/home/m262f99/A=B/Maple/zdemo1.mws

Math 262a, Fall 1999, Glenn Tesler
Zeilberger's Algorithm demo
10/17/99

```
>
> read EKHAD;
```

Version of Feb 25, 1999

This version is much faster than previous versions, thanks to

a remark of Frederic Chyzak. We thank him SO MUCH!

The penultimate version, Feb. 1997,

corrected a subtle bug discovered by Helmut Prodinger

Previous versions benefited from comments by Paula Cohen,

Lyle Ramshaw, and Bob Sulanke.

This is EKHAD, One of the Maple packages

accompanying the book

"A=B"

(published by A.K. Peters, Welesley, 1996)

by Marko Petkovsek, Herb Wilf, and Doron Zeilberger.

The most current version is available on WWW at:

http://www.math.temple.edu/~zeilberg .

Information about the book, and how to order it, can be found in

http://www.central.cis.upenn.edu/~wilf/AeqB.html .

Please report all bugs to: zeilberg@math.temple.edu .

All bugs or other comments used will be acknowledged in future versions.

For general help, and a list of the available functions,

type "ezra();". For specific help type "ezra(procedure_name)"

```
> ezra();
```

EKHAD

A Maple package for proving Hypergemetric (Binomial Coeff.)

and other kinds of identities

This version (Feb, 25, 1999) is much faster than the previous version, thanks to a SLIGHT (yet POWERFUL) modification suggested by FREDERIC CHYZAK

For help with a specific procedure, type "ezra(procedure_name);"

Contains procedures:

findrec,ct,zeil,zeilpap,zeillim,AZd,AZc,AZpapd,AZpapc,celine

```
> ?zeilpap
> ezra(zeilpap);
```

zeilpap(SUMMAND,k,n) or zeilpap(SUMMAND,k,n,NAME,REF)

Just like zeil but writes a paper with the proof

NAME and REF are optional name and reference

Warning: It assumes that the definite summation w.r.t. k extends over all k where it is non-zero, and that it is zero for other k

For non-natural summation limits, use zeillim

```
> ezra(zeil);
```

Like ct,

this is a Maple inplementation of the algorithm described in Ch. 6 of the book A=B, first proposed in : D. Zeilberger, "The method of

But it is not necessary to guess the ORDER

zeil(SUMMAND,k,n,N) or zeil(SUMMAND,k,n,N,MAXORDER) or zeil(SUMMAND,k,n,N,MAXORDER,parameter_list)

finds a linear recurrence equation for SUMMAND, with polynomial coefficients

of ORDER<=MAXORDER, where the default of MAXORDER is 6

in the parameter n, the shift operator in n being N

of the form ope(N,n)SUMMAND=G(n,k+1)-G(n,k)

*where G(n,k):=R(n,k)*SUMMAND, and R(n,k) is the 2nd item of output.*

The output is ope(N,n),R(n,k) .

For example zeil(binomial(n,k),k,n,N) would yield

N-2, k/(k-n-1)

in which N-2 is the "ope" operator, and k/(k-n-1) is R(n,k)

SUMMAND should be a product of factorials and/or binomial coeffs

and/or rising factorials, where (a)_k is denoted by rf(a,k)
and/or powers in k and n, and, optionally, a polynomial factor.
The last optional parameter, is the list of other parameters,
if present. Giving them causes considerable speedup. For example
*zeil(binomial(n,k)*binomial(a,k)*binomial(b,k),k,n,N,6,[a,b])*

```
>
>
> zeil(binomial(n,k),k,n,En);
```

$$-2 + En, \frac{k}{-n-1+k}$$

```
> zeilpap(binomial(n,k),k,n);
```

```
A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let   F(n,k)   be given by
```

$$\text{binomial}(n, k)$$

```
and let   SUM(n)   be the sum of   F(n,k)    with
respect to   k   .

SUM(n)    satisfies the following linear recurrence equation
```

$$-2\ \text{SUM}(n) + \text{SUM}(n+1)$$

=0.

```
PROOF: We cleverly construct   G(n,k)   :=
```

$$\frac{k\ \text{binomial}(n, k)}{-n-1+k}$$

```
with the motive that
```

$$-2\ \text{F}(n, k) + \text{F}(n+1, k)$$

```
=   G(n,k+1)-G(n,k)   (check!)

and the theorem follows upon summing with respect to   k   .QED.
```

