```
> currentdir("/Users/grad/Desktop/GT");
```

/Applications/Maple 9.5/Maple 9.5.app/Contents/MacOS/bin.APPLE_PPC_OSX

```
> read `stripdemo.maple`;
# stripdemo.maple
# Copyright (C) 2007 by Glenn Tesler, gptesler@math.ucsd.edu
#
# Last updated: November 9, 2007
#
# This software computes many of the formulas in the paper
#    "Distribution of Segment Lengths in Genome Rearrangements"
#     by Glenn Tesler
#     (submitted)
#
# Tested with Maple 9.5
#
# To run the demo, start maple and type
#       read `stripdemo.maple`;

> read `stripcounts.maple`;
Warning, the protected name Chi has been redefined and unprotected
Warning, the assigned name Group now has a global binding

> interface(echo=2);
```

$$1$$

```
############################################################################
# Sample values to use to demonstrate the software
############################################################################

# Permutation size (# genes or blocks)
> n1 := 6;
```

$$n1 := 6$$

```
# Number of strips
> k1 := 3;
```

$$k1 := 3$$

```
# Number of permutations (genomes)
> g1 := 2;
```

$$g1 := 2$$

```
# Unordered type (integer Partition)
> lambda1 := [3,3,1];
```

$$\lambda 1 := [3, 3, 1]$$

```
# Ordered type (integer composition)
> alpha1 := [3,1,3];
```

$$\alpha 1 := [3, 1, 3]$$

```
############################################################################
# Partition and composition functions
############################################################################

# P_n = list of partitions of n, with parts in decreasing order
> partitions(n1);
```

[[1, 1, 1, 1, 1, 1], [2, 1, 1, 1, 1], [2, 2, 1, 1], [2, 2, 2], [3, 1, 1, 1], [3, 2, 1], [3, 3], [4, 1, 1], [4, 2], [5, 1], [6]]

```
# C_{n,k} = list of compositions of n with k parts
# This is from the combinat package
> composition(n1,k1);
```

{[2, 2, 2], [1, 2, 3], [1, 1, 4], [3, 2, 1], [4, 1, 1], [2, 3, 1], [1, 4, 1], [3, 1, 2], [1, 3, 2], [2, 1, 3]}

```
# \ell(mu) = number of parts of composition or partition mu
> nops(lambda1);
```

3

```
> nops(alpha1);
```

3

```
# n = |mu| = sum of parts of a composition or partition
> csize(lambda1);
```

7

```
> csize(alpha1);
```

7

```
# M(\mu), mu a partition, gives # distinct linear perms. of parts of mu
> linmult(lambda1);
```

3

```
# \mathring{M}(\mu), Sec. 8
# mu a partition, gives # subsets of {1,...,n} (where n=|\mu|)
# whose unordered circular type is mu.  That is,
#  S = {j(1)<j(2)<...<j(k)}
```

```
# and the differences j(2)-j(1), j(3)-j(2), ... j(k)-j(k-1),  j(1)-j(k)+n
# are a permutation of the parts of mu
> circmult(lambda1);
```

7

```
################################################################################
# Count number of (n,g)-arrangements with k linear strips,
# for unsigned/signed, linear/circular
#
# The symbols in the paper with circle accents (\mathring in LaTeX) are
# represented in this software by appending a lowercase "c" to the name:
#   paper's \mathring{a} = ac     paper's \mathring{A} = Ac
#   paper's \mathring{b} = bc     paper's \mathring{B} = Bc
################################################################################

# Number of (n,g)-arrangements with k linear strips

# Unsigned linear arrangements, a_{n,k}^{(g)}
> cof_a_nk(n1,k1,g1);
```

120

```
# Signed linear arrangements, b_{n,k}^{(g)}
> cof_b_nk(n1,k1,g1);
```

340

```
# Unsigned circular arrangements, ac_{n,k}^{(g)}
> cof_ac_nk(n1,k1,g1);
```

20

```
# Signed circular arrangements bc_{n,k}^{(g)}
> cof_bc_nk(n1,k1,g1);
```

80

```
# Special cases for incompressible signed permutations:
# cof_b_kk(k,g) = cof_b_nk(n,k,g)    (linear arrangements)
# cof_bc_kk(k,g) = cof_bc_nk(n,k,g)  (circular arrangements)
# The b_{n,k}^{(g)} formula is derived from the b_{k,k}^{(g)} formula,
# so computing directly b_{k,k}^{(g)} is faster if that's what's needed.
# Same for bc
# These pairs should be equal:
> cof_b_kk(k1,g1), cof_b_nk(k1,k1,g1);
```

34, 34

```
> cof_bc_kk(k1,g1), cof_bc_nk(k1,k1,g1);
                                        4, 4

# Special case for n=k with g=2 genomes: can use simple formula with
# multiplication and integer floor instead of summation.
# cof_b_kk2(k) should be faster than cof_b_nk(k,k,2)
# These should be equal:
> cof_b_kk2(k1), cof_b_nk2(k1,k1), cof_b_nk(k1,k1,2);
                                        34, 34, 34

############################################################################
# Count number of arrangements with specified type
# a_\lambda^{(g)}, etc.
# unordered type (partition lambda) or ordered type (composition alpha)
# must be specific values
# g=integer or indeterminate
############################################################################

# unordered type, unsigned linear, a_{lambda}^{(g)}
> cof_a_utype(lambda1,g1);
                                        38

# ordered type, unsigned linear, A_{alpha}^{(g)}
> cof_A_otype(alpha1,g1);
                                        10

# unordered type, signed linear, b_{lambda}^{(g)}
> cof_b_utype(lambda1,g1);
                                        102

# ordered type, signed linear, B_{alpha}^{(g)}
> cof_B_otype(alpha1,g1);
                                        34

# unordered type, unsigned circular, ac_{lambda}^{(g)}
> cof_ac_utype(lambda1,g1);
                                        7

# ordered type, unsigned circular, Ac_{alpha}^{(g)}
> cof_Ac_otype(alpha1,g1);
                                        0
```

```
# unordered type, signed circular, bc_{lambda}^{(g)}
> cof_bc_utype(lambda1,g1);
```

$$28$$

```
# ordered type, signed circular, Bc_{alpha}^{(g)}
> cof_Bc_otype(alpha1,g1);
```

$$28$$

```
# Demo with g=indeterminate
# unordered type, unsigned linear, a_{lambda}^{(g)}
> cof_a_utype(lambda1,g);
```

$$-\frac{1}{2}4^g + 3\,2^{(g-1)} + \frac{1}{8}24^g - \frac{1}{2}8^g$$

```
###########################################################################
# Distribution of (un)ordered types for all (n,g)-arrangements
# unsigned or signed, linear or circular, at specified n & g
# n must be an integer
# g = integer or indeterminate
###########################################################################

# unsigned linear, unordered types:
#   set of {a_\lambda^{(g)} = ...} over *all* partitions |lambda|=n
> cofs_a_all(n1,g1);
```

$\{a_{2,1,1,1,1} = 230, a_6 = 2, a_{5,1} = 4, a_{3,3} = 6, a_{4,2} = 12, a_{1,1,1,1,1,1} = 90, a_{4,1,1} = 10, a_{2,2,2} = 34, a_{3,2,1} = 76, a_{3,1,1,1} = 40, a_{2,2,1,1} = 216\}$

```
# unsigned linear, ordered types:
#   set of {A_\alpha^{(g)} = ...} over *all* compositions |alpha|=n
> cofs_A_all(n1,g1);
```

$\{A_{2,2,2} = 34, A_{1,4,1} = 6, A_{4,2} = 6, A_{1,1,2,2} = 30, A_{1,3,1,1} = 14, A_{2,1,1,1,1} = 34, A_{3,2,1} = 14, A_{1,1,1,1,2} = 34, A_{1,2,1,1,1} = 50, A_{1,1,2,1,1} = 62, A_{3,3} = 6, A_{2,2,1,1} = 30, A_{1,1,1,1,1,1} = 90, A_{1,1,1,2,1} = 50, A_{3,1,2} = 10, A_{1,3,2} = 14, A_{2,3,1} = 14, A_{2,4} = 6, A_{2,1,3} = 10, A_{3,1,1,1} = 6, A_{1,1,1,3} = 6, A_6 = 2, A_{4,1,1} = 2, A_{2,1,1,2} = 22, A_{5,1} = 2, A_{1,2,2,1} = 50, A_{1,2,3} = 14, A_{1,5} = 2, A_{1,2,1,2} = 42, A_{1,1,4} = 2, A_{1,1,3,1} = 14, A_{2,1,2,1} = 42\}$

```
# signed linear, unordered types:
#   set of {b_\lambda^{(g)} = ...} over *all* partitions |lambda|=n
> cofs_b_all(n1,g1);
```

