derived from two facts: (a) in an n-gon there are $n-2$ triangles—pointing out that each vertex but two contributes one triangle—and (b) the sum of the measures of the angles in a triangle is 180^0.

Following this student's presentation, the teacher presented a proof by MI of the same statement. The students protested that the MI proof is superfluous and that their classmate's proof makes more sense.

Insert Figure 4 About Here

Although the classroom discussion on problems such as 10-12 helped students better see the underlying structure of all the problems they had dealt with in the three phases, it did not enhance their understanding beyond the level of MI as a transformational proof scheme. It did confirm, however, students' view of the non-essentiality of MI in solving certain kinds of problems.

Conclusion: The DNR System As a Conceptual Foundations for the Alternative Instructional Treatment to MI

MI is a significant proof technique in all fields of mathematics. Beyond its mathematical significance it provides a context to enhance students' proof schemes. Hence, I believe it must be part of secondary school mathematics curricula, especially those taught to college-intended students.

This paper addresses two questions. The first two will be summarized, in turn, in the next two subsections. To these I will add a third question about the conceptual foundation of the alternative instructional treatment to MI. The latter will be addressed in the third, and final, subsection.

Question 1: How Are Students' Conceptions of and Difficulties with MI in a Standard Instructional Treatment a Manifestation of Their Proof Schemes?

The three most prevalent proof schemes among students are the authoritative proof scheme, the symbolic, non-quantitative proof scheme, or the empirical proof scheme. These ways of thinking impact and even determine students' understanding of MI, which, as we have seen, is conceived mainly as a result pattern generalization. Process pattern generalization—an expression of the transformational proof scheme—is virtually absent among students.

One possible cause of this situation is the standard instructional treatment of MI. In particular, the introduction of the principle of MI in this treatment is abrupt. Students do not see how the principle is born out of a need, an intellectual need that is, to solve problems that are intrinsic[12] to them. Consequently, they use the principle of MI as a prescription; thus reinforcing the authoritative and symbolic, non-quantitative proof schemes.

MI problems in standard textbooks are sequenced according to what their authors seem to perceive as an increasing order of difficulty, not in accordance to students' conceptual development. As a consequence, the task to use MI in all problems discriminately—including those for whom the students have alternative solutions—is alien[13] to these students and exacerbates their poor view of mathematics.

Question 2: What Are Students' Difficulties with and Conceptual Development of MI in an Alternative Instructional Treatment?

The new instructional treatment of MI takes into consideration the possible causes for the failure of the standard approach. In particular, it consists of phases corresponding to levels of conceptual development: It begins with the formation of quasi-induction, continues with its internalization and interiorization, and concludes with its abstraction into MI. For the NT experiment students, therefore, the concept of quasi-induction seems vital in the process of learning the principle of MI. This observation is consistent with the historical development of MI, as it will be reported in Harel at al. (in preparation).

The most significant result reported in this paper is that in this alternative treatment students altered their current ways of thinking, primarily from mere empirical reasoning—in the form of result pattern generalization—into transformational reasoning—in the form of process pattern generalization.

Question 3: What Is the Theoretical Basis for the Alternative Instructional Treatment Implemented in the NT Experiment?

The DNR system of pedagogical principles for designing, developing, and implementing mathematics curricula is the conceptual foundation for the alternative instructional treatment offered here. In the remainder of this paper I will present a synopsis of the system. A full description of the system appears in Harel (in press). The three essential pedagogical principles constituting the DNR system are the duality principle, necessity principle, and repeated reasoning principle.

The duality principle. Fundamental to the duality principle is the distinction between ways of thinking and ways of understanding (Harel, 1998, p. 497):

> Ways of thinking are students' apparatuses for filtering and interpreting what we intend to teach them. … They are distinguished from ways of understanding. A way of understanding is the meaning(s) students have for a specific concept. For example, students may understand the "derivative of a function" as the slope of a line tangent to the graph of a function, as the best linear approximation to a function near a point, etc., but they may understand it superficially (e.g., ``derivative is x^{n-1} for x^n) or even incorrectly (e.g., "derivative is the quotient $\frac{f(x+h)-f(x)}{h}$").

The importance of this distinction is primarily in the mutual cognitive impact of ways of thinking and ways of understanding, which is the content of the duality principle (Harel, 1998):

> Students' ways of thinking impact their ways of understanding mathematical concepts. Conversely, how students come to understand mathematical content influences their ways of thinking.

