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Chapter 7

FROM NAIVE-INTERPRETIST TO OPERATION-CONSERVER

Guershon Harel

An interesting phenomenon has been observed in research on the concepts of multiplication and division: Many children, and even adults, do not understand that problems that share the same story line and differ only in their numerical data can be solved by the same arithmetical operation. Consider, for example, the following observation made by Af Ekenstam and Greger (1983) and reported in Greer (1988, p. 283). Twelve-to-thirteen-year-old subjects were presented with two problems:

A_1. A cheese weighs 5 kg; 1 kg costs 28 kr. Find out the price of the cheese. Which operation would you have to perform?

$28 \div 5 \quad 5 \times 28 \quad 5 \div 28 \quad 28 + 28 + 28 + 28 + 28$

A_2. A cheese weighs 0.923 kg; 1 kg costs 27.50 kr. Find out the price of the cheese. Which operation would you have to perform?

$27.50 + 0.923 \quad 27.50 \div 0.923 \quad 27.50 \times 0.923 \quad 27.50 - 0.923$

In a written test, 83% of the students correctly chose multiplication for Problem A_1, but only 29% chose it for Problem A_2. In interviews, when these problems were presented in succession, few students recognized the relationship between them, even when the interviewer drew attention to it.

If a person recognizes that a solution operation for one problem can be conserved as a solution operation for another problem, we say that the two problems are equivalent for that person. In Af Ekenstam and Greger's study, for example, only a few of the subjects recognized that Problems A_1 and A_2 are equivalent; according to other studies, this would be the case with adult subjects as well (see, for example, Graeber, Tirosh, & Glover, 1989; Harel, Behr, Post, & Lesh, 1994). Analogous to the Piagetian nonconservations, Greer (1988) calls this phenomenon, "nonconservation of operation."

The consensus is that the origin of this phenomenon lies in the initial conception children build when they learn multiplication. When first introduced, and long thereafter, multiplication is taught in a restricted context typified by such problems as:

B_1. There are 4 teams, 12 players on each team. How many players are there altogether?

B_2. One pound of meat costs $4.25. Dan bought 6 lb of meat. How much did he pay?

Notice that in these two problems the *multiplier* (i.e., the quantity representing the number of equivalent collections, such as "4 teams" and "6 lb") is a whole number. Because for a long period of time the only multiplication situations children deal with are those with a whole number multiplier, they build a restricted conception in which the *multiplicand* (i.e., the quantity representing the size of each collection, such as "12 players" and "$4.75") must be repeated a whole number of times. Fischbein, Deri, Nello, and Marino (1985) call this a *repeated addition* conception. This explains why the subjects in Af Ekenstam and Greger's study couldn't see Problems A_1 and A_2 as equivalent problems and why they were much more successful in solving the former than the latter: In Problem A_1, the multiplier is represented by a whole number, 5, and, therefore, they could additively repeat the multiplicand, 28 kr, a whole number of times: 28 kr + 28 kr + 28 kr + 28 kr + 28 kr. In Problem A_2, on the other hand, the multiplier is a decimal fraction, 0.923 kg, in which case subjects' conception of multiplication—the repeated addition conception—was inapplicable.

The nonconservation-of-operation phenomenon, which is consistent with numerous other observations (indicating that numbers affect the interpretation of word problems), is particularly important because, as Greer (1988) has indicated, "It can be taken as symptomatic of the difficulty of reconceptualizing multiplication and division on moving from the integer domain to the domain of positive real numbers, the difficulty which lies at the heart of the problems that schoolchildren have with multiplication and division" (p. 289). Researchers in mathematics education have debated the questions of how the concept of multiplication and division develop and how they should be taught. Fischbein et al. (1985), for example, have argued that multiplication as repeated addition "correspond[s] to features of human mental behavior that are primary, natural, and basic" (p. 15). Greer (1988) pointed out that this is a pessimistic view that, as Fischbein agrees, restricts the role of instruction to merely providing "learners with efficient mental strategies to control the impact of [their intuition]" (Fischbein et al., 1985, p. 16).

Rather than speaking of mental strategies to *control* intuition, instruction should help children build on their intuition in modifying and refining their conceptions. To provide such instruction one needs an understanding of children's learning processes. Accordingly, this chapter aims to share with teachers some cognitive analyses of the reasoning employed by children in solving multiplicative problems and analyses of research data concerning systematic actions taken by subjects in solving them. These analyses have helped us better understand what multiplication is and how the transition from additive reasoning to multiplicative reasoning may take place. I hope that this chapter will stimulate teachers' interest in probing their students' learning processes of multiplicativity and in developing their own instructional approaches to enhance its acquisition.

The paper consists of two parts. In the first part I examine two behaviors associated with the nonconservation of operation. I suggest that one of these behaviors is an unavoidable conceptual stage a child *must* go through in moving from additive reasoning to multiplicative reasoning, whereas the other is primarily a consequence of faulty instruction. In the second part of the paper I discuss the concept of multiplication in the whole number domain and its extension to the rational number domain.

NAIVE-INTERPRETISM AND OPERATION NONCONSERVATION

The nonconservation of operation involves two phenomena that are worth distinguishing. When subjects are presented with two equivalent (to us) problems in succession—one dovetailing with the repeated addition conception (e.g., Problem A_1) and the other conflicting with it (e.g., Problem A_2)—two phenomena can be observed:

1. Children provide *different* solution operations to the two problems.
2. One of these solutions is meaningful and *right*; the other is based on superficial considerations and produces a *wrong* answer.