$\{b_{1,1,1,1,1,1} = 30278, b_{2,1,1,1,1} = 12810, b_{2,2,1,1} = 1572, b_{2,2,2} = 34, b_{3,1,1,1} = 1048, b_{3,3} = 6,$

$b_{3,2,1} = 204, b_{4,2} = 12, b_{4,1,1} = 102, b_{5,1} = 12, b_6 = 2\}$

```
# signed linear, ordered types:
#   set of {B_\alpha^{(g)} = ...} over *all* compositions |alpha|=n
> cofs_B_all(n1,g1);
```

$\{B_6 = 2, B_{3,3} = 6, B_{2,4} = 6, B_{1,5} = 6, B_{4,2} = 6, B_{5,1} = 6, B_{2,2,2} = 34, B_{1,1,4} = 34, B_{3,2,1} = 34, B_{4,1,1} = 34, B_{2,3,1} = 34, B_{1,2,3} = 34, B_{3,1,2} = 34, B_{1,3,2} = 34, B_{2,1,3} = 34, B_{1,4,1} = 34, B_{2,1,2,1} = 262, B_{1,2,2,1} = 262, B_{1,3,1,1} = 262, B_{1,1,3,1} = 262, B_{1,2,1,2} = 262, B_{2,1,1,2} = 262, B_{3,1,1,1} = 262, B_{1,1,2,1,1} = 2562, B_{1,1,1,3} = 262, B_{2,2,1,1} = 262, B_{1,1,2,2} = 262, B_{1,2,1,1,1} = 2562, B_{1,1,1,2,1} = 2562, B_{1,1,1,1,2} = 2562, B_{2,1,1,1,1} = 2562, B_{1,1,1,1,1,1} = 30278\}$

```
# unsigned circular, unordered types:
#   set of {ac_\lambda^{(g)} = ...} over *all* partitions |lambda|=n
> cofs_ac_all(n1,g1);
```

$\{ac_{3,1,1,1} = 0, ac_{4,1,1} = 0, ac_{5,1} = 0, ac_6 = 0, ac_{2,2,1,1} = 15, ac_{2,1,1,1,1} = 12, ac_{1,1,1,1,1,1} = 3, ac_{4,2} = 6, ac_{3,3} = 3, ac_{2,2,2} = 8, ac_{3,2,1} = 12, ac_{C_6} = 1\}$

```
# unsigned circular, ordered types:
#   set of {Ac_\alpha^{(g)} = ...} over *all* circular compositions
|alpha|=n
> cofs_Ac_all(n1,g1);
```

$\{Ac_{4,1,1} = 0, Ac_6 = 0, Ac_{5,1} = 0, Ac_{3,1,1,1} = 0, Ac_{2,1,2,1} = 9, Ac_{2,2,1,1} = 6, Ac_{C_6} = 1, Ac_{2,1,1,1,1} = 12, Ac_{1,1,1,1,1,1} = 3, Ac_{4,2} = 6, Ac_{3,3} = 3, Ac_{3,1,2} = 6, Ac_{2,2,2} = 8, Ac_{3,2,1} = 6\}$

```
# signed circular, unordered types:
#   set of {bc_\lambda^{(g)} = ...} over *all* partitions |lambda|=n
> cofs_bc_all(n1,g1);
```

$\{bc_{2,2,2} = 8, bc_{3,1,1,1} = 150, bc_{3,2,1} = 48, bc_{3,3} = 3, bc_{C_6} = 1, bc_{1,1,1,1,1,1} = 2121, bc_{2,1,1,1,1} = 1248, bc_{2,2,1,1} = 225, bc_{4,1,1} = 24, bc_{4,2} = 6, bc_{5,1} = 6, bc_6 = 0\}$

```
# signed circular, ordered types:
#   set of {Bc_\alpha^{(g)} = ...} over *all* circular compositions
|alpha|=n
> cofs_Bc_all(n1,g1);
```

$\{Bc_{3,3} = 3, Bc_{4,2} = 6, Bc_{5,1} = 6, Bc_{2,2,2} = 8, Bc_{3,2,1} = 24, Bc_{4,1,1} = 24, Bc_{3,1,2} = 24, Bc_{C_6} = 1, Bc_6 = 0, Bc_{2,2,1,1} = 150, Bc_{3,1,1,1} = 150, Bc_{2,1,1,1,1} = 1248, Bc_{1,1,1,1,1,1} = 2121, Bc_{2,1,2,1} = 75\}$

```
# Demo leaving g as an indeterminate
# unsigned linear, unordered types:
#   set of {a_\lambda^{(g)} = ...} over *all* partitions |lambda|=n
> cofs_a_all(n1,g);
```

$$\left\{ a_{2,1,1,1,1} = -5\,48^{(g-1)} - 2\,4^{g} + 5\,2^{(g-1)} + \frac{3}{4}\,24^{g} - \frac{3}{2}\,8^{g} - \frac{1}{8}\,96^{g} + \frac{3}{4}\,12^{g} + \frac{1}{48}\,240^{g}, \right.$$
$$a_{3,2,1} = -4^{g} + \frac{1}{4}\,24^{g} + 3\,2^{g} - 8^{g},\ a_{4,1,1} = -4^{g} + 3\,2^{(g-1)} - \frac{1}{4}\,8^{g} + \frac{1}{4}\,12^{g},\ a_{3,3} = -2^{(g-1)} + 8^{(g-1)},$$
$$a_{2,2,1,1} = -\frac{1}{16}\,48^{g} + \frac{3}{2}\,4^{g} - \frac{1}{2}\,24^{g} - 3\,2^{g} + \frac{3}{2}\,8^{g} + \frac{1}{16}\,96^{g} - \frac{1}{4}\,12^{g},$$
$$a_{3,1,1,1} = \frac{1}{12}\,48^{g} + \frac{3}{2}\,4^{g} - 2^{(g+1)} - \frac{1}{4}\,24^{g} + \frac{3}{4}\,8^{g} - \frac{1}{2}\,12^{g},\ a_{5,1} = \frac{1}{2}\,4^{g} - 2^{g},$$
$$a_{2,2,2} = 2^{(g-1)} + 48^{(g-1)} - \frac{1}{4}\,8^{g},$$
$$a_{1,1,1,1,1,1} = \frac{1}{16}\,48^{g} + \frac{1}{2}\,4^{g} + 720^{(g-1)} - 2^{(g-1)} + 3\,8^{(g-1)} - \frac{1}{4}\,24^{g} + \frac{1}{16}\,96^{g} - \frac{1}{4}\,12^{g} - \frac{1}{48}\,240^{g},$$
$$\left. a_{6} = 2^{(g-1)},\ a_{4,2} = -2^{g} + \frac{1}{4}\,8^{g} \right\}$$

```
################################################################################
# Generating functions for number of strips in (n,g)-arrangements
# at fixed (n,g).
# n=integer
# g=integer or indeterminate
#
# ogf_a_n_z(n,g)
#  = a_n^{(g)}(z) = sum_{k=0..n} a_{n,k}^{(g)} z^k
# and similar for b, ac, bc
################################################################################

##########################################################################
# First method: using the direct formulas in the paper for these
# gen fn for # strips in unsigned linear arrangements, a_n^{(g)}(z).
# This is the best way to compute it.
##########################################################################

# strips in unsigned linear arrangements, a_n^{(g)}(z)
> ogf_a_n_z(n1,g1);
```

$$90\,z^{6} + 230\,z^{5} + 256\,z^{4} + 120\,z^{3} + 22\,z^{2} + 2\,z$$

```
# strips in signed linear arrangements, b_n^{(g)}(z)
> ogf_b_n_z(n1,g1);
```