By intervening in the students' ways of understanding a specific content, in our case that of implicit recursion problems, the students altered their narrow ways of thinking about what constitutes proof—from authoritative, symbolic, non-quantitative, and empirical proof schemes into quasi-induction—a form of a transformational proof scheme. In turn, this new way of thinking impacted students' ways of understanding explicit recursion problems. This dual effect between ways of thinking and ways understanding is an example of how the duality principle guided the design and implementation of the alternative instructional treatment of MI.

The necessity principle. The necessity principle—the second underlying conceptual basis for the alternative instructional treatments—asserts:

> Students are likely to learn if they see need for we intend to teach them, where by 'need' it is meant intellectual need as opposed to social or economic need (Harel, 1985, 1990, 1998).

The term intellectual need refers to a behavior that manifests itself internally with learners when they encounter a situation that is incompatible with, or presents a problem that is unsolvable by, their existing knowledge. Such an encounter is intrinsic to the learners for it stimulates a desire with them to search for a resolution or a solution, whereby they might construct new knowledge. There is no guarantee that the learners construct the knowledge sought or any knowledge at all, but whatever knowledge they construct, it is meaningful to them in that it is integrated within their existing cognitive schemes, because it is a product of an effort that stems from and is driven by their personal intellectual need.

The implementation of the necessity principle involves (a) recognizing what constitutes an intellectual need for a particular population of students relative to the concept to be learned, (b) developing a system of problem situations that correspond to their intellectual need, and from whose solutions the concept can be elicited, and (c) creating an instructional environment in which the student can elicit the concept through engagement with the system (Harel, 1998). These are not three steps of a recipe to be carried out chronologically. Rather, these constitute, respectively, the essence of three inseparable aspects of research in learning, curriculum development, and teaching.

In Harel (1998), I distinguished among three kinds of intellectual needs in the learning of mathematics: the need for computation, the need for formulation, and the need for elegance. Relevant to the work here are the first two needs.

The need for computation means finding unknown objects or determining their causes. Examples of this need includes (a) finding the solution of a system of equations, the *GCD* of two integers, the probability of an event, the cardinality of a set, and (b) determining the cause[14] (i.e., reason) for a system $Ax = b$ to have a solution or a unique solution.

The need for formulation is the need to communicate ideas, including convincing others of the truth of an observation. In the NT teaching experiment, a didactical contract (in the sense of Brousseau, 1997) was established according to which persuasion, not only ascertainment, is an essential part of proving. This practice induced students to reflect on and formulate quasi induction—a significant step in the development of students' conception of MI.

The need for computation and the need for formulation were combined in the NT teaching experiment as a means to construct the concept of MI. For example, the task to find an upper bound fits well into what the NT students conceive as doing mathematics. From this reason the need to compute an unknown quantity, or determine conditions for its existence, is well understood and appreciated by them. Once this need was established, the need for formulation—that is, the need to persuade others—facilitated the institutionalization of the superiority of the process pattern generalization over the result pattern generalization.

The repeated reasoning principle. While the necessity principle aims at the initial formation of a concept, the repeated reasoning principle addresses its internalization and interiorization. The principle asserts:

> Students must practice reasoning in order to internalize and interiorized specific ways of thinking and ways of understanding.

The repeated attempt by the NT experiment students to attend the underlying common structure to the implicit recursion problems and the explicit recursion problems helped them first internalize and then interiorized quasi induction. In turn, the repeated reasoning in applying quasi induction to solve these two types of problems help them assimilate the principle of MI almost effortlessly. Thus, repeated reasoning, not mere drill and practice of routine problems, is an important mechanism for solidifying concepts.

The DNR as a system. The three principles, duality, necessity, and repeated reasoning, constitute a system in the sense that they interdependently address three fundamental aspects of the learning-teaching process (Figure 5).

1. Instructional objectives. In designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding have been our ultimate cognitive objectives. We have addressed them simultaneously, for one affects the other.
2. Formation of Concepts. Meaningful concepts can only be elicited through solutions to problems that correspond to students' intellectual need (the necessity principle). The concepts elicited as well as the ways in which they are elicited constitute and at the same time impact students' ways of thinking and ways of understanding.
3. Internalization/Interiorization of concepts. Ways of thinking and ways of understanding are internalized and interiorized through repeated solutions to fundamental problems.

