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Conceptual Units Analysis of Preservice Elementary School Teachers' Strategies on a Rational-Number-as-Operator Task

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This study explores preservice teachers' understanding of the operator construct of rational number. Three related problems, given in 1-on-1 clinical interviews, consisted of finding $^3/4$ of a pile of 8 bundles of 4 counting sticks. Problem conditions were suggestive of showing $^3/4$ of the number of bundles (duplicator/partition-reducer [DPR] subconstruct) and $^3/4$ of the size of each bundle (stretcher/shrinker [SS] subconstruct). This study provides confirming instances that students use these 2 rational number operator subconstructs. The SS strategies are identified when the rational number, as an operator, is distributed over a uniting operation. With these SS strategies, rational number is conceptualized as a rate. However, the SS strategies were used less often than the DPR strategies. Detailed cognitive models of these strategies in terms of the underlying conceptual units, their structures, and their modifications, were produced, and a "mathematics of quantity" notational system was used as an analytical tool to describe and model the embedded abstractions.

Children's and teachers' understanding of multiplicative concepts—multiplication, division, ratio, rational number, and others—is important to their ability to gain mathematical understanding. Although much research has been accomplished on the knowing, learning, and teaching of these concepts among these populations during the last decade, much work remains to be done.

Although the question of what a rational number is can be easily answered from the perspective of mathematics, the same question considered from a psychological or developmental perspective is less clear. One point of view from the psychological perspective originally put forth by Kieren (1976) is that the concept of rational number consists of a number of possible subconstructs—part-whole, quotient, ratio number, operator, and measure. Kieren (1976, 1988) and other writers (Behr, Lesh, Post, & Silver, 1983; Freudenthal, 1983; Vergnaud, 1983) have suggested that a complete understanding of rational number requires an understanding of each of these subconstructs separately and also an understanding of the relationships among the subconstructs. Researchers (e.g., Kieren & Southwell, 1979; Rahim, 1986) used the

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rational number test developed by Kieren (reported in Harrison, Bye, & Brinkley, 1980) to get information on the kind of knowledge certain populations have about the separate rational number subconstructs. Little additional work has been done to investigate learners' understanding of these separate rational number subconstructs. Behr, Harel, Post, and Lesh (1992a) have argued that explicit information is lacking among researchers and teachers with regard to the concepts that underlie the understanding of the separate subconstructs.

This study investigates the separate rational number operator subconstruct of preservice elementary school teachers. For the purposes of this article, we will refer from now on to the understanding of the operator rational number as the "operator construct." Behr et al. (1992b) identified five potential, hypothetical "personalities," or subconstructs, of rational number as an operator. The two main personalities were the duplicator/partition-reducer (DPR) and the stretcher/shrinker (SS).

This study explores, analyzes, and describes the rational number operator subconstructs of a group of preservice elementary school teachers, within a problem-solving context involving finding three fourths of a quantity presented as eight bundles of four sticks. In describing the rational number operator subconstructs of preservice elementary school teachers, this study focuses on (a) identifying preservice teachers' strategies with the two proposed hypothetical rational number operator subconstructs—the duplicator/partition-reducer (DPR) and the stretcher/shrinker (SS); (b) describing and analyzing the underlying conceptual unit structures, and their modifications, as the various partitive number-exchange and/or quotitive size-exchange strategies are used during problem solving; and (c) implementing the "mathematics of quantity" notational system, proposed by Behr et al. (1992a, 1992b), as an analytical tool and a means of communication, to represent the embedded abstractions of the constructed conceptual unit structures and to model the dynamics of the underlying thought processes.

RATIONAL NUMBERS AND THE CONSTRUCTION OF CONCEPTUAL UNITS

As a logical mathematical construction, a rational number can be thought of as a number that derives its meaning from the pair of numbers used in its denotation. However, the phenomenological foundation for knowers' understanding of rational number is based on physical embodiments (Behr, Wachsmuth, Post, & Lesh, 1984) and involves a "fractioning" (Freudenthal, 1983) of some physical or mental object—a unit. The nature of the unit that a person uses and transforms as the argument of a fractioning procedure is of central importance in attempting to describe and model his or her concept of rational number and operations with rational numbers. A unit "is that by virtue of which each of the things that exist is called one" (Euclid, *Elements*, Book VII, cited in von Glasersfeld & Richards, 1983), and may itself be composed of units. Construction of units is an act of abstraction. "There is a first act of abstraction that produces units from sensory-motor material, i.e., unitary things, corresponding to what Piaget calls 'simple' or 'empirical' abstraction;

and there is a second act of abstraction that takes these units as the material for the construction of a unit that comprises them" (von Glasersfeld & Richards, 1983).

We see this notion of a unit as a special case of what Greeno (1983) referred to as a "conceptual entity"—a cognitive object that can be reasoned about directly, a cognitive object for which the system has procedures that take the object as the argument. Greeno (1983) pointed out that a characteristic of a cognitive entity is that it is continuously included from an initial representation of a problem through to its conclusion, although it is modified as the situation changes. This continuous inclusion and modification of a conceptual entity is evident in fractioning procedures. In this research we seek to describe and model the nature of this unit and the modifications of it that preservice elementary school teachers construct in the process of finding $3/4$ of a quantity presented as a pile of eight bundles of four sticks.

The need for learners to construct conceptual units in their construction of mathematical concepts is becoming more widely recognized. Various researchers have considered various aspects of unit iterations and modifications for conceptual understanding (e.g., von Glasersfeld & Richards, 1983, and Steffe, von Glasersfeld, Richards, & Cobb, 1983, in counting; Piaget, Inhelder, & Szeminska, 1960, and Sáenz-Ludlow, 1994, in fraction understanding). According to Gal'perin and Georgiev (1969) "all elementary mathematical concepts, regardless of the limitations of their content, assume the notion of unit" (p. 1).

A question of importance in the current context is what behavior manifestations an observer might look for as the basis of an inference that a person has identified some sequence of experiential items as a unit. A basis for this was provided by von Glasersfeld (1981). He pointed out that "we do divide our visual, auditory, and tactual fields of experience into separate parts which ... then become individual items or 'things' .... we differentiate or 'cut' things out of a background and perceive each one of them as an entity or whole" (p. 86). He went on to say that "unitizing operations consist in the differential distribution of focused and unfocused attentional pulses. A group of co-occurring sensory-motor signals becomes a 'whole' or 'thing' or 'object' when an unbroken sequence is framed or bonded by an unfocused pulse at both ends" (p. 87). For this research we were concerned with the matter of identifying behavioral manifestations to take as evidence that the whole pile, some subcollection of the pile, individual sticks, or groups of individual sticks had been segregated from the background field and conceptualized as a unit. We have taken the position that these focused and unfocused demarcations could be evidenced by such behavioral manifestations as a pointing gesture to ends of a collection of sticks or bundles, an encompassing gesture to a collection, a picking-up or separating-out motion, head nods toward the ends of a collection, and eye or head movements across or toward the ends of a collection.