A plausible explanation for the first phenomenon is that subjects do *not* see an underlying common structure to the two problems because their interpretation of one problem situation is different from their interpretation of the other. As a result, they do not employ the same kind of reasoning and, therefore, they provide—as should be expected—*different* kinds of solutions to the two problems. In the most innocent and nonpejorative sense of the term, we call these subjects "naive-interpretist." We suggest the term *naive-interpretism* because it points to the underlying source of this phenomenon. The term *operation nonconservation*, on the other hand, refers to its outcome.

The second phenomenon, however, raises a puzzling question. In all cases of operation nonconservation reported in the literature, subjects provide a *correct* operation to a problem whose numerical data conform to the repeated addition conception (e.g., when the problem multiplier is a whole number) but *incorrect* operations to an (equivalent) problem whose numerical data conflict with this conception (e.g., when the problem multiplier is a decimal fraction). Further, this research has indicated that the kind of reasoning employed by children (and adults) to produce these incorrect operations is based merely on considerations of numerical relationships. For example, most of the subjects in the study presented earlier chose 27.50 ÷ 0.923 for Problem A_2, with the explanation that "the piece of cheese must cost less than 27.50, so I must divide to get a smaller number" (Af Ekenstam and Greger, 1983; quoted in Greer, 1988, p. 283). Sowder (1988) found that subjects' choice of operation can be based on even more superficial considerations. For example, some of his subjects believed that for a problem with one "big number and a little number," division is the most likely correct solution (p. 229). One gets the impression from these findings that the second phenomenon is a *necessary* consequence of the first; namely, a naive-interpretist is incapable of solving meaningfully multiplicative problems (such as Problem A_2) that involve non-whole number multipliers. My question is: Must this be the case?

I argue that the two phenomena are independent of each other in that while being a naive-interpretist is an unavoidable conceptual stage a child *must* go through in moving from additive reasoning to multiplicative reasoning, the behavior of applying superficial considerations in solving problems is primarily a consequence of faulty instruction. In the previous chapter, Larry Sowder pointed to the computation-centered school mathematics curricula and low demand for meaning and reasoning as aspects of instruction responsible for the superficial considerations children employ in solving problems. In conjunction with these sources of difficulty are the instrumental strategies (that is, "rules without reason," according to Skemp, 1978) taught merely to enable children to get the right answer. The key-word strategy is an example. It is widely recognized and strongly condemned by educators. Another equally harmful strategy is the conservation-

formula strategy (see below). It has not been debated in the mathematics education literature, but it is particularly relevant to the conceptual transition from additivity to multiplicativity and, therefore, will be given greater attention in this paper. These kinds of instrumental strategies do indeed help children get the right answer, but they are of great harm to children's mathematical development in that they delay and even block the transition to multiplicative conceptions.

The Key-Word Strategy. Despite its condemnation by mathematics educators, the key-word strategy is still, unfortunately, in use in many schools. For example, in a recent official district-wide learning-objectives document, the following objectives appear under the category "Sentences and Problem Solving":

> Learn key words for solving addition and subtraction problems.
> Learn key words for solving multiplication and division problems.

With such instructional objectives, one wonders if the following dialogue reported in Harel and Behr (in press) should be of surprise:

Interviewer: Each package of typing paper weighs 0.55 kg. Adam used 0.35 of a package for his research paper. How many kilograms of paper did he use?

Teacher: One package weighs 0.55 kilograms. And Adam used 35 hundredths of that one package. How many did he use? In order to find out how many he used, *I've got to subtract.*

Interviewer: How would you explain your solution to a child?

Teacher: What I would say to them, I'd have them read the problem to me and my question would be, *"Do you have to add, subtract, multiply, or divide?"* All right. And from the *last question, "How much did he use?"* they know they have to take away. *That's what they're looking at, is the last question.* So then I'd say to them, "How much did he start out with?" Fifty-five kilograms. "How much did he use up?" Thirty-five kilograms. And then everything's set up all right, so then we subtract.

Conservation-of-Operation Formula. The second equally harmful strategy is the ready-made conservation formula, which might be formulated as follows:

When you encounter a word problem with "nasty" numbers,

(a) replace the "nasty" numbers with "friendly" numbers;
(b) solve the problem with the "friendly" numbers;
(c) transform back your solution to the problem with the "nasty" numbers.

For example, using the conservation formula, one can "solve" Problem A_2 above by

(a) replacing the multiplier, 0.923 kg, with a "friendly" multiplier, say 4 kg;
(b) solve the problem with the new multiplier by the operation 4×27.50;
(c) transform this operation into 0.923×27.50 (by replacing 4 by 0.923), which will be your solution for the original problem.

While there is a wide consensus against the use of the key-word strategy, the use of the conservation formula may be tolerated and even encouraged by some educators, who may argue that unlike the key-word strategy, the conservation formula always leads to a correct solution. This, of course, is a true argument. But, it constitutes no justification for those who believe that students' actions must be judged not by their outcomes, correct or wrong, but by the reasoning processes they involve. In my perspective, should students choose to apply the conservation formula, they *must* have a relational understanding of it (i.e., knowing why, not just how, the formula works [Skemp, 1978]).

The question is: Can the conservation formula be learned relationally? My answer to this question is: Students cannot learn the conservation formula relationally before they have become multiplicative reasoners, but by the time they become multiplicative reasoners they will have constructed the conservation formula. In other words, the construction of the conservation formula is a culminating stage in children's understanding of multiplication, not the beginning of it. Consequently, the use of the conservation formula in an early stage in the child's construction of multiplicativity can only be instrumental, not relational.