Insert Figure 5 About Here

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Figure 1

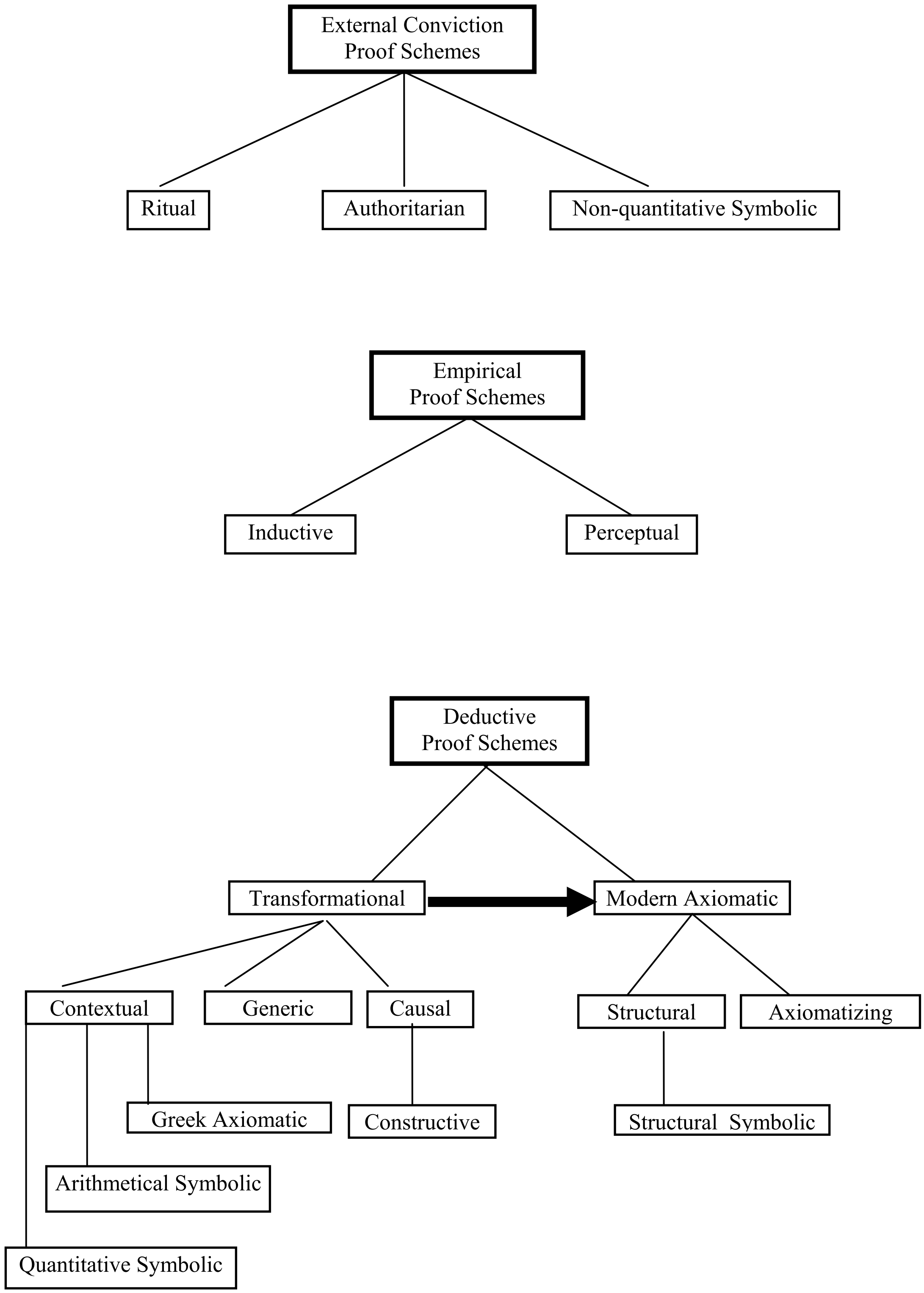
External Conviction Proof Schemes
Ritual
Authoritarian
Non-quantitative Symbolic
Empirical Proof Schemes
Inductive
Perceptual
Deductive Proof Schemes
Transformational
Modern Axiomatic
Contextual
Generic
Causal
Structural
Axiomatizing
Greek Axiomatic
Constructive
Structural Symbolic
Arithmetical Symbolic
Quantitative Symbolic