$$2\,z + 30\,z^{2} + 340\,z^{3} + 2620\,z^{4} + 12810\,z^{5} + 30278\,z^{6}$$

```
# strips in unsigned circular arrangements, ac_n^{(g)}(z)
> ogf_ac_n_z(n1,g1);
```

$$1 + 20\,z^{3} + 15\,z^{4} + 3\,z^{6} + 12\,z^{5} + 9\,z^{2}$$

```
# strips in signed circular arrangements, bc_n^{(g)}(z)
> ogf_bc_n_z(n1,g1);
```

$$2121\,z^6 + 1248\,z^5 + 375\,z^4 + 80\,z^3 + 15\,z^2 + 1$$

```
########################################################################
# Second method: directly compute coefficient of t^n from the formula
# given in the paper for a(t,z), b(t,z), ac(t,z), bc(t,z)
# This is provided just to confirm the formulas give the same result.
########################################################################

> ogf_a_n_z2(n1,g1);
```

$$90\,z^6 + 230\,z^5 + 256\,z^4 + 120\,z^3 + 22\,z^2 + 2\,z$$

```
> ogf_b_n_z2(n1,g1);
```

$$2\,z + 30\,z^2 + 340\,z^3 + 2620\,z^4 + 12810\,z^5 + 30278\,z^6$$

```
> ogf_ac_n_z2(n1,g1);
```

$$1 + 20\,z^3 + 15\,z^4 + 3\,z^6 + 12\,z^5 + 9\,z^2$$

```
> ogf_bc_n_z2(n1,g1);
```

$$2121\,z^6 + 1248\,z^5 + 375\,z^4 + 80\,z^3 + 15\,z^2 + 1$$

```
########################################################################
# Third method: sum_{k=0..n} a_{n,k}^{(g)}*z^k
# using the formula given in the paper for a_{n,k} (or b,ac,bc)
# This is provided just to confirm the formulas give the same result.
########################################################################

> ogf_a_n_z3(n1,g1);
```

$$90\,z^6 + 230\,z^5 + 256\,z^4 + 120\,z^3 + 22\,z^2 + 2\,z$$

```
> ogf_b_n_z3(n1,g1);
```

$$2\,z + 30\,z^2 + 340\,z^3 + 2620\,z^4 + 12810\,z^5 + 30278\,z^6$$

```
> ogf_ac_n_z3(n1,g1);
```

$$1 + 20\,z^3 + 15\,z^4 + 3\,z^6 + 12\,z^5 + 9\,z^2$$

```
> ogf_bc_n_z3(n1,g1);
```

$$2121\,z^6 + 1248\,z^5 + 375\,z^4 + 80\,z^3 + 15\,z^2 + 1$$

```
###############################################################################
# Weight generating functions -- all (n,g) arrangements, specified (n,g)
# n=integer
```

```
# g=integer or indeterminate
# wgf_a(n,g)
#   = a_n^{(g)}(\vec v) = a_n^{(g)}(v[1],v[2],v[3],...)
#   = sum over all unsigned linear (n,g)-arrangements of their
#     unordered weight as a polynomial in v's
#
# Also b, ac, bc, A, B, Ac, Bc versions:
#
# unsigned unordered weight (a): u_\lambda = u[lambda1]*u[lambda2]*...
# unsigned ordered weight (A): U_\lambda = U[lambda1,lambda2,...]
# signed (b,B) is same but with v or V
#
# Circular versions (ac,Ac,bc,Bc) use the same weight variables (u,U,v,V)
# and also additional variables for the weight of the circular identity.
# The circular identity of length n has weight uc[n],Uc[n],vc[n], or Vc[n],#
respectively.
################################################################################

# unsigned linear, unordered weights u[i]'s
> wgf_a(n1,g1);
```

$$90 u_1^6 + 230 u_1^4 u_2 + 40 u_1^3 u_3 + 216 u_1^2 u_2^2 + 10 u_1^2 u_4 + 76 u_1 u_2 u_3 + 34 u_2^3 + 4 u_1 u_5 + 12 u_2 u_4 + 6 u_3^2 + 2 u_6$$

```
# unsigned linear, ordered weights U[alpha]'s
> wgf_A(n1,g1);
```

$$10 U_{2,1,3} + 10 U_{3,1,2} + 34 U_{2,1,1,1,1} + 6 U_{3,1,1,1} + 2 U_{4,1,1} + 22 U_{2,1,1,2} + 14 U_{1,2,3} + 50 U_{1,2,2,1} + 62 U_{1,1,2,1,1} + 2 U_{1,5} + 2 U_{1,1,4} + 6 U_{1,1,1,3} + 6 U_{1,4,1} + 50 U_{1,2,1,1,1} + 34 U_{1,1,1,1,2} + 34 U_{2,2,2} + 14 U_{1,1,3,1} + 50 U_{1,1,1,2,1} + 42 U_{2,1,2,1} + 6 U_{4,2} + 90 U_{1,1,1,1,1,1} + 14 U_{1,3,2} + 2 U_{5,1} + 14 U_{3,2,1} + 14 U_{2,3,1} + 30 U_{2,2,1,1} + 42 U_{1,2,1,2} + 14 U_{1,3,1,1} + 6 U_{2,4} + 2 U_6 + 6 U_{3,3} + 30 U_{1,1,2,2}$$

```
# signed linear, unordered weights v[i]'s
> wgf_b(n1,g1);
```

$$30278 v_1^6 + 12810 v_1^4 v_2 + 1048 v_3 v_1^3 + 1572 v_1^2 v_2^2 + 204 v_3 v_1 v_2 + 102 v_1^2 v_4 + 34 v_2^3 + 12 v_5 v_1 + 6 v_3^2 + 12 v_2 v_4 + 2 v_6$$

```
# signed linear, ordered weights V[alpha]'s
> wgf_B(n1,g1);
```

$$34 V_{4,1,1} + 262 V_{3,1,1,1} + 2 V_6 + 6 V_{3,3} + 6 V_{2,4} + 262 V_{2,1,2,1} + 6 V_{5,1} + 2562 V_{2,1,1,1,1} + 6 V_{1,5} + 6 V_{4,2} + 34 V_{2,2,2} + 34 V_{1,2,3} + 34 V_{1,1,4} + 34 V_{3,2,1} + 34 V_{1,4,1} + 34 V_{2,3,1} + 34 V_{3,1,2} + 34 V_{1,3,2} + 34 V_{2,1,3} + 262 V_{1,2,2,1} + 262 V_{1,1,3,1} + 262 V_{1,2,1,2} + 262 V_{2,1,1,2} + 262 V_{1,1,2,2} + 262 V_{1,1,1,3} + 262 V_{2,2,1,1} + 262 V_{1,3,1,1} + 30278 V_{1,1,1,1,1,1} + 2562 V_{1,1,2,1,1} + 2562 V_{1,1,1,2,1} + 2562 V_{1,1,1,1,2} + 2562 V_{1,2,1,1,1}$$

```
# unsigned circular, unordered weights u[i]'s, uc[i]'s
> wgf_ac(n1,g1);
```

$$3\,u_1^{\,6} + 12\,u_1^{\,4}\,u_2 + 15\,u_1^{\,2}\,u_2^{\,2} + 12\,u_1\,u_2\,u_3 + 8\,u_2^{\,3} + 6\,u_2\,u_4 + 3\,u_3^{\,2} + uc_6$$

```
# unsigned circular, ordered weights U[alpha]'s, Uc[i]'s
> wgf_Ac(n1,g1);
```

$$6\,U_{3,1,2} + 12\,U_{2,1,1,1,1} + 8\,U_{2,2,2} + 9\,U_{2,1,2,1} + 6\,U_{4,2} + 3\,U_{1,1,1,1,1,1} + 6\,U_{3,2,1} + 6\,U_{2,2,1,1} + 3\,U_{3,3} + Uc_6$$

```
# signed circular, unordered weights v[i]'s, vc[i]'s
> wgf_bc(n1,g1);
```

$$2121\,v_1^{\,6} + 1248\,v_1^{\,4}\,v_2 + 150\,v_3\,v_1^{\,3} + 225\,v_1^{\,2}\,v_2^{\,2} + 48\,v_3\,v_1\,v_2 + 24\,v_1^{\,2}\,v_4 + 8\,v_2^{\,3} + 6\,v_5\,v_1 + 3\,v_3^{\,2} + 6\,v_2\,v_4 + vc_6$$

```
# signed circular, ordered weights V[alpha]'s, Vc[i]'s
> wgf_Bc(n1,g1);
```