Let’s verify it:

```
> FF := (n,k) -> binomial(n,k);
  GG := (n,k) -> k*binomial(n,k)/(-n-1+k);
  lh := -2*FF(n,k)+FF(n+1,k);
  rh := GG(n,k+1)-GG(n,k);
```

$$FF := \text{binomial}$$

$$GG := (n, k) \rightarrow \frac{k\ \text{binomial}(n, k)}{-n-1+k}$$

$$lh := -2\,\mathrm{binomial}(n, k) + \mathrm{binomial}(n+1, k)$$

$$rh := \frac{(k+1)\,\mathrm{binomial}(n, k+1)}{-n+k} - \frac{k\,\mathrm{binomial}(n, k)}{-n-1+k}$$

```
>
```

Dividing through by F(n,k) and simplifying gives rational functions on both sides.

```
> lh := sumtools[simpcomb](lh/FF(n,k));
  rh := sumtools[simpcomb](rh/FF(n,k));
```

$$lh := -\frac{-n-1+2\,k}{-n-1+k}$$

$$rh := -\frac{-n-1+2\,k}{-n-1+k}$$

```
> simplify(lh-rh);
```

$$0$$

```
>
>
>
> zeilpap(binomial(n,k)^2,k,n);
```

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let   F(n,k)   be given by

$$\mathrm{binomial}(n, k)^2$$

and let   SUM(n)   be the sum of   F(n,k)   with
respect to   k   .

SUM(n)   satisfies the following linear recurrence equation

$$(-4\,n-2)\,\mathrm{SUM}(n) + (n+1)\,\mathrm{SUM}(n+1)$$

$$=0.$$

PROOF: We cleverly construct   G(n,k)   :=

$$\frac{(-3\,n-3+2\,k)\,k^2\,\mathrm{binomial}(n, k)^2}{(-n-1+k)^2}$$

with the motive that

$$(-4\,n-2)\,\mathrm{F}(n, k) + (n+1)\,\mathrm{F}(n+1, k)$$

=   G(n,k+1)-G(n,k)   (check!)

and the theorem follows upon summing with respect to   k   .QED.

```
> zeilpap(binomial(n,k)^3,k,n);
```

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let F(n,k) be given by

$$\mathrm{binomial}(n,k)^3$$

and let SUM(n) be the sum of F(n,k) with respect to k .

SUM(n) satisfies the following linear recurrence equation

$$-8\,(n+1)^2\,\mathrm{SUM}(n)+(-7\,n^2-21\,n-16)\,\mathrm{SUM}(n+1)+(n+2)^2\,\mathrm{SUM}(n+2)$$
$$=0.$$

PROOF: We cleverly construct G(n,k) :=

$$(n^2+2\,n+1)\,(-14\,n^3-74\,n^2-128\,n-72+78\,k+27\,n^2\,k+93\,n\,k-18\,n\,k^2-30\,k^2+4\,k^3)\,k^3\,\mathrm{binomial}(n,k)^3 \Big/ ((-n-1+k)^3\,(-n-2+k)^3)$$

with the motive that

$$-8\,(n+1)^2\,\mathrm{F}(n,k)+(-7\,n^2-21\,n-16)\,\mathrm{F}(n+1,k)+(n+2)^2\,\mathrm{F}(n+2,k)$$

= G(n,k+1)-G(n,k) (check!)

and the theorem follows upon summing with respect to k .QED.

>

Gauss's 2F1 identity

```
> zeilpap(GAMMA(k-n)*GAMMA(k+b)/(GAMMA(k+c)*k!),k,n);
```

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let F(n,k) be given by

$$\frac{\Gamma(-n+k)\,\Gamma(k+b)}{\Gamma(k+c)\,k!}$$

and let SUM(n) be the sum of F(n,k) with respect to k .

SUM(n) satisfies the following linear recurrence equation

$$(-n+b-c)\,\mathrm{SUM}(n)-(n+1)\,(n+c)\,\mathrm{SUM}(n+1)$$
$$=0.$$

PROOF: We cleverly construct G(n,k) :=

$$\frac{(k-1+c)\,k\,\Gamma(-n+k)\,\Gamma(k+b)}{(-n-1+k)\,\Gamma(k+c)\,k!}$$

with the motive that

$$(-n+b-c)\,F(n,k)-(n+1)\,(n+c)\,F(n+1,k)$$

= G(n,k+1)-G(n,k) (check!)

and the theorem follows upon summing with respect to k .QED.

Bailey's 4F3. Last identity on Koepf p. 84.

```
> zeilpap(GAMMA(k+a)*GAMMA(k+1+a/2)*GAMMA(k+b)*GAMMA(k-n)/
  (GAMMA(k+a/2)*GAMMA(k+1+a-b)*GAMMA(2+2*b-n)),k,n);
```

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let F(n,k) be given by

$$\frac{\Gamma(k+a)\,\Gamma\left(k+1+\frac{1}{2}a\right)\Gamma(k+b)\,\Gamma(-n+k)}{\Gamma\left(k+\frac{1}{2}a\right)\Gamma(k+1+a-b)\,\Gamma(2+2\,b-n)}$$

and let SUM(n) be the sum of F(n,k) with respect to k .