The instrumental use of the conservation formula can cause serious harm to children's mathematical development. One conspicuous consequence is that by using formulas they don't understand, children build a distorted conception of mathematics as a subject that does not require intrinsic justification (Harel, 1994). Another consequence pertaining to the development of multiplicative reasoning is this: It has been established by Piaget and others (see, for example, Thompson, 1985) that the main tool for modifying existing conceptions is true problem-solving activity where the learner applies existing conceptions to solve the problems and modifies these conceptions when encountering cognitive conflicts. For example, if a child chooses the expression $27.50 \div 0.923$ as the solution of Problem A_2, computes the result, and evaluates its reasonableness, he or she may encounter a conflict between the operation chosen and the outcome expected. Such a conflict, or a "disequilibrium" as Piaget calls it, can lead the child to question her or his prior action and seek new ways to resolve it. It is these cognitive conflicts and the resolutions invented by the child that define learning and constitute a gradual transition from additivity to multiplicativity. The use of the conservation formula sterilizes these conflicts and gives both the child and the teacher the illusion of accomplishment, where in fact the child is *not* experiencing any problematic situations that can bring her or him to invent ways of thinking multiplicatively.

Naive, Yet Meaningful, Interpretism

This discussion brings us back to the question: Can children cope with multiplicative problems relationally while they are naive-interpretists? Our argument was that a naive-interpretist stage does not necessarily imply inability to employ meaningful reasoning in solving multiplicative problems. The following two examples present cases supporting this argument and give some insight into how children employ cognitive tools available to them in dealing with multiplicative problems during the naive-interpretist stage.

Example 1.

The first example is from an interview conducted simultaneously with two children, a 13-year-old girl, Tami, and an 8-year-old boy, Dan.

Interviewer: One pound of candy cost $7. How much would 3 pounds of candy cost?
Tami: Three times seven, 21.
Dan: I agree, three times seven.
Interviewer: What if I changed the 3 into 0.31? What if the problem were: One pound of candy cost $7; how much would 0.31 of a pound cost?
Tami: The same. It is the same problem, you have just changed the number, 0.31 times 7.
Dan: No way! It isn't the same. Can't be [angrily]. It isn't times. Why did you [speaking to the interviewer] agree with her?
Interviewer: I didn't agree with her, I'm just listening to both of you. How would *you* solve the problem?
Dan: [After a short pause], you take 1 and you divide by 0.31. You take that number, whatever that number is, and you divide 7 by that number.

It took the interviewer a long moment to realize, not before applying some algebraic manipulations, that Dan's solution was correct. Dan's solution was: first perform the operation $1 \div 0.31$, then take the result of this operation and divide 7 by it, namely: $7 \div (1 \div 0.31)$. Indeed, $7 \div (1/0.31) = 7 \times (0.31/1) = 7 \times 0.31$.

Clearly, Dan was a naive-interpretist. He saw the two questions asked as different problems and therefore rejected the idea that the solution of the first problem could be conserved as a solution to the second. Despite this he was able to offer a correct and, more importantly, a meaningful solution. Although this is merely an anecdotal case, it does demonstrate that, while they are naive-interpretist, children can reason meaningfully about problems with decimal fraction multipliers.

As is common with children Dan's age, it was difficult to persuade him to explain his solution because he viewed it as self-evident; when he did provide an explanation, it wasn't easy to interpret. The following description of what possibly was his solution process is based on his fragmented responses and my fill-in-the-gap interpretations. The description of his solution is accompanied by geometric representations that can better illustrate his solution process.

1. You would pay $7 for a whole 1 lb of candy (see Figure 7.1).

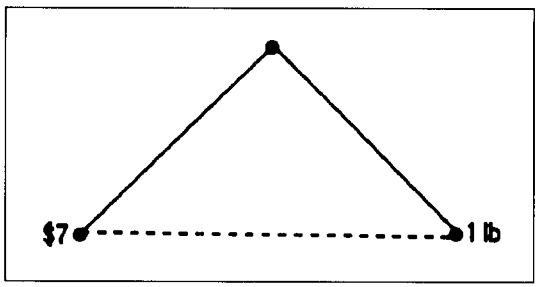

Figure 7.1

2. But, you are not to buy a whole pound; you are to buy only a portion of 1 lb, 0.31 lb It is only natural then that you would pay only a portion of $7 (see Figure 7.2).

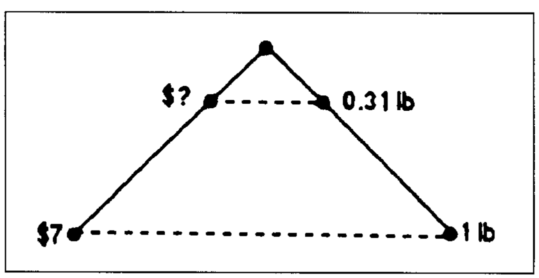

Figure 7.2

3. To find the exact amount you will have to pay, imagine that you distribute the 1 lb of candy into packages of 0.31 lb each (see Figure 7.3).

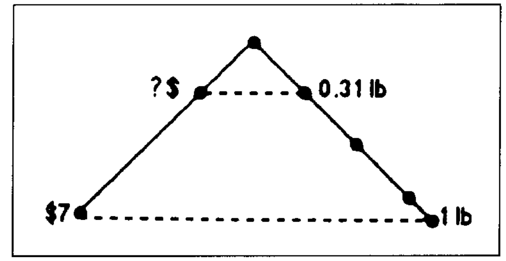

Figure 7.3

4. By dividing 1 by 0.31 (1÷ 0.31), you will find the number of 0.31-lb packages in 1 lb [Dan wasn't bothered by the fact that 1 lb is distributed into a non-whole number of packages] (see Figure 7.4).