NOTATIONAL REPRESENTATION OF THE STRUCTURE OF CONCEPTUAL UNITS

Behr et al. (1992a, 1992b) have developed two notational systems to represent a hypothesized structure of the units that underlie aspects of children's mathematics.

These systems exemplify the structure of the units according to singleton units, composite units of known numerousness, and units of embedded units. They are like the attentional model presented by von Glasersfeld (1981) in this regard but unlike it in that his model is able to account for the level of abstraction to which the unit might have been constructed. The notational systems developed by Behr et al. (1992a, 1992b, 1994) consist of a generic manipulative aid representation and a generalized mathematics of quantity representation; we present only enough of the systems here to meet the purpose of their application in this article.

In the generic manipulative aid system, symbols such as o, *, #, /, and others are used to denote experiential items or objects such as an apple, a bead, or a counting stick. To indicate that such an experiential item is conceptualized as a unit, we enclose its symbol in parentheses, (/). A composite unit of four experiential items is denoted by (////). The notation used also reflects the depth of the "embeddedness" of units within units (the rank of the unit), although the analysis in this article does not refer to the ranks of the composed units.

The generalized mathematics of quantity and the generic manipulative aid representations reflect the abstraction process and indicate the number and the size of the experiential items or conceptual units that are united to make a unit. The following is an example:

(/)	corresponds to	one (l-unit) or (l-unit),
(/) (/) (/)	corresponds to	three (1-unit)s,
(////)	corresponds to	one (4-unit) or (4-unit),
((/) (/) (/) (/))	corresponds to	one (4 (1-unit)s-unit) or (4 (1-unit)s-unit),
(((////)(////)) ((////)(////)) ((////)(////)) ((////)(////)))	corresponds to	one (4 (2 (4-unit)s-unit)s-unit).

OPERATOR SUBCONSTRUCTS OF RATIONAL NUMBERS: DPR AND SS

This analysis focuses on an in-depth description of strategies that exemplify the two operator subconstructs of rational numbers—duplicator/partition-reducer (DPR) and stretcher/shrinker (SS), as identified by Behr et al. (1992b). The application of a rational number as a DPR on an operand unit of quantity has the effect of partitive division of that unit into parts equal in *number* to the denominator and mutually equal in size, a reduction of this number of parts to one, and a duplication (iteration) of this one part to a number of parts equal to the numerator. As SS, the effect is that of quotitive division of the operand quantity unit into parts equal in size to the denominator, a shrinking of each part to the size of one, and a stretching of each part to be equal in size to the numerator. An application of the $3/4$ operator to an operand of eight bundles of four sticks is shown for the DPR and SS subconstructs in Table 1.

The effect of the DPR is on the *number* of embedded units in the unit of the operand quantity, whereas the effect of the SS is on the *size* of the embedded units. The sequence of operations for both the DPR and SS has the final effect of being the composite (as composition of functions) of two transformations, exchange functions, a 1-for-denominator exchange and a numerator-for-1 exchange. Ultimately, this composite can be seen as a denominator-for-numerator exchange, which helps to suggest that "rational number" is a single entity, a single number.

Table 1
Applications of the Three-Fourths Operator to an Operand of Eight Bundles of Four Sticks for Two Operator Subconstructs

Operator Subconstruct	
Duplicator/Partition-Reducer	Stretcher/Shrinker
1. *//// //// //// //// //// //// //// ////* The operand quantity is represented as eight bundles of four sticks.	1. *//// //// //// //// //// //// //// ////* The operand quantity is represented as eight bundles of four sticks.
2. *(////)(////)(////)(////)(////)(////)(////)(////)* The operand quantity is represented as eight units of four sticks.	2. *(////)(////)(////)(////)(////)(////)(////)(////)* The operand quantity is represented as eight units of four sticks.
3. (*(////)(////)(////)(////)(////)(////)(////)(////)*) The operand quantity is reunitized (composed) to a unit of units—a unit of eight units of four sticks.	3. (*(////)(////)(////)(////)(////)(////)(////)(////)*) The operand quantity is reunitized (composed) to a unit of units—a unit of eight units of four sticks.
4. (*((////)(////)) ((////)(////)) ((////)(////)) ((////)(////))*) Through partitive division, the operand unit quantity is reunitized (composed) to a unit for which the *number* of embedded units is *4* (the denominator of $^3/4$).	4. (*((////)(////)(////)(////)) ((////)(////) (////)(////))*) Through quotitive division, the operand unit quantity is reunitized (composed) to a unit for which the *size* of embedded units is four (the denominator of $^3/4$).
5. (*((////)(////)) ((////)(////)) ((////)(////))*) A 3-for-4 exchange is carried out on the operand quantity—the number, 4, of two units of four sticks in the operand quantity is exchanged for a unit of size three.	5. (*((////)(////)(////)) ((////)(////)(////))*) A 3-for-4 exchange is carried out on the operand quantity—each of the two units of size 4 in the operand quantity is exchanged for a unit of size 3.
6. (*(////) (////) (////) (////) (////) (////)*) The operand quantity is reunitized (decomposed) to a unit of six units of four sticks each.	6. (*(////) (////) (////) (////) (////) (////)*) The operand quantity is reunitized (decomposed) to a unit of six units of four sticks each.
7. *(////) (////) (////) (////) (////) (////)* The operand quantity is reunitized (decomposed) to six units of four sticks.	7. *(////) (////) (////) (////) (////) (////)* The operand quantity is reunitized (decomposed) to six units of four sticks.
8. *//// //// //// //// //// ////* The operand quantity is decomposed to six bundles of four sticks.	8. *//// //// //// //// //// ////* The operand quantity is decomposed to six bundles of four sticks.

The cognitive structure of the operand quantity on which a rational number as operator acts is that of (a) a unit, (b) a unit of units, (c) a unit of units of units, or (d) an even more deeply embedded unit of units of units of units. Our distinction between DPR and SS depends on whether the operator acts on the "outermost" *number* of embedded units (DPR) or on the *size* of the contents of the embedded units (SS). In order for a rational number operator to be classified as SS, there must be a distribution of the rational number operator over one or more uniting operation(s).

The task used in this study is the following: "Find three fourths of a *pile* of eight *bundles* of four *sticks*." By comparing Step 1 with the last step in Table 1, we see that a multiple of the denominator number of bundles (8) is exchanged for, or transformed to, a multiple of the numerator number of bundles (6) for both the DPR and

SS subconstructs. However, the process to accomplish this exchange as a DPR operator (Table 1) involves a recomposition through partitive division of the eight-bundle unit to a unit of *four* 2-bundle units (i.e., a unit of units), and then the 3-for-4 exchange is an exchange on the *number* of embedded two-bundle units to result with three 2-bundle units. The process to accomplish the 3-for-4 exchange as SS (Table 1) involves the recomposition, through quotitive division, of the eight-bundle unit to a unit of *two* 4-bundle units (i.e., a unit of units); the 3-for-4 exchange is, in effect, an exchange on the *size* of units—in which *each* of the two 4-bundle units is exchanged for a three-bundle unit. This SS interpretation depends on a distribution of the 3/4 operator over a uniting operation on the subunits of quantity. A major motivation for this study depended on the question of the relative salience of number-exchange strategies compared to size-exchange strategies for students operating with rational numbers; therefore, careful consideration of this question guided the classification of strategies into these two categories.