Figure 2

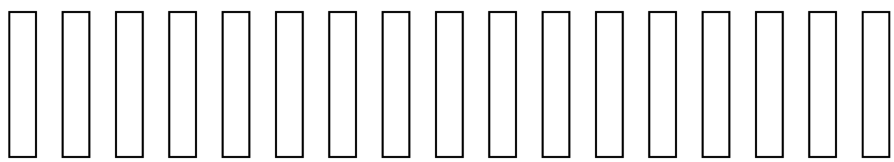

Figure 3

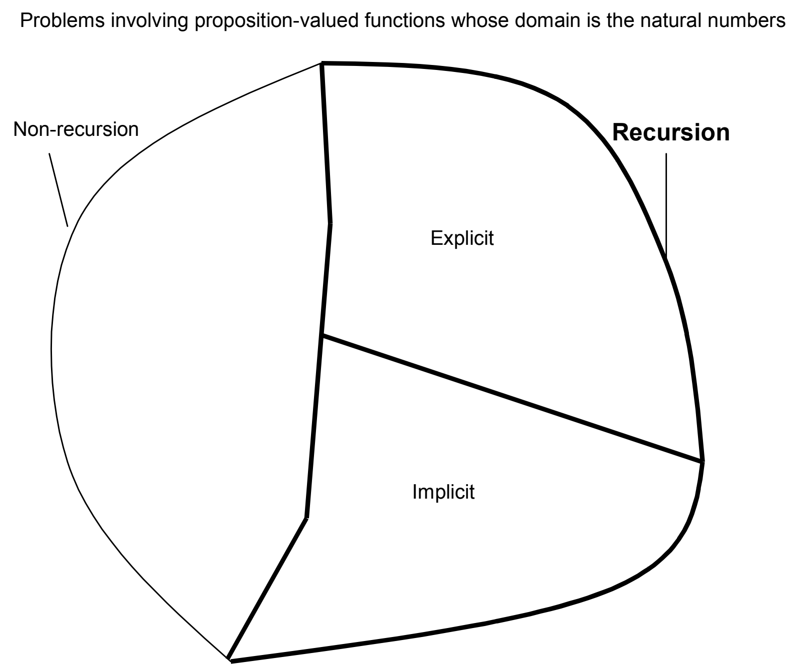
Problems involving proposition-valued functions whose domain is the natural numbers
Non-recursion
Recursion
Explicit
Implicit

