/home/m262f99/A=B/Maple/int3.mws

Math 262a, Fall 1999, Glenn Tesler
Creative Telescoping for a triple integral
10/27/99

```
> read TRIPLE_INTEGRAL;
  P:=1:
  F := 1/(1-z*(w+1/w+x+1/x))/(x*w)/z^(n+1);

  ORDER:=3:
  #resh:=[n,ca1,ca2,k]:
  S1:=1/x^2/w^2/z^2:
  S2:=S1:
  S3:=S1:
  resh := [n]:
  gu:=ikar(P,F,n,z,w,x,ORDER,resh,N,S1,S2,S3);
```

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Warning, 'mekh' is implicitly declared local
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$$F := \frac{1}{\left(1 - z\left(w + \frac{1}{w} + x + \frac{1}{x}\right)\right) x\, w\, z^{(n+1)}}$$

We seek a polynomial P(N,n) and rational functions Rx,Rw,Rz of (n,x,w,z), such that
P(N,n) F(n,x,w,z) = (d/dx)(Rx F) + (d/dw) (Rw F) + (d/dz) (Rz F).
Expanding derivatives and dividing by F gives
(P F)/F = (d/dx Rx + d/dw Rw + d/dz Rz) + (Rx * dF/dx + Rw * dF/dw + Rz * dF/dz)
Here the variables are numbered: z, w, x are vars. 1, 2, 3; Rz, Rw, Rx are R1, R2, R3; d/dz Rz is R1Y1; etc.

$$eq\ is, \left(1 + \frac{a1}{z} + \frac{a2}{z^2} + \frac{a3}{z^3}\right) Smol - Y1R1 + R1$$
$$(2\,z\,x\,w^2 + 2\,z\,x + 2\,z\,x^2\,w + 2\,z\,w - n\,x\,w + n\,z\,x\,w^2 + n\,z\,x + n\,z\,x^2\,w + n\,z\,w - x\,w)$$
$$\Big/ \left((-x\,w + z\,x\,w^2 + z\,x + z\,x^2\,w + z\,w)\,z\right) - Y2R2$$
$$+ \frac{R2\,(-x + 2\,z\,x\,w + z\,x^2 + z)}{-x\,w + z\,x\,w^2 + z\,x + z\,x^2\,w + z\,w} - Y3R3 + \frac{R3\,(-w + z\,w^2 + z + 2\,z\,x\,w)}{-x\,w + z\,x\,w^2 + z\,x + z\,x^2\,w + z\,w}$$

No good theorems for the exact form of Rx, Rw, Rz are presently known. You "guess"

the denominators by hand, and form rational functions Sx, Sw, Sz containing the denominators (required) and any numerator factors that you also manage to guess (optional, just to speed things up if you have the knack for it).
This makes the rational functions Rx, Rw, Rz have the form Sx, Sw, Sz times mystery polynomials in x,w,z, whose coefficients are in Q(n). The degrees of these mystery polynomials are all unknown. Also, P is a polynomial in N (whose order is specified by the input variable ORDER, here 3) with unknown coefficients in Q(n). The program gets a lower bound on the degree D to try for the mystery polynomials, and then using the method of undetermined coefficients, lets these polynomials be generic polynomials with all terms of total degree <=D. If this fails, it tries again with D+1, then D+2, etc., until it succeeds or the program is stopped or crashes.

Now it tries various degrees, and cryptically tells us it failed ("0") or succeeded ("1").

$$gu\ is,\ 0$$

$$gu\ is,\ 1$$

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Theorem:

Let G(   z   ,   w   ,   x   ,   n   ) be
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$$\frac{1}{\left(1 - z\left(w + \frac{1}{w} + x + \frac{1}{x}\right)\right) x\, w\, z^{(n+1)}},$$

```
and a(   n   ) be its triple integral w.r.t to   z   w   and   x   .

Let   N   be the forward shift operator in   n   .

The sequence a(   n   ) satisfies the recurrence
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$$(16\,(n+1)^2 - (n+2)^2\,N^2)\ \mathrm{a}(n) = 0.$$

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Proof: It is routinely verifiable that
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$$(16\,(n+1)^2 - (n+2)^2\,N^2)\ \mathrm{G}(z, w, x, n)$$

```
= D_   z   (
```

$$-(16\,z^2\,w^2\,x^2\,n - w^2\,x^2\,n + 4\,z\,x + 24\,z^2\,w^2\,x^2 - 12\,z^2 + 4\,z^2\,x^2 - 2\,w^2\,x^2)$$
$$\mathrm{G}(z, w, x, n) \Big/ (w^2\,x^2\,z)$$

```
        )

+D_   w   (
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$$(-w^2\,x^2\,n + 8\,z^2\,w^2\,x^2\,n - 2\,x\,w\,n + 2\,z\,w\,n + 4\,z^2\,x^2\,n + 8\,z^2\,w\,x\,n + 4\,z\,x^2\,w\,n$$

$$- 4 z^2 n + 2 z x n - 4 x w - 4 z^2 + 8 z x^2 w + 4 z w + 8 z^2 w x - 2 w^2 x^2 + 4 z^2 x^2 + 8 z^2 w^2 x^2) \mathrm{G}(z, w, x, n) / (z^2 x^2 w)$$

+D_ x (

$$(-2 z w n + 8 z^2 w^2 x^2 n - 4 z^2 x^2 n + 2 z x n - 4 z^2 n - w^2 x^2 n - 4 z x^2 w n + 2 x w n + 8 z x - 4 z w + 8 z^2 w^2 x^2 - 4 z^2 x^2 + 4 x w - 4 z^2 - 2 w^2 x^2 - 8 z x^2 w) \mathrm{G}(z, w, x, n) / (z^2 w^2 x)$$

),

and the result follows by triple integrating w.r.t to
z , w and x .

$$gu := 16 (n+1)^2 - (n+2)^2 N^2, -\frac{16 z^2 w^2 x^2 n - w^2 x^2 n + 4 z x + 24 z^2 w^2 x^2 - 12 z^2 + 4 z^2 x^2 - 2 w^2 x^2}{w^2 x^2 z}, (-w^2 x^2 n + 8 z^2 w^2 x^2 n - 2 x w n + 2 z w n + 4 z^2 x^2 n + 8 z^2 w x n + 4 z x^2 w n - 4 z^2 n + 2 z x n - 4 x w - 4 z^2 + 8 z x^2 w + 4 z w + 8 z^2 w x - 2 w^2 x^2 + 4 z^2 x^2 + 8 z^2 w^2 x^2) / (z^2 x^2 w), (-2 z w n + 8 z^2 w^2 x^2 n - 4 z^2 x^2 n + 2 z x n - 4 z^2 n - w^2 x^2 n - 4 z x^2 w n + 2 x w n + 8 z x - 4 z w + 8 z^2 w^2 x^2 - 4 z^2 x^2 + 4 x w - 4 z^2 - 2 w^2 x^2 - 8 z x^2 w) / (z^2 w^2 x), 40.450$$

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