METHOD

Students

A total of 30 preservice elementary school teachers enrolled in an elementary mathematics methods course participated in the study. All participants were undergraduate college seniors, and all but 2 were female. Students were asked to volunteer to participate in a study about the problem-solving strategies they apply on a selected mathematical problem. All 30 students agreed to be interviewed individually and to be videotaped during the interview.

The Bundles of Sticks Problems

The Bundles of Sticks problems were designed by the authors in such a way as to allow for a variety of either partitive number-exchange or quotitive size-exchange strategies to be used for problem solution. These problems were used to investigate the ways that students construct and transform conceptual units within one problem-solving context and how these constructions and transformations correspond to the hypothesized application of the DPR and SS interpretations of the operator construct of rational number. Problems were designed within the same contextual situation so that students could construct three fourths of a unit of eight bundles of four sticks each—that is, 3/4(8(4-unit)s-unit) by transforming the eight composite 4-units to either (a) a composite of six (4-unit)s, or (b) a composite of eight (3-unit)s.

A warm-up exercise to the Bundles of Sticks problems was first administered to make sure that students could form one fourth, two fourths, and three fourths of a set of (a) four singleton cubes, (b) eight singleton cubes, and (c) twelve singleton cubes. All participants in this study were successful at all parts of the warm-up exercise.

The Bundles of Sticks task consisted of three related problems. Problem 1 was presented as follows.

> Here is a pile of sticks that are in eight bundles of four sticks. I want you to show me a pile of sticks that has three fourths as many sticks as the whole pile has. You can use these bundles in any way you want to.

As a follow-up to the student's response, the interviewer asked,

> Is there another way you can show me a pile of sticks that has three fourths as many sticks as this pile?

Problem 2 consisted of having the interviewer say,

> Now let's pretend that these eight bundles of four sticks are bundles of boards, and that carpenter's helpers use these bundles of boards to do a job. They work on the job together and evenly. You are the person to rearrange, if needed, the bundles of boards, and then you are to give these to each of the carpenter's helpers.
>
> One day only three fourths of the carpenter's helpers come to work, so you are to prepare bundles of boards so that the helpers get three fourths of the pile of eight bundles of four boards for that day. How would you arrange the boards for the helpers? Remember, there are only three fourths of the usual eight helpers. Show me what you would do with this pile of eight bundles of four boards to prepare the boards for the carpenter's helpers.

After the interviewer finished probing the student's response to the second problem, he or she presented Problem 3 by saying,

> On another day, all eight helpers come to work, but they come late. It is decided to use all eight of the helpers but to do only three fourths of the job. The boss tells you to arrange this pile of eight bundles of four boards so there will be three fourths the amount of boards. There are eight helpers and each helper gets the same amount of boards. Show me what you would do with this pile of eight bundles each of four boards to arrange the boards for the eight carpenter's helpers?

After the student responded to the third problem, probing questions were presented either to have the student further explain the problem-solving strategy used or to get the student to suggest other alternative strategies to solve the same problem.

The Interview

All interviews were conducted by the first two authors. When two students had scheduled their interviews at the same time, one of the interviews was audiotaped and the other videotaped; otherwise, all interviews were videotaped. A graduate student in mathematics education taped each interviewing session. The average interviewing time for the Bundles of Sticks problem was about 20 minutes.

Prior to the interviews, students were informed that their responses during the interviews would not have any impact on the evaluation of their performance in the elementary mathematics methods course in which they were enrolled. The interview session started with the warm-up exercise, and then the story situations of the Bundles of Sticks problems were given. Students were asked to explain, justify, and discuss their strategies as well as to suggest multiple strategies for solving each problem. Probing questions were asked whenever the interviewer felt that further observable evidence of students' actions, intentions, units formations, justifications, or problem-solving strategies was needed.

Students' Protocols

In order to get an initial sense of the types of responses students gave, a subset of the transcribed interviews was first selected to explore and identify some of the strategies and conceptual unit structures that were used by the students on each of the three problems. Then all transcribed interviews were divided between two teams of researchers. Each of the two teams included one of the first two authors. Each team read the transcripts independently and used the strategies identified earlier, or identified new ones, to classify and describe students' rational number constructs. Whenever a new strategy, which had not been identified earlier, was used by a student, it was added to the classification list. Later, the two teams met and discussed in depth all the strategies identified, including the sequencing of the composition and decomposition of the conceptual units and the related notation of the transformed units. Efforts were made to ensure that sufficient observable evidence existed to document the occurrence of each step in the identified strategies.

The two teams then exchanged the transcibed protocols, to ensure that each student's interview was evaluated by both teams. The two teams met again to compare and to discuss the very few classification differences of the students' strategies, until 100% agreement was reached in the classification of all strategies. Some students used more than one strategy to solve the problem. Thus, although the total number of preservice teachers was 30, the total number of classified strategies for each problem was more than 30. Because a major motivation for the study depended on documenting students' use of number-exchange strategies (DPR) compared to size-exchange strategies (SS), careful consideration of this was given in the classification of the strategies into these two categories. The ultimate criterion in differentiating between the two categories was whether the protocol gave evidence of a distribution of the 3/4 operator over an operation of uniting units. Evidence of this distribution was a necessary condition for a strategy to be classified as a size-exchange strategy.

RESULTS: RATIONAL NUMBER OPERATOR STRATEGIES

Two major types of strategies were used by the preservice teachers in solving the Bundles of Sticks problems. These two types of strategies are consistent with either the duplicator/partition-reducer interpretation of rational number or the stretcher/shrinker interpretation of rational number. As the students apply the 3/4 operator to a quantity of eight bundles of four sticks, the DPR strategies have the effect of changing the number of bundles of sticks, whereas the SS strategies have the effect of changing the size, or content, of each of the bundles of sticks.

The strategies used by the students are organized into those used in Problems 1 and 2 and those used in Problem 3. Strategies on Problems 1 and 2 are classified as DPR, SS, or Other Strategies. Responses to Problem 2 added no new strategies to those observed in Problem 1. Although the intent of Problem 3 was to elicit size-exchange strategies, only 14 of 32 responses fell into this category; the majority of the remaining strategies were of the number-exchange type. For this reason, the strategies observed in Problem 3 are also arranged into the same DPR, SS, and Other Strategies subcategories.