SUM(n) satisfies the following linear recurrence equation

$$(n-2\,b+1)\,(n-2\,b)\,(n-2\,b-1)\,(2\,n\,a+a^2+6\,a-4)\,\mathrm{SUM}(n)-(n-2\,b+1)\,(n-2\,b)\,(4\,n^2\,a+4\,n\,a^2+2\,n\,a\,b+a^3+a^2\,b+18\,n\,a+9\,a^2+6\,a\,b-8\,n+14\,a-4\,b-16)\,\mathrm{SUM}(n+1)+(n-2\,b+1)\,(2\,n^3\,a+3\,n^2\,a^2+2\,n^2\,a\,b+n\,a^3+3\,n\,a^2\,b+a^3\,b+14\,n^2\,a+14\,n\,a^2+10\,n\,a\,b+2\,a^3+8\,a^2\,b-4\,n^2+26\,n\,a-4\,n\,b+15\,a^2+8\,a\,b-20\,n+8\,a-12\,b-20)\,\mathrm{SUM}(n+2)+(2\,n\,a+a^2+4\,a-4)\,(n+a-b+3)\,\mathrm{SUM}(n+3)$$

$$=0.$$

PROOF: We cleverly construct G(n,k) :=

$$(n-2\,b-1)\,(n^2-4\,n\,b+n+4\,b^2-2\,b)\,(2\,n\,a^2+4\,a^2+a^3-16\,a+2\,k\,a^2+12\,k\,a+4\,n\,k\,a-8\,k-4\,n\,a)\,(k+a-b)\,\Gamma(k+a)\,\Gamma\left(k+1+\frac{1}{2}a\right)\Gamma(k+b)\,\Gamma(-n+k)\Big/\left((-n-1+k)\,(2\,k+a)\,(-n-2+k)\,(-n-3+k)\,\Gamma\left(k+\frac{1}{2}a\right)\Gamma(k+1+a-b)\,\Gamma(2+2\,b-n)\right)$$

with the motive that

$$(n-2\,b+1)\,(n-2\,b)\,(n-2\,b-1)\,(2\,n\,a+a^2+6\,a-4)\,F(n,k)-(n-2\,b+1)$$

$$(n-2\,b)$$
$$(4\,n^2\,a+4\,n\,a^2+2\,n\,a\,b+a^3+a^2\,b+18\,n\,a+9\,a^2+6\,a\,b-8\,n+14\,a-4\,b-16)$$
$$\mathrm{F}(n+1,k)+(n-2\,b+1)\,(2\,n^3\,a+3\,n^2\,a^2+2\,n^2\,a\,b+n\,a^3+3\,n\,a^2\,b+a^3\,b$$
$$+14\,n^2\,a+14\,n\,a^2+10\,n\,a\,b+2\,a^3+8\,a^2\,b-4\,n^2+26\,n\,a-4\,n\,b+15\,a^2+8\,a\,b$$
$$-20\,n+8\,a-12\,b-20)\,\mathrm{F}(n+2,k)$$
$$+(2\,n\,a+a^2+4\,a-4)\,(n+a-b+3)\,\mathrm{F}(n+3,k)$$

= G(n,k+1)-G(n,k) (check!)

and the theorem follows upon summing with respect to k .QED.

```
> AZpapd(1/(1-x)/x^(n+1),x,n);
```

A PROOF OF A LINEAR RECURRENCE SATISFIED BY AN INTEGRAL

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

I will give a short proof of the following result.

Theorem:Let F(n,x) be given by

$$\frac{1}{(1-x)\,x^{(n+1)}}$$

and let INTEGRAL(n) be the integral of F(n,x) with respect to x .

INTEGRAL(n) satisfies the following linear recurrence equation

$$(-n-1)\,\mathrm{INTEGRAL}(n)+(n+1)\,\mathrm{INTEGRAL}(n+1)$$
$$=0.$$

PROOF: We cleverly construct G(n,x) :=

$$\frac{-1+x}{(1-x)\,x^{(n+1)}}$$

with the motive that

$$(-n-1)\,\mathrm{F}(n,x)+(n+1)\,\mathrm{F}(n+1,x)$$

= diff(G(n,x),x)

and the theorem follows upon integrating with respect to x .QED.

```
>
```