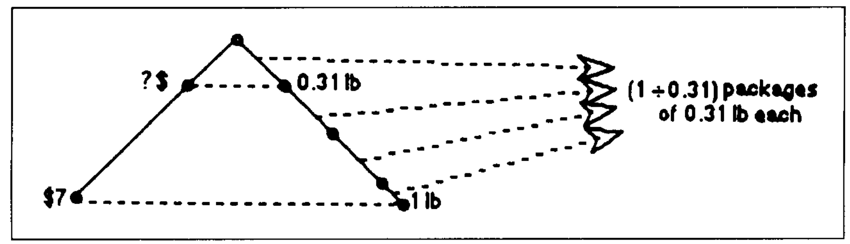

Figure 7.4

5. The cost of 1 lb should be distributed equally among the number of 0.31-lb packages; that is $7 \div (1 \div 0.31)$ is the price for one package of 0.31 lb (see Figure 7.5).

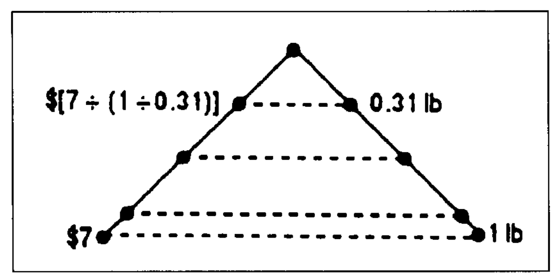

Figure 7.5

The figures accompanying the solution process may remind us of the proportionality of similar triangles. Of course, it is unlikely that Dan was thinking in these terms, but perhaps we can say that Dan was conceptually ready for the concept of proportion; using Vygotsky's

(1978) term, the concept of proportion was, at the time of the interview, in Dan's zone of proximal knowledge.

Example 2.

A more typical naive-interpretist solution to a problem with a non-whole number can be seen in the following example. Gina, a sixth-grade child, was asked to solve the following problem:

C. The shipment of 1 m^3 of cargo from the U.S. to Europe is $185. Don wants to ship four different pieces of cargo from New York to London. The volume of Cargo A is 3 m^3; Cargo B, $^4/_7$ m^3; Cargo C, 4.17 m^3; and Cargo D, 0.23 m^3. Help Don fill out the following shipment form:

Cargo	Volume	Cost
A	3 m^3	
B	$^4/_7$ m^3	
C	2.7 m^3	
D	0.17 m^3	

Figures 7.6–7.9 represent pictorially Gina's solution process.