Rational Number Operator Strategies on Problems 1 and 2

Number-Exchange Strategies (DPR)

This major category contains strategies for which the effect of applying the 3/4 operator to (8(4-unit)s-unit), a pile of eight bundles of four sticks each, is that of an exchange on the number, eight, of the composite (4-unit)s. That is, a sequence of steps transforms a unit of *eight* (4-unit)s to a unit of *six* (4-unit)s. The size, or content, of each (4-unit) is unchanged in the process. Thus, this category of strategies suggests an interpretation of rational number on the part of the preservice teachers that is consistent with the duplicator/partition-reducer personality of rational number. DPR strategies were used 18 times by the 30 preservice teachers in solving Problems 1 and 2 of the Bundles of Sticks problems. Two DPR strategies were observed: (a) a numerator-for-denominator number-exchange strategy with an initial student emphasis on the numerical symbols and (b) a numerator-for-denominator number-exchange strategy with an initial emphasis on the manipulatives.

Numerator-for-denominator number-exchange strategy: Nonmanipulative. With this strategy, the student seems to take each of the eight (4-unit)s as an entity by taking three fourths of the (8(4-unit)s-unit) as if taking three fourths of eight units. The following excerpt from a student's protocol illustrates the above.

S: ... *And three fourths of eight bundles would still be six bundles* [S seems to take each of the eight bundles as one unit, and consider each as a cognitive entity, without regard at this point to its content, then applies the $^3/_4$ operator on this group of eight bundles, which may be considered a unit of units.] *So, each* (worker) *still has one bundle, so four pieces of sticks each* [At this point S looks into the contents of the unit of four sticks].

The student quoted above is able to treat a bundle of four sticks as a unit and each bundle as four singleton sticks. The protocol does not give information about whether or not the student treated the pile as a collection of singleton objects. However, because the problem presentation referred to a pile of sticks, one can assume that the student's first focus was on singleton sticks. Thus, this strategy seems to imply flexibility in unit composition and recomposition. The major point here is that the student has the ability to treat a collection of objects, in this case a bundle of four sticks, as a singleton and use the conception of this singleton object as an input to a cognitive process. Thus, this student apparently had an a priori goal of making four groups of bundles, each of four sticks, without using the sticks. This goal is consistent with a rational number construct for which $^3/_4$ applied to some number means a 3-for-4 number exchange.

Numerator-for-denominator number-exchange strategy: Manipulatives. With this class of DPR strategies, students demonstrated a goal of forming a composite unit of units so that the total *number* of embedded units equaled the denominator of the fraction operator and then carried out a strategy the effect of which was to exchange that number of embedded units with a number of units equal to the numerator. With this class of strategies, students initially used the bundled sticks and at some point in the sequence carried out partitive division by the denominator on the unit of (4-unit)s to form a number of embedded units equal to the denominator of the fraction. The students' concern was with the number of subunits. The size (content) of the subunits is of concern only to the extent of their equality. The following student's protocol, from Problem 2, provides an example of the application of this strategy.

S: *Three fourths* [she organizes the bundles into four groups, each consisting of two bundles; i.e., she reconceptualizes 3/4(8 bundles of 4 sticks) as 3/4(4 groups of 2 bundles of 4 sticks)].... *So,* [moves one group of two bundles aside and encloses the other group of three separate groups of two bundles within cupped palms and gestures to each group of two bundles; i.e., she identifies (3(2(4-unit)s-unit)s-unit) as 3/4(4(2(4-unit)s-unit)s-unit)] *only six workers showed up, so each one carries one pile* [touches each group of two bundles of the three separate groups; i.e., she recognizes a decomposition of 1(3(2(4-unit)s-unit)s-unit) to 3(2(4-unit)s-unit)s] ... [reaches out and touches the one group of two bundles she placed aside; i.e., suggesting that this 1(2(4-unit)s-unit) is still conceptually connected with the 3(2(4-unit)s-unit)s to form the (4(2(4-unit)s-unit)s-unit)]. *Three fourths* [again enclosing the three separate groups of two bundles within cupped palms; i.e., she apparently recognizes 3(2(4-unit)s-unit)s as 1(3(2(4-unit)s-unit)s-unit)] *out of eight piles* (bundles) *equals six.... And there are six workers, so each one is going to carry one of the piles* [she begins to separate each group of two bundles into two separate bundles; i.e., suggesting a plan to reconceptualize 3(2(4-unit)s-unit)s as 6(4-unit)s]. *Three fourths of the workers are six because each one* [she continues and finishes separating groups of two bundles of four boards into single bundles of four boards each, showing a realization of her plan to reconceptualize the units] *had to carry one before ... only six piles had to be taken.*

To demonstrate the use of the generic manipulative aid as an analytical tool, we present an example of this class of DPR strategies. This strategy involves partitive division on the eight (4-unit)s.

1. (////) (////) (////) (////) (////) (////) (////) (////)	1. The eight bundles of four sticks are conceptualized as eight (4-unit)s.
2. ((////) (////) (////) (////) (////) (////) (////) (////))	2. The eight (4-unit)s are composed to a unit of units—(8(4-unit)s unit).
3. (((////) (////)) ((////) (////)) ((////) (////)) ((////) (////)))	3. Composition of units through partitive division by 4 gives a unit of units of units—(4(2(4-unit)s-unit)s-unit).
4. (((////) (////)) ((////) (////)) ((////) (////)))	4. Recomposition of the unit through a 3-for-4 exchange gives another unit of units of units—(3(2(4-unit)s-unit)s-unit).
5. ((////) (////) (////) (////) (////) (////))	5. Decomposition of each (2(4-unit)s-unit) to two (4-unit)s gives (6(4-unit)s-unit).
6. (////) (////) (////) (////) (////) (////)	6. Decomposition of the (6(4-unit)s-unit) gives six (4-unit)s.

A representation of the preceding DPR strategy using the generalized mathematics of quantity notational system, as an analytical tool, is given below.

3/4(8(4-unit)s)	= 3/4((8(4-unit)s-unit)	corresponds to Step 2
	= 3/4((4(2(4-unit)s-unit)s-unit))	corresponds to Step 3

= (3(2(4-unit)s-unit)s-unit)	corresponds to Step 4
= (6(4-unit)s-unit)	corresponds to Step 5
= 6(4-unit)s	corresponds to Step 6

The students' use, in this class of strategies, of partitive division by 4 in the context of applying a rational number operator with a denominator of 4 is significant. The physical realization of partitive division by 4 is to make four groups. The students apparently had a plan to make four groups. We infer from this that the students used a rational number operator subconstruct that was in essence a 3-for-4 number exchange. Students' a priori reasoning for making four groups facilitated this exchange.

Some students first reunitized the 8 bundles of 4 sticks to 32 (1-unit)s, carried out the partitive division by 4 on the pile of 32 (1-unit)s, and then applied the 3-for-4 exchange on the 4 bundles of 8 sticks. It is interesting that with this strategy students did not go directly from 8 bundles of 4 sticks to 4 bundles of 8 sticks.