$$24\,V_{4,1,1} + 150\,V_{3,1,1,1} + 3\,V_{3,3} + 75\,V_{2,1,2,1} + 6\,V_{5,1} + 1248\,V_{2,1,1,1,1} + 6\,V_{4,2} + 8\,V_{2,2,2} + 24\,V_{3,2,1} + 24\,V_{3,1,2} + 150\,V_{2,2,1,1} + Vc_6 + 2121\,V_{1,1,1,1,1,1}$$

```
############################################################################
# Algebraic manipulation of weight generating functions
############################################################################

# u2v = paper's function phi(f)
# v2u = paper's function phi^{-1}(f)

> u2v(u[3]);
```

$$(G^2 + 3\,G + 1)\,G\,v_1^{\,3} + 2\,G\,v_1\,v_2 + v_3$$

```
> u2v(U[3]);
```

$$G\,V_{1,2} + G\,V_{2,1} + (G^2 + 3\,G + 1)\,G\,V_{1,1,1} + V_3$$

```
> v2u(v[3]);
```

$$\frac{(G^2 + G - 1)\,u_1^{\,3}\,G}{(G+1)^3} - \frac{2\,u_1\,G\,u_2}{G+1} + u_3$$

```
> v2u(V[3]);
```

$$-\frac{G\,U_{2,1}}{G+1} + \frac{(G^2 + G - 1)\,G\,U_{1,1,1}}{(G+1)^3} - \frac{G\,U_{1,2}}{G+1} + U_3$$

```
> v2u(u2v(U[3]));
```

$$U_3$$

```
> u2v(10*U[3,2]-8*U[1,4]);
```

$$-8 (G+1) V_{1,1,2,1} G^2 + 10 (G+2) V_{2,1,1,1} G^2 + 2 (G^3 + 5 G^2 + 7 G - 2) G^2 V_{1,1,1,1,1} + 10 (G+2) V_{3,1,1} G + 10 V_{2,1,2} G + 10 G V_{1,2,2} + 2 (G^2 + 2 G - 4) G V_{1,2,1,1} - 8 (G+1) G V_{1,3,1} + 2 (G^2 + 7 G + 1) G V_{1,1,1,2} - 8 (G+1) G V_{1,1,3} + 10 V_{3,2} + (-8 G - 8) V_{1,4}$$

```
# non-commutative multiplication
> mult_nc(10*U[3,2]-8*U[1,4], 2*U[5]-U[2,2,1],U);
```

$$20 U_{3,2,5} - 10 U_{3,2,2,2,1} - 16 U_{1,4,5} + 8 U_{1,4,2,2,1}$$

```
> mult_nc(10*V[3,2]-8*V[1,4], 2*V[5]-V[2,2,1],V);
```

$$20 V_{3,2,5} - 10 V_{3,2,2,2,1} - 16 V_{1,4,5} + 8 V_{1,4,2,2,1}$$

```
# In the non-commutative circular case,
# *after* performing all multiplications we must "straighten" the products
> straighten_circ(mult_nc(10*U[3,2]-8*U[1,4], 2*U[5]-U[2,2,1], U));
```

$$20 U_{5,3,2} - 10 U_{3,2,2,2,1} - 16 U_{5,1,4} + 8 U_{4,2,2,1,1}$$

```
# Example of duality:
# Let
#   G = 2^(g-1)-1
#   G2 = paper's \widetilde{G} = 2^(1-g)-1
#   Note that G2=-G/(1+G), G=-G2/(1+G2) and (G+1)(G2+1)=1
# If fv is an expression in v,V,vc,Vc's,
# and fu is the same expression with v's replaced by u's,
# then v2u(fv) and u2v(fu) will be very similar, but with
# u's & G2's in the former swapped with v's & G's in the latter
```

```
> v2u(V[3,2]);
```

$$-\frac{(G+2)(G^2+G-1) U_{1,1,1,1,1} G^2}{(G+1)^5} + \frac{(G+2) U_{1,2,1,1} G^2}{(G+1)^3} + \frac{(G+2) G^2 U_{2,1,1,1}}{(G+1)^3} - \frac{U_{2,1,2} G}{G+1} - \frac{(G+2) U_{3,1,1} G}{(G+1)^2} + \frac{(G^2+G-1) U_{1,1,1,2} G}{(G+1)^3} - \frac{G U_{1,2,2}}{G+1} + U_{3,2}$$

```
> icollect(subs(G=-G2/(1+G2),v2u(V[3,2])));
```

$$(G2+2)(G2^2+3 G2+1) U_{1,1,1,1,1} G2^2 + (G2+2) U_{1,2,1,1} G2^2 + (G2+2) G2^2 U_{2,1,1,1} + U_{2,1,2} G2 + (G2+2) U_{3,1,1} G2 + (G2^2+3 G2+1) U_{1,1,1,2} G2 + G2 U_{1,2,2} + U_{3,2}$$

```
> u2v(U[3,2]);
```

$$(G+2)\,V_{2,1,1,1}\,G^2+(G+2)\,G^2\,V_{1,2,1,1}+(G+2)\,(G^2+3\,G+1)\,G^2\,V_{1,1,1,1,1}+(G+2)\,V_{3,1,1}\,G+V_{2,1,2}\,G+G\,V_{1,2,2}+(G^2+3\,G+1)\,G\,V_{1,1,1,2}+V_{3,2}$$

```
# In the commutative case, we can demonstrate multiplicativity
# and icollect for cleaning up the expression.
# Note that in the noncommutative case,
# the paper's \phi, \phi^{-1} are multiplicative, but
# the representation in Maple isn't suited to an easy demo.

> u2v_u32 := u2v(u[3])*u2v(u[2]);
```

$$u2v\_u32 := ((G^2+3\,G+1)\,G\,v_1^3+2\,G\,v_1\,v_2+v_3)\,((G+2)\,G\,v_1^2+v_2)$$

```
> icollect(u2v_u32);
```

$$(G+2)\,(G^2+3\,G+1)\,G^2\,v_1^5+(3\,G^2+7\,G+1)\,G\,v_1^3\,v_2+(G+2)\,v_3\,G\,v_1^2+2\,G\,v_1\,v_2^2+v_3\,v_2$$

```
> u2v(u[3]*u[2]);
```

$$(G+2)\,(G^2+3\,G+1)\,G^2\,v_1^5+(3\,G^2+7\,G+1)\,G\,v_1^3\,v_2+(G+2)\,v_3\,G\,v_1^2+2\,G\,v_1\,v_2^2+v_3\,v_2$$

```
> v2u_v32 := v2u(v[3])*v2u(v[2]);
```

$$v2u\_v32 := \left(\frac{(G^2+G-1)\,u_1^3\,G}{(G+1)^3}-\frac{2\,u_1\,G\,u_2}{G+1}+u_3\right)\left(-\frac{(G+2)\,u_1^2\,G}{(G+1)^2}+u_2\right)$$

```
> icollect(u2v_u32);
```

$$(G+2)\,(G^2+3\,G+1)\,G^2\,v_1^5+(3\,G^2+7\,G+1)\,G\,v_1^3\,v_2+(G+2)\,v_3\,G\,v_1^2+2\,G\,v_1\,v_2^2+v_3\,v_2$$

```
> icollect(subs(G=-G2/(1+G2),v2u_v32));
```

$$(G2+2)\,(G2^2+3\,G2+1)\,u_1^5\,G2^2+(3\,G2^2+7\,G2+1)\,u_1^3\,G2\,u_2+(G2+2)\,u_1^2\,G2\,u_3+2\,u_1\,G2\,u_2^2+u_2\,u_3$$

```
> v2u(v[3]*v[2]);
```

$$-\frac{(G+2)\,(G^2+G-1)\,u_1^5\,G^2}{(G+1)^5}+\frac{(3\,G^2+5\,G-1)\,u_1^3\,G\,u_2}{(G+1)^3}-\frac{(G+2)\,u_1^2\,G\,u_3}{(G+1)^2}-\frac{2\,u_1\,G\,u_2^2}{G+1}+u_2\,u_3$$

```
##########################################################################
# Specializations
##########################################################################

# Commutative specialization: converts U,Uc,V,Vc to lowercase multiplicative
# versions
> nc2comm(10*U[3,2]-8*U[1,4]);
```