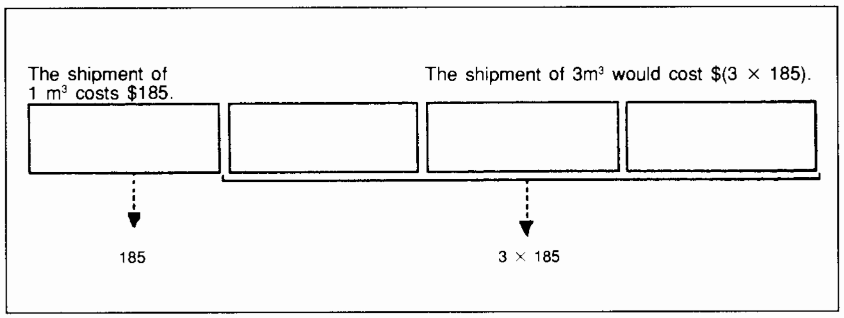

Figure 7.6

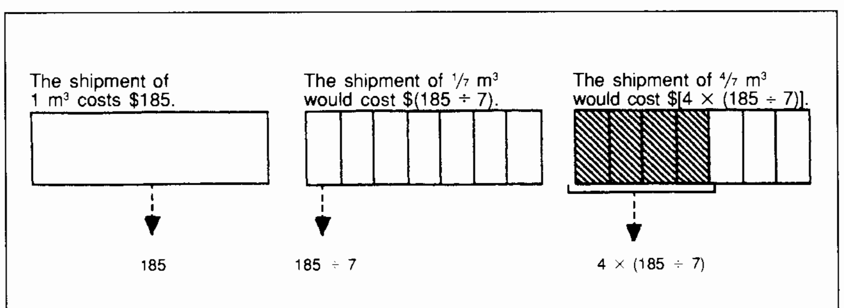

Figure 7.7

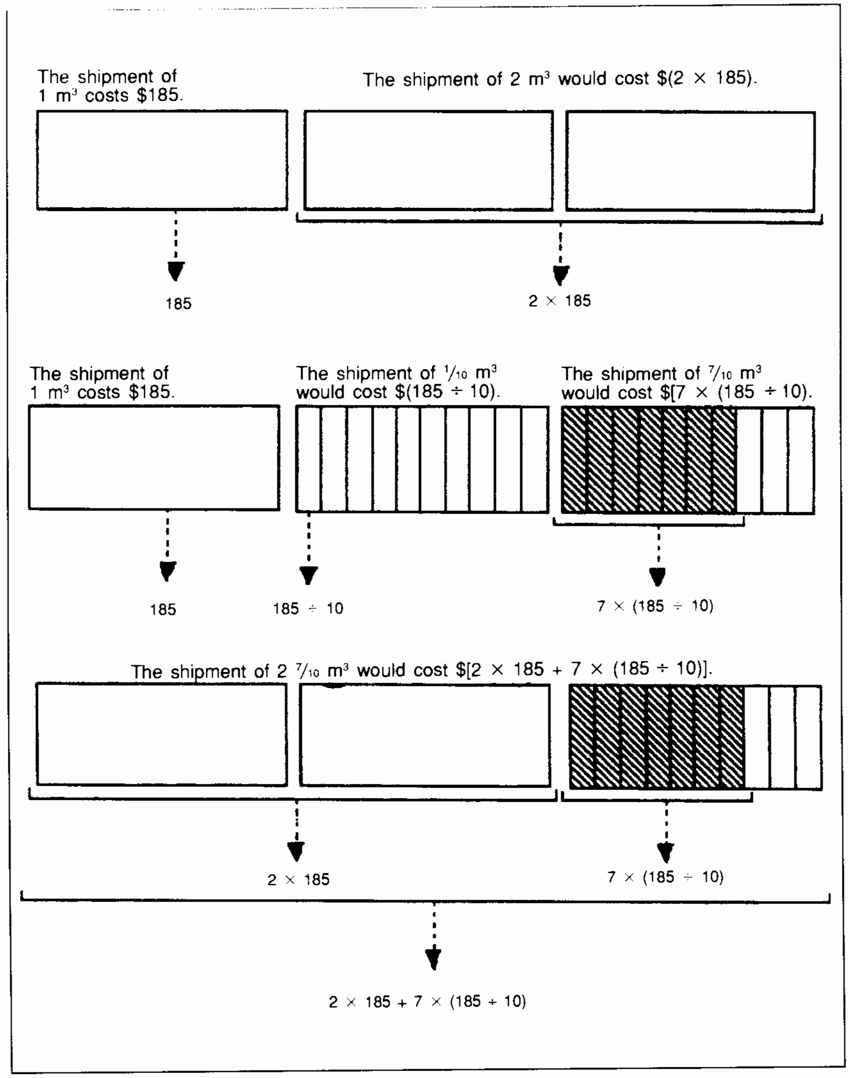

Figure 7.8

A year later Gina was presented again with Problem C. At this time Gina viewed Problem C's four cases as instantiations of *one* multiplication case. Table 7.1 summarizes the two kinds of solutions offered by Gina, first as a naive-interpretist and then (in the later interview) as an operation-conserver.

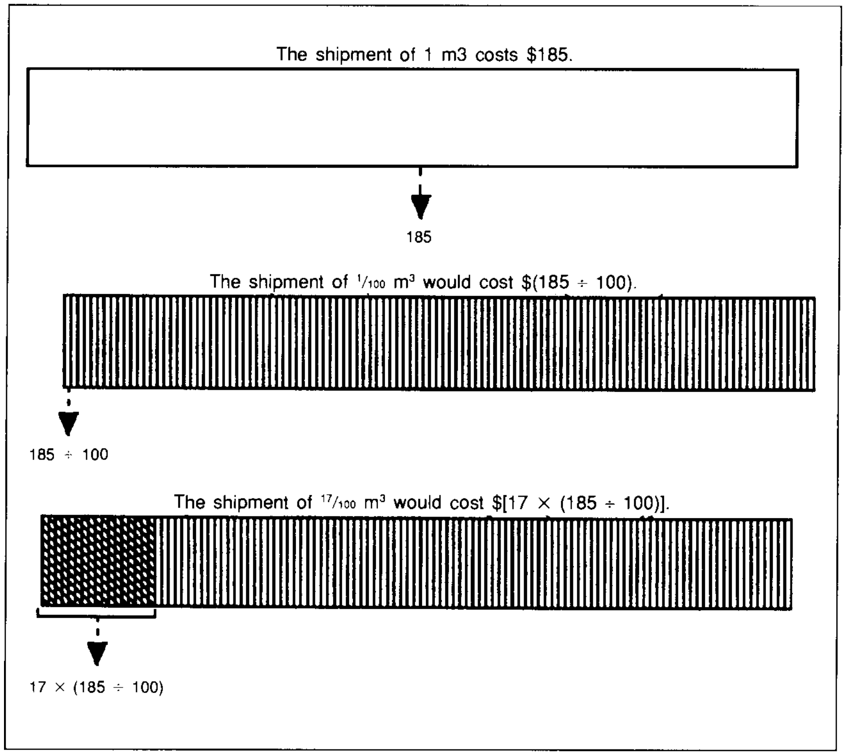

Figure 7.9

TABLE 7.1

Gina's Two Types of Solutions for Problem C.

Cargo	Volume	Cost Gina's solutions	
A	3 m^3	$(3 × 185) = 555	$(3 × 185) = 555
B	4/7 m^3	$[4 × (185 ÷ 7)] = 105.71	$(4/7 × 185) = 105.71
C	2.7 m^3	$[2 × 185 + 7 × (185 ÷ 10)] = 499.50	$(2.7 × 185) = 499.50
D	0.17 m^3	$[17 × (185 ÷ 100)] = 31.45	$(0.17 × 185) = 31.45

The two examples just presented demonstrate that children can and should be able to use their existing conceptions to cope with multiplicative problems with non-whole number multipliers. The question of great interest to us is: How do children extend their conception of multiplication from the whole number domain to the rational number domain? For example, how did Gina build a conception that enabled her to see the four cases in Problem C as instantiations of one multiplication case? A more fundamental question is: What, from a cognitive point of view, is whole number multiplication? The second part of this chapter deals with these questions.