The fact that one of the DPR strategies involves division of the total number of bundles whereas the other involves division of the total number of sticks is also significant. In the first case, students showed greater ability to compose and decompose units. They were able to deal with bundles of sticks as inputs for a division process. In the second case, students did not demonstrate this ability. A physical realization of division involves counting. Counting, in turn requires conceptual units (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Thus, the students who decomposed the eight bundles (i.e., eight (4-unit)s) to 32 sticks (i.e., 32 (1-unit)s) were at best unaware of using bundles as a counting unit for counting the pile of sticks and at worst unable to do so. In the case of unawareness, students might be brought to using collections as singletons by brief experiences in which attention was called to the issue. In the second case of inability, longer-term experiences with unit composition and decomposition might be necessary.

Size-Exchange Strategies (SS)

Size-exchange strategies are consistent with the stretcher/shrinker interpretation of rational number. An effort was made, during the interviews, on Problems 1 and 2 of the Bundles of Sticks problems to elicit size-exchange strategies from the students during their problem solving. Only four preservice teachers used one SS strategy on Problems 1 and 2. The descriptions and analyses of their responses follow.

Numerator-for-denominator size exchange with quotitive division strategy. With this strategy, the student demonstrates a goal of forming a composite unit of units in which the *size* of the embedded units equals the denominator of the fraction and then carries out a procedure the effect of which is to exchange the embedded units with units equal in size to the numerator. The strategy begins with quotitive division of the collection of eight (4-unit)s into a number of parts (or embedded units) each equal in size to the denominator of the fraction; for example, the size of *each* of the two units—(4 (4-unit)s-unit)—is *four*. The student's concern is with the size of the part. The number of the parts—that is, the number, or frequency, of (4 (4-unit)s-unit)s—is of concern only to the extent that each is an operand of the $3/4$ operator. After parts of this size are formed, a procedure is carried out for which the effect is that each (4(4-unit)s-unit) is replaced by or transformed to one equal in size to the numerator, 3; that is, each (*4*(4-unit)s-unit) is replaced by, or transformed to, a (*3*(4-unit)s-unit). Application of this strategy may reflect understanding of the distributivity of the rational number operator over the uniting operation on the 2(*4*(4-unit)s-unit)s. The following student's protocol provides an example of the application of this strategy.

I: [Places eight bundles of sticks, with four sticks in each bundle, on the table, in front of the student.]

S: *Three fourths?* [There was a 9-second pause, then S rearranged the pile in two groups of four bundles of four sticks in each group. Then, for each of the two groups of four bundles, S moved three bundles aside (i.e., considered 2($3/4$(4(4-unit)s-unit)-unit)s, or considered three fourths of the entire pile to be two times three fourths of each group of four bundles.]

S: *That would be six bundles of sticks.*

I: How did you figure that out? I noticed you put four on one side and four on the other.

S: *It helps me think* [laughs]. *I have to see it in order to do it.*

I: So what did you see?

S: *Well, three fourths would be one three bundles of <u>one</u> four* [authors' emphasis], *or of four bundles of sticks* [points to the three bundles of the closest group and then to the second group of three bundles out of the other group of four bundles].

I: And then you added three and three to get to the six?

S: [Nods] *The six.*

A generalized mathematics of quantity representation of the aforementioned strategy is shown below.

$3/4$(8(4-unit)s) = $3/4$((8(4-unit)s-unit))
= $3/4$((4(4-unit)s-unit) + (4(4-unit)s-unit))
= $3/4$((2(4(4-unit)s-unit)s-unit))
= 2($3/4$((4(4-unit)s-unit)s-unit)s)
= 2(3(4-unit)s-unit)s
= 6(4-unit)s

The above strategy suggests considerable ability and flexibility in composition and decomposition of units. There was a deliberate plan to form units of size 4, which was accomplished through quotitive division. This division was carried out on bundles of sticks, which necessitated conceptualization of each bundle as a unit, a 4-unit. Moreover, the quotitive division established a two-tier embedding of units: (4-unit)s in (4(4-unit)s-unit) and these in (8(4-unit)s-unit). The evidence of an a priori plan to make groups of size 4 suggests a conceptualization of rational number as a size-exchange function. Furthermore, the student quoted above distributed the $3/4$ fraction operator over the uniting operation of two (4(4-unit)s-unit)s, then applied the size-exchange strategy, and ended with two (3(4-unit)s-unit)s. When the two (3(4-unit)s-unit)s were united, the recomposed unit of six bundles—that is, (6(4-unit)s-unit)s—was given as an answer.

Other Strategies

Recursive halving and uniting strategy. With this strategy, the student demonstrates a goal of recursive halving by selecting half and half of a half as three fourths. Then, the student selects one half of the 8 bundles of 4 sticks each, and unites it with the half of the other 1/2 of the 8 bundles of 4 sticks. The frequency of occurrence of this strategy was six. This strategy suggests that the student was able to perceive units in variable compositions.

Symbolic multiplier and divider strategy. Application of this strategy involves transformation of the composite unit of eight (4-unit)s to 32 singleton units by a symbolic multiplication of 8 times 4. This is followed by symbolic application of the 3/4 operator to the number of singleton units. Finally, a transformation back to units of size 4 was accomplished by a mental partitive division by 6. Information to divide by 6 was obtained from the problem context, which established that three fourths of the workers was six workers. The frequency of occurrence of this strategy was 7.

This strategy suggests a very limited ability to decompose and compose units of quantity. There is no evidence in the students' work that they operated with the bundles of sticks as units. The only use of a bundle was in decomposing it to singleton sticks. This unbundling of sticks could take place strictly at the enactive level, without a conceptualization of bundles or sticks as units. The students' conception of rational number is likely one of memorized procedures.

Rational Number Operator Strategies on Problem 3

Size-Exchange Strategies (SS)

This category contains strategies for which the effect of applying the 3/4 operator to (8(4-unit)s-unit) is that of an exchange on the size, or content, of the embedded units. The effect of this category of strategies is therefore consistent with the stretcher/shrinker personality of rational number (Behr et al., 1992b). The effect of applying 3/4 to an operand quantity as an SS is, for example, exchanging, shrinking, or transforming each 4-unit to a 3-unit, or possibly each (4(4-unit)s-unit) to a (3(4-unit)s-unit). The application of SS strategies seems to reflect an understanding of distributivity of the rational number operator over the uniting operation. The total frequency of occurrence of the two types of SS strategies on Problem 3 was 12.

Numerator-for-denominator size-exchange strategy. With this strategy, the student composes four singleton units within each (4-unit) to form eight composite units of units—8(4(1-unit)s-unit)s—and distributes the 3/4 operator over the uniting operation on the eight composite units of units. The student then applies the 3/4 operator to each of the units of units to get the (3(1-unit)s-unit)s, recognizing that three fourths of the whole is equal to the result of uniting three fourths of each of the individual parts. The frequency of occurrence of this strategy was 9. The following student's protocol provides an example of the application of this strategy.