$$-8\,u_1\,u_4+10\,u_2\,u_3$$

```
# need functions to work with first
> A_wt := wgf_A(n1,g1);
```

$$A\_wt := 10\,U_{2,1,3} + 10\,U_{3,1,2} + 34\,U_{2,1,1,1,1} + 6\,U_{3,1,1,1} + 2\,U_{4,1,1} + 22\,U_{2,1,1,2} + 14\,U_{1,2,3} + 50\,U_{1,2,2,1} + 62\,U_{1,1,2,1,1} + 2\,U_{1,5} + 2\,U_{1,1,4} + 6\,U_{1,1,1,3} + 6\,U_{1,4,1} + 50\,U_{1,2,1,1,1} + 34\,U_{1,1,1,1,2} + 34\,U_{2,2,2} + 14\,U_{1,1,3,1} + 50\,U_{1,1,1,2,1} + 42\,U_{2,1,2,1} + 6\,U_{4,2} + 90\,U_{1,1,1,1,1,1} + 14\,U_{1,3,2} + 2\,U_{5,1} + 14\,U_{3,2,1} + 14\,U_{2,3,1} + 30\,U_{2,2,1,1} + 42\,U_{1,2,1,2} + 14\,U_{1,3,1,1} + 6\,U_{2,4} + 2\,U_{6} + 6\,U_{3,3} + 30\,U_{1,1,2,2}$$

```
> a_wt := wgf_a(n1,g1);
```

$$a\_wt := 90\,u_1^6 + 230\,u_1^4\,u_2 + 40\,u_1^3\,u_3 + 216\,u_1^2\,u_2^2 + 10\,u_1^2\,u_4 + 76\,u_1\,u_2\,u_3 + 34\,u_2^3 + 4\,u_1\,u_5 + 12\,u_2\,u_4 + 6\,u_3^2 + 2\,u_6$$

```
# Commutative specialization on ordered weight gives unordered weight
# This should equal a_wt
> nc2comm(A_wt);
```

$$90\,u_1^6 + 230\,u_1^4\,u_2 + 40\,u_1^3\,u_3 + 216\,u_1^2\,u_2^2 + 10\,u_1^2\,u_4 + 76\,u_1\,u_2\,u_3 + 34\,u_2^3 + 4\,u_1\,u_5 + 12\,u_2\,u_4 + 6\,u_3^2 + 2\,u_6$$

```
# 1-specialization: u[i],U[i],uc[i],Uc[i], & v versions, all -> 1
# so we get an integer = the total # arrangements
# These should be equal
> spec1(A_wt), spec1(a_wt), n1!^(g1-1);
```

$$720, 720, 720$$

```
# z-specialization: u[i],U[i],v[i],V[i] all -> z
# uc[i],Uc[i],vc[i],Vc[i] all -> 1
# so we get z^k for k strips
# specz(wgf_a(n,g)) and specz(wgf_A(n,g)) should agree with ogf_a_n_z(n,g)
# similar for b/B,ac/Ac,bc/Bc cases
# These should be equal
> specz(A_wt), specz(a_wt);
```

$$90\,z^6 + 230\,z^5 + 256\,z^4 + 120\,z^3 + 22\,z^2 + 2\,z,\ 90\,z^6 + 230\,z^5 + 256\,z^4 + 120\,z^3 + 22\,z^2 + 2\,z$$

```
# Incompressible specialization: u[1],U[1],v[1],V[1] -> 1
# u[i],U[i],v[i],V[i] (for i!=1) -> 0
# uc[i],Uc[i],vc[i],Vc[i] (for all i) -> 0
# so we get an integer = count of the incompressible arrangements
# These should be equal:
> speci(A_wt), speci(a_wt), cof_a_nk(n1,n1,g1);
```

$$90, 90, 90$$

```
#########################################################################
# RECURSIONS & DIFFERENTIAL EQUATIONS
#
# rec_a(...): symbollic recursion & evaluation via recursion in n (or n,k)
of
#          a_n^{(g)}(z)
#          a_{n,k}^{(g)}
# recd_a(...): similar but with mixed recursion in n / diffeq in ze
#
# Also b,ac,bc versions
#
# g=integer value
#########################################################################

# rec_a("rec",g)
# Compute the recursion equation in n, for
#     paper's a_n^{(g)}(z) = maple's a[n,g](z)
# Must specify g=integer in all of these recursion functions
#
# The function a[n,g](z) is expressed in terms of a[n-i,g](z)
# for various i's.
# Upon plugging in a specific value of n, the equation includes
# initial conditions:
#    a[n-i,g](z)=0 for n-i <= 0
#    delta[0]=1, delta[x]=0 when x!=0


# recursion for g=g1
> rec_a("rec",g1);
```

$$a_{n,2}(z) = (n z - 1 + z)\, a_{n-1,2}(z) - (z-1)\,(n z - 2 z + 3)\, a_{n-2,2}(z) - (z-1)^2\,(n z - 5 z + 1)\, a_{n-3,2}(z) + (z-1)^3\,(n-3)\, z\, a_{n-4,2}(z) + \delta_n + (-z+1)\,\delta_{n-1} - 3\,(z-1)^2\,\delta_{n-2} - (z-1)^3\,\delta_{n-3}$$

```
# rec_a(n,g)   (n,g=integers)
# Compute a_n^{(g)}(z) as an explicit polynomial by iterating the recursion
# Should give the same results as the direct formula in ogf_a_n_z(n,g)
> rec_a(n1,g1);
```

$$90\, z^6 + 230\, z^5 + 256\, z^4 + 120\, z^3 + 22\, z^2 + 2\, z$$

```
> ogf_a_n_z(n1,g1);
```

$$90\, z^6 + 230\, z^5 + 256\, z^4 + 120\, z^3 + 22\, z^2 + 2\, z$$

```
# rec_a(n,k,g)  (n,k,g=integers)
# Compute a_{n,k}^{(g)} by computing a_n^{(g)} from recursion and
```

```
# extracting coefficient of z^k

# a_{n,k}^{(g)}
> rec_a(n1,k1,g1);
```

$$120$$

```
# For fixed g, compute the symbollic recursion equation in n, k
# for a_{n,k}^{(g)}
# Expressed in terms of a[n-i,k-j,g]
#  where n,k are symbollic and i>=0, j>=0 are specific integers
> rec_a("recnk",g1);
```

$$a_{n,k,2} = -a_{n-1,k,2} + (n+1)\, a_{n-1,k-1,2} + 3\, a_{n-2,k,2} + (n-5)\, a_{n-2,k-1,2} + (-n+2)\, a_{n-2,k-2,2} - a_{n-3,k,2} + (-n+7)\, a_{n-3,k-1,2} + (-11+2\,n)\, a_{n-3,k-2,2} + (5-n)\, a_{n-3,k-3,2} + (-n+3)\, a_{n-4,k-1,2} + (-9+3\,n)\, a_{n-4,k-2,2} + (9-3\,n)\, a_{n-4,k-3,2} + (n-3)\, a_{n-4,k-4,2} + \delta_n\, \delta_k + (\delta_k - \delta_{k-1})\, \delta_{n-1} + (-3\,\delta_k + 6\,\delta_{k-1} - 3\,\delta_{k-2})\, \delta_{n-2} + (\delta_k - 3\,\delta_{k-1} + 3\,\delta_{k-2} - \delta_{k-3})\, \delta_{n-3}$$

```
########################################################################
# recd_a("rec",g)
# Compute the mixed recursion in n / differential equation in z, for
#     paper's a_n^{(g)}(z) = maple's a[n,g](z)
########################################################################

# Mixed recursion in n / diffeq in z, for g=g1
> recd_a("rec",g1);
```