MULTIPLICATIVE CONCEPTUAL SUBFIELD

Vergnaud (1983, 1988) suggested that the concepts of multiplication, division, fractions, ratio and proportions, and many other advanced concepts in mathematics develop not in isolation but in connection with each other. The term multiplicative conceptual field (MCF) was introduced by Vergnaud, and it is used by mathematics educators to refer to these concepts and their connectedness. To a large extent, until recently (see, for example, the works of several researchers reported in Harel and Confrey, 1994), research has examined the development of individual multiplicative concepts, without much effort given to dealing with the interrelations and dependencies within, between, and among these concepts. For example, research on the learning of the rational number concept did not take into account children's conceptions of multiplication and division, and research on the learning of the decimal system was somehow separated from research on fractions and proportionality.

Within the enormous structure of the MCF one can identify smaller structures that, although interconnected, are, to some extent, autonomous. The development of the concept of multiplication is one such smaller structure. My perspective on this development is that it occurs in three stages: from an early stage of whole number multiplication, to a naive-interpretist stage, and then to an operation-conserver stage. We call this structure a *multiplicative conceptual subfield* (MCS) because it represents a closed unit within the greater structure of the MCF.

In the first stage, children learn how to think of whole numbers as multipliers. This way of thinking will be discussed in some detail in the next section. The other two stages require a broad background from current research in multiplicative structures—particularly, the works of Kieren (1988), Steffe (1994), Behr, Harel, Post, and Lesh (1992), Thompson (1992, 1994), and Kaput and Maxwell-West (1994)—that is beyond the scope and the space limitations of this paper. Roughly speaking, in the second stage children treat fractions as pseudo multipliers; we have seen this type of thinking in Gina's first solution of Problem C (see Figures 7.6–7.9). In this solution, Gina used fractions as a composition of a whole number multiplier and a whole number divisor. For example, as was illustrated by Figure 7.7, to find the cost of the shipment of $^4/_7$ m^3, she first found the cost of $^1/_7$ m^3 (using the operation 185 ÷ 7, which is a whole number division operation); then she found the cost of $^4/_7$ m^3 (using 4 × [185 ÷ 7], which is a whole number multiplication operation).

The conception of fractions as pseudo multipliers is affected by several conceptions, two of which are the quotitive division operation and the partitive division operation. To define these operations, consider, for example, the two steps in Dan's solution. The first step was to find how many 0.31 lb quantities are in 1 lb. *An action that is taken by an individual to find how many times a given quantity is contained in another quantity is called "quotitive division."* The second step in Dan's solution was to divide the $7 equally among the number of 0.31-lb-packages in order to determine the cost of each package. *An action that is taken by an individual to find the size of an object resulting from equal sharing is called "partitive division."* But as Thompson (1994) has indicated:

> A decision to divide need not be based always on considerations of partition or quotition. It could also be made relationally—as when one conceives of a situation multiplicatively and the information being sought pertains to an initial condition, such as "How long must one travel at 4 miles/hour to go 12 miles?"

That is, one can create a multiplier for the purpose of finding the problem unknown by means of building relations to a corresponding multiplication situation. (This operation will be discussed in the next section.) For example, a child may approach this problem by extrapolating the multiplier from a sequence of multiplication problems: If you travel 5 hours at 4 miles/hour, you would go 20 miles; if you travel 2 hours at 4 miles/hour, you would go 8 miles; if you travel 4 hours at 4 miles/hour, you would go 16 miles; if you travel 3 hours at 4 miles/hour, you would go 12 miles.

The third and culminating major stage in this MCS is the conception of fractions as multipliers. We have seen such a conception in Tami's solution (viewing 0.31 as a multiplier in the same way she viewed 7 as a multiplier) and in Gina's second solution (viewing the four cases of Problem C as instantiations of one multiplication case).

We will return now to discuss in some detail the first stage of this MCS, the stage in which whole numbers are conceived as multipliers.

Whole Numbers as Multipliers

This section consists of two parts. In the first part I examine research data on subjects' performance on non-whole number multiplication problems and derive from this examination a new perspective on the impact of the type of multiplier on the relative difficulty of these problems. On the basis of this perspective, we will offer in the second part of this section a cognitive definition of whole number multiplication.

The Impact of Multiplier Types: A Different Perspective. As we have indicated earlier, under the conception of multiplication as repeated addition, a multiplier must be a whole number; when it isn't, subjects can encounter difficulties in solving the problem correctly. Consider, for example, a sample of results from Harel et al. (1994). As can be seen from Table 7.2, the percentage of correct responses on problems with a whole number multiplier is very high, whereas the performance on problems whose multiplier is a decimal fraction drops by about 27%. This, as was explained in the beginning of this chapter, is consistent with the effect of the repeated addition conception theorized by Fischbein et al. (1985).

TABLE 7.2

Distribution of Responses to Multiplication Problems across Two Types of Multipliers: Whole Number and Decimal Fraction.

Type of Multiplier	% Correct Responses
Whole number	96.3
Decimal fraction	69.6

A further analysis reveals, however, that not all decimal multipliers have the same effect on problem difficulty. To illustrate, consider Table 7.3, in which the decimal fraction multiplier results from Table 7.2 are broken down into two categories: The first category consists of results on problems with a decimal multiplier *greater than one*, such as in the problem:

D_1. A cheese weighs 5.15 kg; 1 kg costs 4 dollars. Find out the price of the cheese.