In Problem 1, S initially used mental computation to find 3/4(8(4-unit)s-unit) by multiplying 8 times 4 and then 3/4 times 32 to get 24. When I probed for a second way, she did a recursive halving procedure, saying, "half and half of a half is three fourths." Further probing did not lead to any conceptually different procedures. In Problem 3, S first separated out six bundles of four sticks, mentally multiplied to get 24, and divided by 8 to get 3. After some discussion about this solution, the following conversation transpired.

I: See if you can think of another way. I want you to think of all the ways you can to show three fourths.

S: [Pause] *three fourths of these* [makes an incomplete gesture with her right hand toward the row of eight parallel bundles of four sticks—that is, although the gesture is brief, truncated, and quick, it might suggest conceptualization of 8(4-unit)s as 1(8(4-unit)s-unit)—then changes the motion midstream and places her hand on her forehead]. [Pause] *There could also be three fourths of each stick* (bundle) [picks up one bundle of four sticks in her right hand, suggesting a conceptualization of 3/4(8(4-unit)s-unit) as 8(3/4(4-unit)s-unit)s—looks at I]. *Three out of the four* [puts bundle back down, suggesting a reversible (composition and decomposition) conceptual relationship between 1(8(4-unit)s-unit) and 8(4-unit)s].

I: Okay, so then if I'm Person 1 ... give me the stack (bundle) that I will get.

S: [Picks up the same bundle of four sticks and fans them out inside the rubber band—i.e., recognizing this (4-unit) as 4(1-unit)s, thus suggesting that each (4-unit) can be reconceptualized as 4(1-unit)s and thus (3/4(4-unit)-unit) as (3/4(4(1-unit)s-unit)-unit)] *it could be three of these* [pause] *three sticks* ... [pulls one stick out and gives the remaining bundle of three sticks to I, who then asks, "And Person 2?"]

S: *They would get the same* [picks up a second bundle of four sticks and makes a motion as if to remove a stick. I says, "And Person 3?"] ... *Same* [gestures with the same bundle of four sticks] ... *They would all get the same* [making a circular encompassing motion in the air with the same bundle of 4 sticks, suggesting that a process of applying the 3/4 operator to a unit whole that consists of parts, (8(4-unit)s-unit), is to distribute the 3/4 operator over the uniting (additive) operation on the parts, the separate (4-unit)s, which she has conceptualized as one (4(1-unit)s-unit)].

I: How is it different (the latter solution from the former)?

S: *Um,* [pause] *all the stacks are used ... there is one left in each thing ... Um, I pulled one out of each one ... the amount stayed the same* (compared to her first solution), *but there's one less in each* [points to the first bundle to her left and then makes two pointing motions from left to right along the row and then finishes with a sweeping motion over the remainder of the row of bundles of three sticks, again conceptualizing 3/4(8(4-unit)s-unit) as 8(3/4(4-unit)-unit)s] *rubber band or bundle.*

The mathematics of quantity representation of the strategy detailed above is shown below.

3/4(8(4-unit)s) = 3/4((8(4-unit)s-unit))
= (8(3/4(4-unit)-unit)s-unit)
= (8(3/4(4(1-unit)s-unit)-unit)s)
= (8(3(1-unit)s-unit)s-unit)
= (8(3-unit)s-unit)
= 8 (3-unit)s

The preceding student protocol demonstrates considerable ability in the composition, decomposition, and recomposition of units. Important to the question at hand is an apparent realization that the 3/4 operator distributes over the uniting operation. In the context of the problem situation, this protocol suggests a conception of three fourths as a 3-for-4 size-exchange function.

Part-part-whole size-exchange strategies. This subcategory of strategies involves the formation of one or more part-part-whole structures, either across the eight (4-unit)s or within each of the eight (4-unit)s. When the structure was formed across the eight (4-unit)s, a 6-to-2 part-to-part ratio resulted. When the structure was formed within each 4-unit, the part-to-part ratio was 3-to-1. This strategy was used three times.

Thus, some students decomposed each 4-unit to a composite unit of units with a 3-to-1 part-part-whole structure—that is, to a (((3-unit) + (1-unit))-unit)—and directly took the three part in each composite unit to get eight 3-units, recognizing that three fourths of the whole is equal to the result of uniting three fourths of each of the individual parts. The application of this strategy reflected an understanding of distributivity of the fraction operator over the uniting operation on the 4-unit subunits. This strategy is different from the previously discussed Numerator-for-Denominator Size-Exchange Strategy in that students are focusing on each composite part-part-whole unit of units of units—that is, (((3-unit) + (1-unit))-unit)—instead of each unit of units—that is, (4(1-unit)s-unit)—before they apply the 3/4 operator over the contents, or the size, of the composite unit.

One student imposed over the eight bundles a 6-to-2 part-part-whole structure. Then a 3-for-4 size-exchange strategy was applied on each (4-unit) bundle in the first part, and then the second part, and finally the (3-unit) bundles were united. This strategy involves formation of a 6-to-2 (3-to-1-equivalent) part-part-whole relation among the eight (4-unit)s. This was followed by a focus on the part of size $3n$ ($n = 2$), with a subsequent focus on the part of size n ($n = 2$). During the focus on either part, a 3-to-1 part-part-whole relation was established on the content (i.e., the size) of each (4-unit) and the part of size 3 of each (4-unit) was selected to give three fourths of the part of focus. There was implicit recognition that three fourths of each unit within the part of focus would result in three fourths of that part and three fourths of the whole (i.e., the (8(4-unit)s-unit)). This was obtained by uniting the three fourths of each of the two parts.

Number-Exchange Strategies (DPR)

Two number-exchange DPR strategies, with a total frequency of 19, were used to solve Problem 3 of the Bundles of Sticks problem. Most of the students who applied the 3/4 operator as DPR in Problem 3 first used a procedure that has the effect of doing a direct, 6 (multiple of the numerator)- for-8 (multiple of the denominator), 3-for-4 equivalent exchange on the number of 4-units. This is followed by decomposition of the six (4-unit)s to singleton units and then partitive division by 8 on these singleton units to find the unit size for each of eight workers. The following student's protocol provides an example of the application of this strategy.

S: *Can we take the bundles loose?...* [Starting with the sixth bundle from the left end of the row of eight and working from there to the left end of the row—recognizing 3/4(8(4-unit)s) as quantitatively equivalent to six (4-units)—she unwraps six bundles of four sticks, holding the singleton sticks in her hand—recognizing six (4-units) as quantitatively equivalent to 24 (1-unit)s] *...That's three fourths.*

I: ...Now what are you going to do?