Figure 4

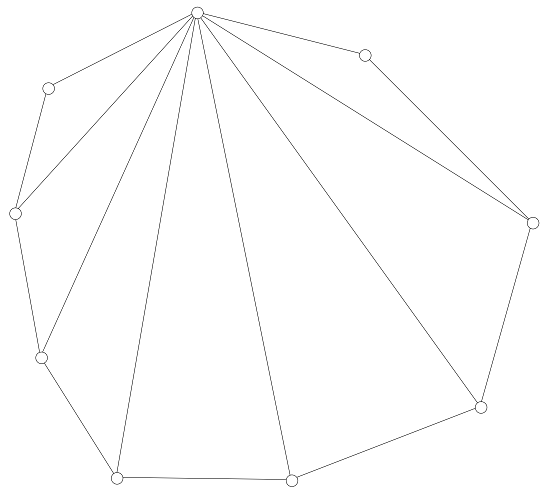

Figure 5

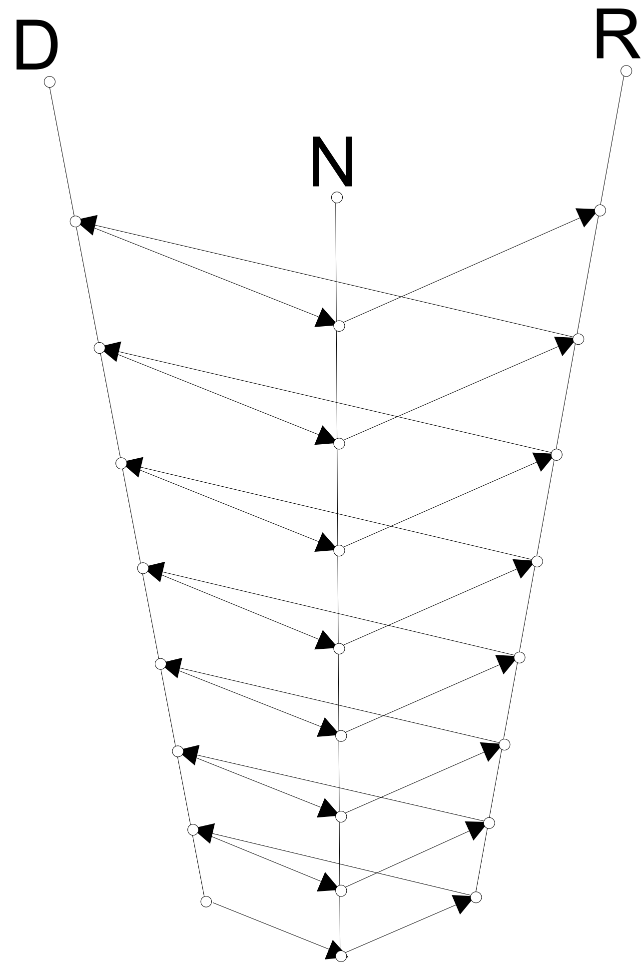

Figure Captions

Figure 1: A conceptual framework for students' proof scheme*s* consisting of three classes of proof schemes. The arrow from the transformational proof scheme category to the modern proof scheme category indicates that transformational reasoning is always present in the modern axiomatic proof reasoning, but the converse is not necessarily true.

Figure 2: A domino analogy to MI.

Figure 3: A classification of problems involving proposition-valued functions whose domain is the natural numbers.

Figure 4: A polygon representing an n-gon drawn by a student to explain why the sum of the measures of the interior angles in a convex n-gon is $180^0(n-2)$ ..

Figure 5: The DNR system.

[1] This paper focuses on cognitive and didactical considerations of mathematical induction in two instructional treatments: one standard, the other new. Initially no historical-epistemological factors were considered in designing and implementing the alternative treatment. However, recent historical analysis of MI has revealed a pleasantly surprising observation: the path of conceptual development in the alternative treatment parallels the historical development of MI. This observation will be reported in a sequel to this paper (Harel, McClure, & Sowder, in preparation).

[2] I thank Alfred Manaster, Maria Alessandra Mariotti, James McClure, and Larry Sowder for the many stimulating conversations on issues addressed in this paper.

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[4] See Footnote 3.

[5] The term "scheme" is used here in the sense of Piaget: "The part that is common to applications or repetitions of the same action" (Vuyk, 1981, p. 63). von Glasersfeld (1991, pp. 55-56) specifies the functionality of schemes: "Schemes are composed of three elements: (1) an initial experiential item or configuration (functionally linked to what the observer would categorize as 'trigger' or 'stimulus'), (2) an activity the subject has associated with it, and (3) a subsequent experience associated with the activity as its outcome or result." The essential actions in a proof scheme are "persuasion" and "ascertainment." These actions are applied in situations one desires to evaluate hypotheses. The kind of persuasion and ascertainment one associates with such situations depends on her or his experience of the outcome actions applied. For example, the inductive proof scheme is most likely the result of the everyday experience in which people's evaluation of hypotheses is probabilistic in nature.

[6] For a discussion on the ineffectiveness of traditional analogy-based instruction, including the domino analogy to MI, see Greer and Harel (1998).

[7] Any algebraic expression containing an ellipsis is a recursive representation of a function.

[8] An example of a solution that does not use induction is the following: Let $f(x) = x - 2^x$. $f(1) < 0$, $f(2) < 0$, and $f(3) < 0$. Since $f'(x) = 1 - 2^x \ln x < 0$ for $x \geq 3$, f is decreasing on $[3, \infty)$. Thus, $f(x) < 0$ on $[3, \infty) \cup \{1, 2\}$.

[9] Some readers of Harel and Sowder (1998) thought that transformational proof scheme refers only to students' arguments using visual, spatial motions. There is no condition in the definition of transformational proof scheme that restricts its applicability to algebraic situations. In fact, both in Harel and Sowder (1998) and here transformational proof scheme is demonstrated with algebraic examples. In the former publication, however, most of the examples dealt with geometry and graphs, which might be the cause of the misinterpretation. This point is further addressed in Harel (in press)

[10] Fibonacci numbers, $f_0, f_1, f_2, \ldots$ are defined by the equations $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.

[11] In presenting the Tower of Hanoi Problem, the instructor suggested to the students to first think in terms of specific number of disks.

[12] Generally speaking, an "intrinsic problem" is one the student understands and appreciates. A specific characterization of this term is in Harel (in press)

[13] Generally speaking, an "alien problem" is one that is not intrinsic. A specific characterization of this term is in Harel (in press)

[14] For more discussion on the notion of causality see Harel (1999).