The second category consists of results on problems with a decimal multiplier *smaller than one*, such in the problem:

D_2. A cheese weighs 0.87 kg; 1 kg costs 4 dollars. Find out the price of the cheese.

As can be seen from Table 7.3, although both categories of problems violate the repeated addition conception, their effect on subjects' performance is not the same. Whereas the effect on the level of difficulty in changing the type of the multiplier from a whole number to a decimal greater than one is moderate (only about 13%), the effect on the level of difficulty in changing the multiplier from a whole number to a decimal smaller than one is excessive (over 40%). The question is: What is the conceptual basis for this finding?

TABLE 7.3

Distribution of Responses to Multiplication Problems Across Three Types of Multipliers: Whole Number, Decimal Fraction Greater Than One, and Decimal Fraction Smaller Than One.

Type of Multiplier	% Correct Responses
Whole number	96.3
Decimal fraction > 1	83.7
Decimal fraction < 1	55.5

To explain this finding, Fischbein et al. (1985) suggested that a decimal multiplier whose whole part is larger than its fractional part may be treated more like a whole number, as though the whole part "masks" or "absorbs" the fractional part. For example, the decimal multiplier 5.15 in Problem D_1 (above) may be treated as 5, in which case the repeated addition conception is satisfied.

But, what does this explanation mean in terms of the subject's cognitive process? Why would a subject treat the multiplier 5.15 kg in Problem D_1 as 5 kg but not treat the multiplier 0.87 kg in Problem D_2 as 1 kg? The metaphorical terms *mask* and *absorb* do not help us to understand what accounts for the relative difficulties of Problems D_1 and D_2. Further, a closer look at data from Fischbein's and others' studies reveals an observation that is not consistent with the "absorption effect" explanation. The observation is that the relative difficulty of multiplication problems does not correspond to whether the multiplier is greater or smaller than one, as has been indicated in the literature, but to whether the multiplier is greater or smaller than two (Harel et al., 1994). For example, if we compare the relative difficulties of Problems D_1, D_2 (presented above), and D_3,

D_3. A cheese weighs 1.32 kg; 1 kg costs 4 dollars. Find out the price of the cheese,

it is likely that Problem D_1, whose multiplier is greater than *two*, would be the least difficult, and Problems D_2 and D_3, whose multipliers are less than *two*, would be of similar difficulty despite the fact that the multiplier of one is smaller than one and of the other is greater than one. This observation per se has only indirect pedagogical importance in that it restructures the relative difficulty of multiplication problems with non-whole number multipliers. But, lurking behind this observation—and this is its main importance—is a clue as to what the concept of multiplication is.

Whole Number Multiplication as an Equal-Quantity Iteration Pattern[1]. A whole number multiplier models actions of accumulation of repeated quantity distribution, repeated quantity duplication (Thompson, 1992), and so on. A child can act out a problem such as

E. There are 12 kids; each kid gets 7 marbles. How many marbles do the kids get altogether?

by imagining herself or himself distributing 12 groups of 7 marbles each. The action of repeated distribution does not, however, have to take place in its entirety in the child's mind; it is sufficient that the child acts out several rounds of the distribution until recognizing its pattern. For example, in the process of acting out Problem E, the child may carry out only a few distribution rounds, thinking, "7 marbles to one kid, 7 marbles to the second kid, 7 marbles to the third kid, *and so on*" thus, $7 + 7 + 7 + \ldots + 7$, twelve times, or 12×7. The "and so on" represents a recognition of a pattern, the pattern of repeated distribution of an equal quantity. But, as we know from our own experience, we can recognize a pattern of an action only after we have repeated the action several times; a single execution of the action is insufficient to recognize its pattern; at least *two* executions are necessary. Accordingly, for a child to extract a pattern of a distribution, he or she must act it out mentally at least twice (e.g., $7 + 7$), for one round of distribution does not necessarily imply a pattern of equal-quantity distribution. Accordingly, when the multiplier is greater than (or equal to) *two* (e.g., 12, as in Problem E), the child can act out the quantity distribution at least twice (e.g., $7 + 7$), which can lead her or him to a pattern of equal-quantity distribution (e.g., $7 + 7 + 7 + 7 + \ldots + 7$, twelve times). On the other hand, when the multiplier is smaller than *two*, a pattern of equal-quantity distribution may not be observed. It should be expected, therefore, that the relative difficulty of multiplication problems would correspond to whether the multiplier is greater or smaller than *two* (as was suggested in Harel et al.'s study (1994)), not *one* as has been held in the literature.

The equal-quantity iteration pattern conception has several important features:

1. It develops through problematic situations. The equal-quantity iteration pattern conception is a way of thinking that children develop gradually while coping with multiplicative situations, especially those with large multipliers. Ways of thinking, as was explained in the previous section, develop when the child feels a need to overcome a difficulty or to resolve a conflict. Such a need is likely to develop when the child encounters multiplication problems with "large" multipliers, because iterating a quantity a large number of times in each multiplication problem is likely to constitute a difficulty that the child would want to resolve. The construction of an equal-quantity iteration pattern seems a natural resolution.

2. *It is an operative thought*. The equal-quantity iteration pattern is a way of thinking, whose ultimate characterization is the ability to visualize the operation—say, putting together 12 groups of 7 marbles—without carrying out the activity itself. That is, the child can build a complete plan of what to do and be certain of the plan's executability without carrying out any of its steps.