$$a_{n,2}(z) = -(z-1)\, z^2 \left(\frac{d}{dz} a_{n-2,2}(z)\right) + z^2 \left(\frac{d}{dz} a_{n-1,2}(z)\right) - (z-1)^2\, z^2 \left(\frac{d}{dz} a_{n-3,2}(z)\right) + (z-1)^3\, z^2 \left(\frac{d}{dz} a_{n-4,2}(z)\right) + (1+2\,z)\, a_{n-1,2}(z) + (1-3\,z)\, a_{n-2,2}(z) + (z-1)\,(2\,z^2 - z + 1)\, a_{n-3,2}(z) + (z-1)^3\, z\, a_{n-4,2}(z) + \delta_n + (-z-1)\, \delta_{n-1} - (3\,z-1)\,(z-1)\, \delta_{n-2} - (z-1)^3\, \delta_{n-3}$$

```
# iterate the g=g1 recursion/diffeq to get a_n1^{(g1)}(z)
# should equal rec_a(n1,g1)
> recd_a(n1,g1);
```

$$90\, z^6 + 230\, z^5 + 256\, z^4 + 120\, z^3 + 22\, z^2 + 2\, z$$

```
# Compute a_{n1,k1}^{(g1)} from that.  Should equal rec_a(n1,k1,g1)
> recd_a(n1,k1,g1);
```

$$120$$

```
# Recursion in n,k corresponding to mixed rec/diffeq.
# Will be different than rec_a("recnk",g1)!
> recd_a("recnk",g1);
```

$$a_{n,k,2} = a_{n-1,k,2} + (1+k)\, a_{n-1,k-1,2} + a_{n-2,k,2} + (k-4)\, a_{n-2,k-1,2} + (2-k)\, a_{n-2,k-2,2} - a_{n-3,k,2} + (3-k)\, a_{n-3,k-1,2} + (-7+2k)\, a_{n-3,k-2,2} + (-k+5)\, a_{n-3,k-3,2} - k\, a_{n-4,k-1,2} + (-3+3k)\, a_{n-4,k-2,2} + (6-3k)\, a_{n-4,k-3,2} + (k-3)\, a_{n-4,k-4,2} + \delta_n \delta_k + (-\delta_k - \delta_{k-1})\, \delta_{n-1} + (-\delta_k + 4\,\delta_{k-1} - 3\,\delta_{k-2})\, \delta_{n-2} + (\delta_k - 3\,\delta_{k-1} + 3\,\delta_{k-2} - \delta_{k-3})\, \delta_{n-3}$$

```
##########################################################################
# All four versions of rec_a and recd_a shown above,
# also have versions for b, ac, bc.
# rec_*(n,k,g) should match cof_*_nk(n,k,g),
# rec_*(n,g) should match ogf_*_n_z(n,g)
##########################################################################

# Evaluation of coefficients a_{n,k}^{(g)}, etc., via
#  recursion on a_n^{(g)}(z); mixed recursion/diffeq on that;
#  and the direct formula from the paper:
> rec_a(n1,k1,g1), recd_a(n1,k1,g1), cof_a_nk(n1,k1,g1);
```

$$120, 120, 120$$

```
> rec_b(n1,k1,g1), recd_b(n1,k1,g1), cof_b_nk(n1,k1,g1);
```

$$340, 340, 340$$

```
> rec_ac(n1,k1,g1), recd_ac(n1,k1,g1), cof_ac_nk(n1,k1,g1);
```

$$20, 20, 20$$

```
> rec_bc(n1,k1,g1), recd_bc(n1,k1,g1), cof_bc_nk(n1,k1,g1);
```

$$80, 80, 80$$

```
# Evaluation of polynomials a_n^{(g)}(z), etc., via
#  recursion on a_n^{(g)}(z); mixed recursion/diffeq on that;
#  and the direct formula from the paper:
> rec_a(n1,g1), recd_a(n1,g1), ogf_a_n_z(n1,g1);
```

$$90 z^6 + 230 z^5 + 256 z^4 + 120 z^3 + 22 z^2 + 2 z,\ 90 z^6 + 230 z^5 + 256 z^4 + 120 z^3 + 22 z^2 + 2 z,\ 90 z^6 + 230 z^5 + 256 z^4 + 120 z^3 + 22 z^2 + 2 z$$

```
> rec_b(n1,g1), recd_b(n1,g1), ogf_b_n_z(n1,g1);
```

$$30278 z^6 + 12810 z^5 + 2620 z^4 + 340 z^3 + 30 z^2 + 2 z,\ 30278 z^6 + 12810 z^5 + 2620 z^4 + 340 z^3 + 30 z^2 + 2 z,\ 30278 z^6 + 12810 z^5 + 2620 z^4 + 340 z^3 + 30 z^2 + 2 z$$

```
> rec_ac(n1,g1), recd_ac(n1,g1), ogf_ac_n_z(n1,g1);
```

$$3 z^6 + 12 z^5 + 15 z^4 + 20 z^3 + 9 z^2 + 1,\ 3 z^6 + 12 z^5 + 15 z^4 + 20 z^3 + 9 z^2 + 1,\ 3 z^6 + 12 z^5 + 15 z^4 + 20 z^3 + 9 z^2 + 1$$

```
> rec_bc(n1,g1), recd_bc(n1,g1), ogf_bc_n_z(n1,g1);
```

$$2121 z^6 + 1248 z^5 + 375 z^4 + 80 z^3 + 15 z^2 + 1,\ 2121 z^6 + 1248 z^5 + 375 z^4 + 80 z^3 + 15 z^2 + 1,\ 2121 z^6 + 1248 z^5 + 375 z^4 + 80 z^3 + 15 z^2 + 1$$

```
# Recursion in n for a_n^{(g)}(z), etc.:
> rec_a("rec",g1);
```

$$a_{n,2}(z) = (n\,z - 1 + z)\, a_{n-1,2}(z) - (z-1)\,(n\,z - 2\,z + 3)\, a_{n-2,2}(z) - (z-1)^2\,(n\,z - 5\,z + 1)\, a_{n-3,2}(z) + (z-1)^3\,(n-3)\, z\, a_{n-4,2}(z) + \delta_n + (-z+1)\,\delta_{n-1} - 3\,(z-1)^2\,\delta_{n-2} - (z-1)^3\,\delta_{n-3}$$

```
> rec_b("rec",g1);
```

$$b_{n,2}(z) = (2\,n\,z - z + 1)\, b_{n-1,2}(z) + 2\,(n-2)\,(z-1)\, z\, b_{n-2,2}(z) + \delta_n + (z-1)\,\delta_{n-1}$$

```
> rec_ac("rec",g1);
```

$$ac_{n,2}(z) = (-2 + n\,z + z)\, ac_{n-1,2}(z) - (z-1)\,(-11\,z + 2\,n\,z + 5)\, ac_{n-2,2}(z) - (z-1)^2\,(-5\,z + 3\,n\,z - 4)\, ac_{n-3,2}(z) + (z-1)^3\,(4\,n\,z - 27\,z + 11)\, ac_{n-4,2}(z) + 3\,(z-1)^4\,(n\,z - 7\,z + 2)\, ac_{n-5,2}(z) - (z-1)^5\,(-9\,z + 2\,n\,z - 1)\, ac_{n-6,2}(z) - (n-7)\,(z-1)^6\, z\, ac_{n-7,2}(z) + \delta_n + (-2\,z + 3)\,\delta_{n-1} - (7\,z - 2)\,(z-1)\,\delta_{n-2} + (18\,z - 17\,z^2 + 4\,z^3 - 6)\,\delta_{n-3} + (z-1)\,(11\,z^3 - 26\,z^2 + 21\,z - 5)\,\delta_{n-4} + (7\,z^3 - 9\,z^2 + 6\,z - 1)\,(z-1)^2\,\delta_{n-5} - (z+1)^2\,(z-1)^3\, z\,\delta_{n-6} + (2\,z - 5)\,(z-1)^4\, z^2\,\delta_{n-7} + (4\,z - 1)\,(z-1)^5\, z^2\,\delta_{n-8} + (z-1)^6\, z^3\,\delta_{n-9}$$

```
> rec_bc("rec",g1);
```

$$bc_{n,2}(z) = (2\,n\,z - 4\,z + 2)\, bc_{n-1,2}(z) + (z-1)\,(4\,n\,z - 7\,z + 1)\, bc_{n-2,2}(z) + 2\,(z-1)^2\,(n-2)\, z\, bc_{n-3,2}(z) + \delta_n + (2\,z - 1)\,\delta_{n-1}$$

```
# Recursions in n,k for a_{n,k}^{(g)}, etc., that follow from
# the recursions for a_n^{(g)}(z), etc.:
> rec_a("recnk",g1);
```