S: [Looking at I] *separate 'em into ... um,* [pause] *eight workers can* [pause] *do it.... Oh well, do it this way* [holding the individual sticks in her left hand, she deals out one stick to each of eight spots in the work area and repeats this until there are three at each spot—i.e., she uses partitive division to establish that 24 (1-unit)s is quantitatively equivalent to eight (3-unit)s.

These there [pushes a group of three forward, repeats this when I asks, "and Worker 2?" and then continues to touch groups of three simultaneously while saying] *three, four, five, six, seven, eight.*

The student quoted above applied the 3/4 operator on the number *8* of the eight (4-unit)s to get six (4-unit)s and then decomposed the six (4-unit)s to 24 singleton units by multiplying 6 times 4. The resulting number of 24 sticks was then divided partitively by eight to get an answer of eight bundles of three sticks in each bundle; that is, (8(3-unit)s-unit). Although there is an ultimate effect of a change on the size of the embedded units, the first operation performed by the student quoted above was on the number of units. This operation was accomplished by applying a memorized fact. A mathematics of quantity representation of this DPR strategy is given below.

3/4(8(4-unit)s) = 3/4((8(4-unit)s-unit))
= (6(4-unit)s-unit)
= (6(4(1-unit)s-unit)s-unit)
= (24(1-unit)s-unit)
= (8(3-unit)s-unit)
= 8 (3-unit)s

One student used a part-part-whole number structure with a DPR number-exchange strategy. This strategy involves formation of a 6 (multiple of numerator)-to-2 (multiple of the complement of the numerator with respect to 8), a 3-to-1 equivalent, part-part-whole relation followed by selection of the part of size $3n$ ($n = 2$) through a distribution of the 3/4 operator over the uniting operation of the eight (4-unit)s to obtain three fourths of the operand quantity. The decomposition to the part-part-whole structure represented by 6(4-unit)s + 2(4-unit)s was followed by what appeared to be a removal of the two (4-unit)s to leave six (4-unit)s, rather than a selection of the six (4-unit)s. This remaining part was then decomposed to singleton units through multiplication. Finally, the unit size for distribution to eight workers was determined through partitive division.

Other Strategies

Symbolic multiplier/divider strategy. This strategy involves transformation of the eight bundles of four sticks each to singleton units by multiplication of the size of each bundle by the number of bundles, followed by application of the 3/4 operator to this, and then partitive division by 8 to determine the unit size for each of the eight units. Most of the steps in this strategy were carried out at the memorized symbolic level. Only one from the sample of interviewed students used this strategy.

Number-exchange, reunitization, and quotitive division strategy. With this strategy, three students carried out a procedure that has the effect of exchanging the eight (4-unit)s for six (4-unit)s, then reunitizing each (4-unit) to a (((3-unit) + (1-unit))-unit), and finally collecting triplets of (1-unit)s to end up with eight (3-unit)s.

OVERALL SUMMARY OF RESULTS

We observed two major types of strategies when a rational number was applied as an operator by the 30 preservice elementary school teachers in the course of solving three Bundles of Sticks problems: number-exchange strategies and size-exchange strategies. With the number-exchange strategies, students tried to operate on the number of units in a unit of *units* when the contextual referents of these units were the bundles of four sticks each. However, with the size-exchange strategies, students were trying to operate on the sizes of the *units* in a unit of *units* when the contextual referents of these units were the bundles of four sticks each. We observed that students tend to be somehow reluctant to distribute the $3/4$ operator over the uniting operation, although they were encouraged by the interviewers to undo the already-bundled sticks as needed. While solving each of the problems, students were asked repeatedly to try to construct more than one solution strategy. Even with prompting, few students applied more than one solution strategy.

When students were asked to find three fourths of (8(4-unit)s-unit) in the first two problems, most of the strategies they used were DPR number-exchange strategies. Size-exchange (SS) strategies occurred only four times as students solved Problems 1 and 2. The number-exchange DPR strategies occurred 18 times. Of these, the non-manipulative Numerator-for-Denominator Number Exchange strategy occurred most frequently, specifically, 11 times. Most of the 11 students using this strategy first tried to solve the problem mentally; then they resorted to discussing their solution strategies using the manipulative objects. Most of these students confirmed that, in using the manipulatives, they were trying to validate the answers they arrived at mentally. In using this number-exchange strategy, students applied the $3/4$ operator on the number 8, which is the number of the bundles of four sticks; then they replaced the number 8 with the number 6, which was arrived at mentally, in most cases, to give an answer of six (4-unit)s. However, with the other number-exchange strategy, students were composing and recomposing units at a deeper level of embeddedness.

Other strategies were also used on Problems 1 and 2. The symbolic multiplier and divider strategy was used by eight students. Of these students, three gave their solutions symbolically on paper first and then tried to validate their work by using the manipulatives, whereas the other students first divided and multiplied mentally and then attempted to validate their work by using the sticks. With the recursive halving and uniting strategy, six students demonstrated a goal of recursive halving by selecting half and half of half as three fourths.

Problem 3 was designed in such a way as to encourage the use of size-exchange strategies during problem solution. Students were asked to find three fourths of (8(4-unit)s-unit) to be assigned to eight individuals. But again, more number-exchange strategies ($f = 19$) than size-exchange strategies ($f = 12$) were used. In spite of the interviewers' heavy probing during the interviews to elicit the use of size-exchange strategies, students tended not to do so, as if the size-exchange strategies were cognitively more demanding.

With size-exchange strategies, students operate on the sizes of the embedded units. Considering three fourths the size of each of the embedded units, each (4-unit) is transformed into a (3-unit), or each (4(4-unit)-unit) is transformed into a (3(4-unit)-unit); then these size-exchanged units are united. The application of the SS strategies by students reflects an understanding of distributivity of the rational number operator over the uniting operation.

DISCUSSION

We first turn our attention to consideration of the question of what concepts, as suggested by the data from this study, are fundamental to an understanding and application of the concept of rational number as operator. These are given in terms of the rational number $3/4$.

Based on the results of the present study, the DPR operator subconstruct requires an understanding of and an ability to apply partitive division or part-part-whole reasoning on a quantity and an exchange on the number of parts. If partitive division by 4 is carried out, it is done with the a priori goal of establishing four units within the operand quantity. The size, or content, of each of the four units may become known in the process, but it is not germane to understanding, or constructing, the DPR subconstruct. If part-part-whole reasoning is applied, it is done to establish a 3-to-1 (or $3 \times n$-to-$1 \times n$) ratio on the number of units in the operand quantity. Some combination of partitive division and part-part-whole operations might also be used.