Such a plan, we must add, can not be devised if the child is unable to think of the multiplicand—the quantity "7 marbles," for example—as a unit, as something that can be replicated or measured by. The ability to unitize was found to be a conceptual prerequisite to the concepts of multiplication (Steffe, 1994; Lamon, 1994) and proportional reasoning (Behr et al., 1992).

The building of such a plan is the conceptual operation of multiplication of two numbers, say 12 and 7 as in Problem E, whereas the computation of the quantity resulting from a distribution of 7 marbles to each of 12 kids is the execution of the plan. Different from plan construction and plan execution is symbolic representation. A symbolic representation is a string of symbols (e.g., $7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7$ or 12×7) produced by a child to represent and communicate what he or she has planned and/or executed. The distinction among plan construction, plan execution, and symbolic representation is important to our thinking as mathematics teachers. An understanding of students' plan construction and plan execution, not just their evaluation of the strings of symbols they produce, can help us assess their learning processes, which in turn can direct our future instructional actions.

To further illustrate the notion of plan construction versus plan execution, consider again Dan's solution. Recall that Dan's solution process did not include any actual computation; it consisted only of a plan that could be executed to obtain the problem answer ("You take 1 and you divide by 0.31. You take that number, whatever that number is, and you divide 7 by that number"). At first, Dan imagined a replication of 0.31 lb: 0.31 lb of candy would constitute one package, another 0.31 lb would constitute a second package, and so on. Because 0.31 is contained in 1 at least *two* times, which, as was discussed earlier, is the minimum number of action repetitions necessary to observe a pattern, Dan was able to extract a pattern of equal-quantity replication. Based on this pattern, he devised a second plan whose goal was to measure the entire 1 lb with the 0.31-lb unit. Although Dan did not execute either of the two plans, he was certain of their executability.

3. *It emerges from more rudimentary ways of thinking*. We might observe a child solving a (to us) multiplication problem and erroneously think that he or she is carrying out a multiplication operation. Consider the following example: Rita, a preschool child, was asked the following question:

F. These monkeys (pointing to three monkey-dolls) were hungry. The zoo-man gave each one of them three bananas to eat. How many bananas did they eat altogether?

Rita put her finger on the first monkey-doll and in synchrony with three finger-taps on the monkey-doll she said, "one, two, three"; then she moved her finger to the middle monkey-doll and in synchrony with three finger-taps she said, "four, five, six"; finally, she moved her finger to the third monkey-doll and in synchrony with three finger-taps she said, "seven, eight, nine." We cannot consider this activity a multiplication operation, for there is no indication that Rita had constructed an equal-quantity iteration pattern. The most we can say is that Rita demonstrated an ability to coordinate her perception of the monkey-dolls, her finger-taps, and the utterance of number words to achieve her goal of finding the total number of bananas eaten by the three monkey-dolls. Such coordination ability is indispensable to the development of multiplication as an equal-quantity pattern formation.

SUMMARY

I began this chapter with a discussion of the nonconservation of operation phenomenon. I suggested a distinction between two behaviors associated with this phenomenon: (a) subjects provide different solution operations to the two multiplicative problems (one with a whole number multiplier and the other with a decimal multiplier), and (b) one of these solutions is meaningful and right, the other is based on superficial considerations and produces a wrong answer. I attributed the first behavior to the fact that subjects in this stage of their conceptual development do not see an underlying common structure to the two problems. I called this stage a "naive-interpretist" stage and argued that it is an unavoidable conceptual stage a child must go through in moving from additivity to multiplicativity. I also argued that the second observation is *not* a necessary consequence of the first, but primarily a consequence of faulty instruction. In particular, I discussed the conservation formula strategy provided to children and highlighted its harmful consequences to the mathematical development of children. I showed that children can and should be able to cope with multiplicative problems having a non-whole number multiplier meaningfully—not necessarily correctly, but with a relational problem-solving approach—even when they are in a naive-interpretist stage. Finally, I suggested a theoretical perspective of the conceptual development of multiplication in three stages: (a) whole numbers as multipliers—the equal-quantity iteration pattern conception, (b) fractions as pseudo multipliers, and (c) fractions as multipliers. During the transition from the first stage to the third stage, children are naive-interpretists; they become operation-conservers only when they reach the third stage.

Through these analyses, I tried to communicate a few ideas that I believe can be of benefit to teachers; they can be summarized in four points:

1. Analyses of students' responses are necessary if we want to understand conceptual development. For example, my definition of multiplication was motivated, as was demonstrated in this chapter, by an analysis of subjects' responses to multiplication problems.
2. We as teachers should be sensitive to our students' cognitive processes. The judgment of what might be going on in the student's head while performing a certain task cannot be made in terms of our own thinking about the task. For example, one might observe Rita solving Problem F and mistakenly think that she is performing a multiplication operation.
3. Modifications of existing conceptions and the construction of new ones can result only from problem situations.
4. An evaluation of learning processes must take into account students' plan construction, plan execution, and symbolic representations.

We have emphasized that students' responses and cognitive processes cannot be evaluated merely by observations of symbolic manipulations; a more intensive interaction with the students is necessary.

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NOTES

1. It is beyond the scope of this paper to deal with the enormous considerations of the concept of multiplication. What follows is a brief analysis representing my own perspective on this concept. This analysis does not address in any substantial way possible relations between my perspective and others' perspectives, such as those introduced by Piaget and Inhelder

(1969), Steffe (1994), Nesher (1988), Vergnaud (1988, 1994), and Schwartz (1988). The latter two references are on the concept of multiplication in the non-integer domain, whereas the first three are on its early construction in the whole number domain.