$$a_{n,k,2} = -a_{n-1,k,2} + (n+1)\, a_{n-1,k-1,2} + 3\, a_{n-2,k,2} + (n-5)\, a_{n-2,k-1,2} + (-n+2)\, a_{n-2,k-2,2} - a_{n-3,k,2} + (-n+7)\, a_{n-3,k-1,2} + (-11 + 2\,n)\, a_{n-3,k-2,2} + (5-n)\, a_{n-3,k-3,2} + (-n+3)\, a_{n-4,k-1,2} + (-9 + 3\,n)\, a_{n-4,k-2,2} + (9 - 3\,n)\, a_{n-4,k-3,2} + (n-3)\, a_{n-4,k-4,2} + \delta_n\,\delta_k + (\delta_k - \delta_{k-1})\,\delta_{n-1} + (-3\,\delta_k + 6\,\delta_{k-1} - 3\,\delta_{k-2})\,\delta_{n-2} + (\delta_k - 3\,\delta_{k-1} + 3\,\delta_{k-2} - \delta_{k-3})\,\delta_{n-3}$$

```
> rec_b("recnk",g1);
```

$$b_{n,k,2} = b_{n-1,k,2} + (-1 + 2\,n)\, b_{n-1,k-1,2} + (4 - 2\,n)\, b_{n-2,k-1,2} + (2\,n - 4)\, b_{n-2,k-2,2} + \delta_n\,\delta_k + (-\delta_k + \delta_{k-1})\,\delta_{n-1}$$

```
> rec_ac("recnk",g1);
```

$$ac_{n,k,2} = -2\, ac_{n-1,k,2} + (n+1)\, ac_{n-1,k-1,2} + 5\, ac_{n-2,k,2} + (2\,n - 16)\, ac_{n-2,k-1,2} + (-2\,n + 11)\, ac_{n-2,k-2,2} + 4\, ac_{n-3,k,2} + (-3 - 3\,n)\, ac_{n-3,k-1,2} + (-6 + 6\,n)\, ac_{n-3,k-2,2} + (5 - 3\,n)\, ac_{n-3,k-3,2} - 11\, ac_{n-4,k,2} + (60 - 4\,n)\, ac_{n-4,k-1,2} + (-114 + 12\,n)\, ac_{n-4,k-2,2} + (92 - 12\,n)\, ac_{n-4,k-3,2} + (-27 + 4\,n)\, ac_{n-4,k-4,2} + 6\, ac_{n-5,k,2} + (-45 + 3\,n)\, ac_{n-5,k-1,2} + (-12\,n + 120)\, ac_{n-5,k-2,2} + (18\,n - 150)\, ac_{n-5,k-3,2} + (90 - 12\,n)\, ac_{n-5,k-4,2} + (3\,n - 21)\, ac_{n-5,k-5,2} - ac_{n-6,k,2} + (2\,n - 4)\, ac_{n-6,k-1,2} + (35 - 10\,n)\, ac_{n-6,k-2,2} + (-80 + 20\,n)\, ac_{n-6,k-3,2} + (85 - 20\,n)\, ac_{n-6,k-4,2} + (10\,n - 44)\, ac_{n-6,k-5,2} + (-2\,n + 9)\, ac_{n-6,k-6,2} + (-n+7)\, ac_{n-7,k-1,2} + (-42 + 6\,n)\, ac_{n-7,k-2,2} + (105 - 15\,n)\, ac_{n-7,k-3,2} + (20\,n - 140)\, ac_{n-7,k-4,2} + (105 - 15\,n)\, ac_{n-7,k-5,2} + (-42 + 6\,n)\, ac_{n-7,k-6,2} + (-n+7)\, ac_{n-7,k-7,2} + \delta_n\,\delta_k + (3\,\delta_k - 2\,\delta_{k-1})\,\delta_{n-1} + (-2\,\delta_k + 9\,\delta_{k-1} - 7\,\delta_{k-2})\,\delta_{n-2} + (-6\,\delta_k + 18\,\delta_{k-1} - 17\,\delta_{k-2} + 4\,\delta_{k-3})\,\delta_{n-3} + (5\,\delta_k - 26\,\delta_{k-1} + 47\,\delta_{k-2} - 37\,\delta_{k-3} + 11\,\delta_{k-4})\,\delta_{n-4}$$

$$+ (-\delta_k + 8\,\delta_{k-1} - 22\,\delta_{k-2} + 31\,\delta_{k-3} - 23\,\delta_{k-4} + 7\,\delta_{k-5})\,\delta_{n-5}$$
$$+ (\delta_{k-1} - \delta_{k-2} - 2\,\delta_{k-3} + 2\,\delta_{k-4} + \delta_{k-5} - \delta_{k-6})\,\delta_{n-6}$$
$$+ (-5\,\delta_{k-2} + 22\,\delta_{k-3} - 38\,\delta_{k-4} + 32\,\delta_{k-5} - 13\,\delta_{k-6} + 2\,\delta_{k-7})\,\delta_{n-7}$$
$$+ (\delta_{k-2} - 9\,\delta_{k-3} + 30\,\delta_{k-4} - 50\,\delta_{k-5} + 45\,\delta_{k-6} - 21\,\delta_{k-7} + 4\,\delta_{k-8})\,\delta_{n-8}$$
$$+ (\delta_{k-3} - 6\,\delta_{k-4} + 15\,\delta_{k-5} - 20\,\delta_{k-6} + 15\,\delta_{k-7} - 6\,\delta_{k-8} + \delta_{k-9})\,\delta_{n-9}$$

```
> rec_bc("recnk",g1);
```

$$bc_{n,k,2} = 2\,bc_{n-1,k,2} + (2\,n-4)\,bc_{n-1,k-1,2} - bc_{n-2,k,2} + (8-4\,n)\,bc_{n-2,k-1,2} + (4\,n-7)\,bc_{n-2,k-2,2} + (2\,n-4)\,bc_{n-3,k-1,2} + (8-4\,n)\,bc_{n-3,k-2,2} + (2\,n-4)\,bc_{n-3,k-3,2} + \delta_n\,\delta_k + (-\delta_k + 2\,\delta_{k-1})\,\delta_{n-1}$$

```
# Mixed diffeq in z / recursion in n for a_n^{(g)}(z), etc.,
> recd_a("rec",g1);
```

$$a_{n,2}(z) = -(z-1)\,z^2\left(\frac{d}{dz}a_{n-2,2}(z)\right) + z^2\left(\frac{d}{dz}a_{n-1,2}(z)\right) - (z-1)^2\,z^2\left(\frac{d}{dz}a_{n-3,2}(z)\right) + (z-1)^3\,z^2\left(\frac{d}{dz}a_{n-4,2}(z)\right) + (1+2\,z)\,a_{n-1,2}(z) + (1-3\,z)\,a_{n-2,2}(z) + (z-1)\,(2\,z^2 - z + 1)\,a_{n-3,2}(z) + (z-1)^3\,z\,a_{n-4,2}(z) + \delta_n + (-z-1)\,\delta_{n-1} - (3\,z-1)\,(z-1)\,\delta_{n-2} - (z-1)^3\,\delta_{n-3}$$

```
> recd_b("rec",g1);
```

$$b_{n,2}(z) = 2\,z^2\left(\frac{d}{dz}b_{n-1,2}(z)\right) + 2\,(z-1)\,z^2\left(\frac{d}{dz}b_{n-2,2}(z)\right) + (2+z)\,b_{n-1,2}(z) + (-z-1)\,b_{n-2,2}(z) + \delta_n + (z-2)\,\delta_{n-1} + (-z+1)\,\delta_{n-2}$$

```
> recd_ac("rec",g1);
```