Behr et al. (1992b) suggested that an application of a rational number to an operand can proceed, as a DPR, in the order of first duplicating the operand and then performing a partition-reduction of the result, or equivalently, performing a partition-reduction of the operand followed by a duplication. Although these two processes may seem to be logically equivalent, in this study no student gave any indication of having thought of duplicating the operand first. Students applied partition-reduction on the given quantity, followed by a duplication. Partitioning of a given quantity may be psychologically less demanding than the duplication of a quantity, which requires an extention of the quantity beyond what is given. This may also be due to the fact that such a duplication would have been exceedingly awkward in the context of this task, considering the amount of material involved in a duplication. Moreover, an adequate number of objects was not even made available. Nevertheless, this observation suggests further investigation. Do students, independent of "instructional coercion," naturally think of applying the denominator operator first and then the numerator to find a "fraction of a quantity," or is this sequence of events task-dependent?

The SS operator subconstruct, judging from the results of this study, depends on an understanding of and an ability to apply quotitive division or part-part-whole reasoning on a quantity and distribute the rational number operator over a uniting operation followed by an exchange of the sizes of parts. If quotitive division is applied, it is done with the preestablished goal of forming units of size 4 within the operand quantity. The number of units could become known during the process, but is not germane to an understanding of the SS subconstruct. If part-part-whole reasoning is applied, it is done to establish a 3-to-1 (or $n \times 3$-to-$n \times 1$) ratio on the size (or content) of each unit, and is likely done after the quotitive division is carried out. After units of size 4 are established within the operand quantity, the rational number is applied individually to each of these units; this constitutes a distribution of the operator over the uniting operation.

One goal of this study was to look for confirming evidence that the hypothetical DPR and SS subconstructs of rational number as proposed by Behr et al. (1992b) are evident in preservice teachers' understanding of rational number. Variations of these subconstructs are reported. The SS subconstruct—which consists of a quotitive division, or segmenting, of a quantity and then an application of a distribution of the rational number operator over the uniting operation on each of the subunits of the operand quantity—is indicative of an understanding of rational number as a rate. This subconstruct was not easy to elicit, even with probing, among many of the preservice teachers who participated in this study.

Overriding the concepts described above is the ability to conceptualize quantities into units and to reunitize according to the demands of a given task. Although the task used in this study involved a discrete quantity, the same ability needed to perform it would be required to operate with a continuous quantity.

Another goal of this study was to conjecture at some concepts that seem to be a common foundation to the several rational number constructs proposed by Kieren (1976). An equally appropriate effort would be to consider some ways in which these constructs interact with and support one another in problem solving. Some writers (e.g., Behr et al., 1983) have suggested that an understanding of rational number requires an understanding of these rational number subconstructs in a somewhat separate manner. What is becoming more apparent, and is supported by the data from this study, is that the several constructs interact during the resolution of a task. Partitioning (Pothier & Sawada, 1983) is believed to be fundamental in the understanding of rational number, especially for the part-whole construct of rational number. Partitive division is an equal sharing of a quantity into a specified number of parts; in this respect, partitioning is a concept common to the part-whole and operator constructs of rational number. The process of quantifying one quantity in terms of another quantity as a unit of measure is the fundamental concern of measurement in general, of quotitive division, and of rational number as a measure number. As measurement underlies quotitive division, which in turn underlies the SS operator subconstruct of rational number, we see measurement as a common thread to operator and measurement constructs of rational number.

A third goal of this study was to give further consideration to the notational systems developed by Behr et al. (1992b) as an analytical tool for exemplifying the cognitive structures that are involved in cognitive operations on a task. Our purpose for giving the amount of detail we did in the protocols in terms of the unit composition and recomposition carried out by students, and in selected instances in terms of a sequence of observed unit compositions, was to communicate about the cognitive structures and to represent them in a compact form. This compact notational representation, to our thinking, represents the cognitive processes of the student when the essential aspects of the process are abstracted from the verbiage of the protocol. In accordance with Thompson's (1994) concern, the notation is used in this study as a medium with which to represent and communicate about students' cognitive processes and not as an instructional tool to help students "get more answers." Each of the notational sequences represents the sequence of the units, as cognitive entities, that the students form, manipulate, and maintain during the solution of the problem.

The particular task used in this study takes on added significance when the relationship between the operator construct of rational number, as reflected by performance on this task and other aspects of understanding rational number and quantity, are considered. A rational number, as distinct from a fraction, represents an intensive quantity. For example, 6 red apples out of 24 apples would more likely be described as $^6/_{24}$ than as $^1/_4$ (J. Kaput and others, personal communication, 1992). The fraction $^6/_{24}$ reflects accurately the 6-out-of-24 relationship between the two extensive quantities, the part and the whole (red apples and apples), but not necessarily the intensive quantity, or rate, inherent in the situation. The number $^1/_4$, with respect to this situation, fails to reflect directly the extensive quantities of the part and the whole; it reflects the relationship between the part and the whole, and potentially the intensive quantity reflected in the situation. In the context of the situation, $^1/_4$ comes closer to expressing the intensive quantity than $^6/_{24}$ does, because the latter draws specific attention to the measures of the extensive quantities. The distinction made here is similar to the distinction Thompson (1994) made between ratio and rate.

The property that three fourths of a whole equals the sum of three fourths of all its parts (the SS operator subconstruct of the rational number $^3/_4$), which is embodied by the task of this study, may be characteristic of intensive quantity, in the following way. One way to think of three fourths is as a random selection of three out of four parts of a whole. Another is to think of three out of four parts embedded in each of the four original parts. Another is as three out of four parts in each of these parts, and so on. Thus, the ability to perform on the task of this study, which is the discrete quantity analog of the first two steps of the sequence above, is a first step in seeing three fourths as a rate. With the SS operator subconstruct, the "three-fourthness" of the whole is equal to the common invariant rate across the three fourths of all the parts of the segmented whole. From this perspective, the identified SS subconstruct of the present study represents an aspect of understanding rational number as an intensive quantity and as a rate. The views expressed by Kaput and West (1994) and Thompson (1994) regarding the understanding of intensive quantities are consistent with our interpretations. We propose that instructional attention to this distributivity of rational number over an equal or random segmenting of a whole, as well as the instructional attention to the reverse process, might very well facilitate learners' understanding of rates, density, and of intensive quantity in general.

On the one hand, the research reported herein has certain limitations, but, on the other hand, it opens up an area of inquiry. The basic task used in this study was of taking three fourths of (8(4-unit)s-unit). Problem situations were used in an attempt to call learners' attention to the fact that the operator 3/4 could be applied to either the unit of eight or to each of the units of four. Because both 8 and 4 are multiples of the denominator of 3/4, the focus may not have been as sharp as it might have been. It would be of interest to investigate students' performance on a task such as taking three fourths of (5(4-unit)s-unit) and then subsequently on the task used in this study. Similarly, it would be of interest to investigate whether the rank of the unit of units would have an effect on students' performance.

The issue of providing instruction so that learners form cognitive entities, units of quantity, does not get the instructional attention it deserves. Teachers should be aware of the cognitive entities that students form, for only then can they consistently ask their students to deal with situations that will help them form and extend the necessary cognitive structures.

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