$$ac_{n,2}(z) = 3\,(z-1)^4\,z^2\left(\frac{d}{dz}ac_{n-5,2}(z)\right) - 3\,(z-1)^2\,z^2\left(\frac{d}{dz}ac_{n-3,2}(z)\right) - 2\,(z-1)^5\,z^2\left(\frac{d}{dz}ac_{n-6,2}(z)\right) - (z-1)^6\,z^2\left(\frac{d}{dz}ac_{n-7,2}(z)\right) + z^2\left(\frac{d}{dz}ac_{n-1,2}(z)\right) - 2\,(z-1)\,z^2\left(\frac{d}{dz}ac_{n-2,2}(z)\right) + 4\,(z-1)^3\,z^2\left(\frac{d}{dz}ac_{n-4,2}(z)\right) + 2\,z\,ac_{n-1,2}(z) + (7\,z^2 + 5 - 10\,z)\,ac_{n-2,2}(z) - 4\,(z+1)\,(z-1)^2\,ac_{n-3,2}(z) - (11\,z-3)\,(z-1)^3\,ac_{n-4,2}(z) - 2\,(3\,z-2)\,(z-1)^4\,ac_{n-5,2}(z) - (3\,z^2 - 2\,z + 1)\,(z-1)^4\,ac_{n-6,2}(z) + \delta_n + (-2\,z+1)\,\delta_{n-1} + (8\,z - 7\,z^2 - 4)\,\delta_{n-2} + (4\,z^2 - 11\,z + 4)\,z\,\delta_{n-3} + (-16\,z - 32\,z^3 + 11\,z^4 + 3 + 33\,z^2)\,\delta_{n-4} + (7\,z^3 - 3\,z^2 + 4\,z - 1)\,(z-1)^2\,\delta_{n-5} - (z-1)^4\,z^2\,\delta_{n-6} + (2\,z-3)\,(z-1)^4\,z^2\,\delta_{n-7} + (2\,z-1)^2\,(z-1)^4\,z^2\,\delta_{n-8} + (z-1)^6\,z^3\,\delta_{n-9}$$

```
> recd_bc("rec",g1);
```

$$bc_{n,2}(z) = 2\,z^2\left(\frac{d}{dz}bc_{n-1,2}(z)\right) + 4\,(z-1)\,z^2\left(\frac{d}{dz}bc_{n-2,2}(z)\right) + 2\,(z-1)^2\,z^2\left(\frac{d}{dz}bc_{n-3,2}(z)\right) + (-2\,z+3)\,bc_{n-1,2}(z) + (z^2 + 4\,z - 3)\,bc_{n-2,2}(z) + (z-1)\,(2\,z-1)\,(z+1)\,bc_{n-3,2}(z) + \delta_n + (2\,z-2)\,\delta_{n-1} + (-2\,z+1)\,\delta_{n-2}$$

```
# and the recursions in n,k for a_{n,k}^{(g)}, etc. that follow from those
> recd_a("recnk",g1);
```

$$a_{n,k,2} = a_{n-1,k,2} + (1+k)\, a_{n-1,k-1,2} + a_{n-2,k,2} + (k-4)\, a_{n-2,k-1,2} + (2-k)\, a_{n-2,k-2,2} - a_{n-3,k,2} + (3-k)\, a_{n-3,k-1,2} + (-7+2k)\, a_{n-3,k-2,2} + (-k+5)\, a_{n-3,k-3,2} - k\, a_{n-4,k-1,2} + (-3+3k)\, a_{n-4,k-2,2} + (6-3k)\, a_{n-4,k-3,2} + (k-3)\, a_{n-4,k-4,2} + \delta_n \delta_k + (-\delta_k - \delta_{k-1})\, \delta_{n-1} + (-\delta_k + 4\,\delta_{k-1} - 3\,\delta_{k-2})\, \delta_{n-2} + (\delta_k - 3\,\delta_{k-1} + 3\,\delta_{k-2} - \delta_{k-3})\, \delta_{n-3}$$

```
> recd_b("recnk",g1);
```

$$b_{n,k,2} = 2\, b_{n-1,k,2} + (2k-1)\, b_{n-1,k-1,2} - b_{n-2,k,2} + (1-2k)\, b_{n-2,k-1,2} + (-4+2k)\, b_{n-2,k-2,2} + \delta_n \delta_k + (-2\,\delta_k + \delta_{k-1})\, \delta_{n-1} + (\delta_k - \delta_{k-1})\, \delta_{n-2}$$

```
> recd_ac("recnk",g1);
```

$$ac_{n,k,2} = (1+k)\, ac_{n-1,k-1,2} + 5\, ac_{n-2,k,2} + (-12+2k)\, ac_{n-2,k-1,2} + (11-2k)\, ac_{n-2,k-2,2} - 4\, ac_{n-3,k,2} + (-3k+7)\, ac_{n-3,k-1,2} + (-8+6k)\, ac_{n-3,k-2,2} + (5-3k)\, ac_{n-3,k-3,2} - 3\, ac_{n-4,k,2} + (-4k+24)\, ac_{n-4,k-1,2} + (-66+12k)\, ac_{n-4,k-2,2} + (72-12k)\, ac_{n-4,k-3,2} + (4k-27)\, ac_{n-4,k-4,2} + 4\, ac_{n-5,k,2} + (3k-25)\, ac_{n-5,k-1,2} + (72-12k)\, ac_{n-5,k-2,2} + (-106+18k)\, ac_{n-5,k-3,2} + (76-12k)\, ac_{n-5,k-4,2} + (3k-21)\, ac_{n-5,k-5,2} - ac_{n-6,k,2} + (4+2k)\, ac_{n-6,k-1,2} + (3-10k)\, ac_{n-6,k-2,2} + (20k-32)\, ac_{n-6,k-3,2} + (53-20k)\, ac_{n-6,k-4,2} + (10k-36)\, ac_{n-6,k-5,2} + (9-2k)\, ac_{n-6,k-6,2} + (-k+1)\, ac_{n-7,k-1,2} + (6k-12)\, ac_{n-7,k-2,2} + (45-15k)\, ac_{n-7,k-3,2} + (20k-80)\, ac_{n-7,k-4,2} + (75-15k)\, ac_{n-7,k-5,2} + (-36+6k)\, ac_{n-7,k-6,2} + (7-k)\, ac_{n-7,k-7,2} + \delta_n \delta_k + (\delta_k - 2\,\delta_{k-1})\, \delta_{n-1} + (-4\,\delta_k + 8\,\delta_{k-1} - 7\,\delta_{k-2})\, \delta_{n-2} + (4\,\delta_{k-1} - 11\,\delta_{k-2} + 4\,\delta_{k-3})\, \delta_{n-3} + (3\,\delta_k - 16\,\delta_{k-1} + 33\,\delta_{k-2} - 32\,\delta_{k-3} + 11\,\delta_{k-4})\, \delta_{n-4} + (-\delta_k + 6\,\delta_{k-1} - 12\,\delta_{k-2} + 17\,\delta_{k-3} - 17\,\delta_{k-4} + 7\,\delta_{k-5})\, \delta_{n-5} + (-\delta_{k-2} + 4\,\delta_{k-3} - 6\,\delta_{k-4} + 4\,\delta_{k-5} - \delta_{k-6})\, \delta_{n-6} + (-3\,\delta_{k-2} + 14\,\delta_{k-3} - 26\,\delta_{k-4} + 24\,\delta_{k-5} - 11\,\delta_{k-6} + 2\,\delta_{k-7})\, \delta_{n-7} + (\delta_{k-2} - 8\,\delta_{k-3} + 26\,\delta_{k-4} - 44\,\delta_{k-5} + 41\,\delta_{k-6} - 20\,\delta_{k-7} + 4\,\delta_{k-8})\, \delta_{n-8} + (\delta_{k-3} - 6\,\delta_{k-4} + 15\,\delta_{k-5} - 20\,\delta_{k-6} + 15\,\delta_{k-7} - 6\,\delta_{k-8} + \delta_{k-9})\, \delta_{n-9}$$

```
> recd_bc("recnk",g1);
```

$$bc_{n,k,2} = 3\, bc_{n-1,k,2} + (-4+2k)\, bc_{n-1,k-1,2} - 3\, bc_{n-2,k,2} + (-4k+8)\, bc_{n-2,k-1,2} + (-7+4k)\, bc_{n-2,k-2,2} + bc_{n-3,k,2} + (-4+2k)\, bc_{n-3,k-1,2} + (-4k+7)\, bc_{n-3,k-2,2} + (-4+2k)\, bc_{n-3,k-3,2} + \delta_n \delta_k + (-2\,\delta_k + 2\,\delta_{k-1})\, \delta_{n-1} + (\delta_k - 2\,\delta_{k-1})\, \delta_{n-2}$$

```
###########################################################################

> interface(echo=1);
```

$$2$$

```
>
```