/home/m262f99/KOEPF/worksheetsV.4/gosperdemo.mws

Math 262a, Fall 1999, Glenn Tesler
Gosper Sum demo
10/17/99

```
> restart;
```

This tells Maple to print out extra info as it does the computation

```
> infolevel[sum] := 3;
```

$$infolevel_{sum} := 3$$

```
> sum(k^2,k);
sum/indefnew:   indefinite summation
sum/indefnew:   indefinite summation finished
```

$$\frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k$$

```
> with(sumtools);
```

[*Hypersum*, *Sumtohyper*, *extended_gosper*, *gosper*, *hyperrecursion*, *hypersum*, *hyperterm*, *simpcomb*, *sumrecursion*, *sumtohyper*]

Koepf wrote this routine that comes with Maple V.4, so the notation agrees with his book.

```
> sumtools[gosper](k^2,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    k^2/(k-1)^2
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   k^2
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   3
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   1/6*k*(k+1)*(2*k+1)
```

$$\frac{1}{6}(k-1)\,k\,(2\,k-1)$$

```
> sumtools[gosper](k!,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   k
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   -1
sumtools[gosper]   Gosper's algorithm: no closed form antidifference exists
```

FAIL

```
> sumtools[gosper](k*k!,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    k^2/(k-1)
```

```
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   k
sumtools[gosper]   q:=   k
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   0
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper’s algorithm successful
sumtools[gosper]   f:=   1
```

$$k!$$

```
> sumtools[gosper](binomial(n,k),k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    -(-n+k-1)/k
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   n-k+1
sumtools[gosper]   r:=   k
sumtools[gosper]   degreebound:=   -1
sumtools[gosper]   Gosper’s algorithm: no closed form antidifference exists
```

$$FAIL$$

```
> sumtools[gosper](binomial(n,k)*(-1)^k,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    (-n+k-1)/k
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   -n+k-1
sumtools[gosper]   r:=   k
sumtools[gosper]   degreebound:=   0
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper’s algorithm successful
sumtools[gosper]   f:=   -1/n
```

$$-\frac{k\,\mathrm{binomial}(n,k)\,(-1)^k}{n}$$

```
> sumtools[gosper]((4*k+1)*k!/(2*k+1)!,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    1/2*(4*k+1)/(4*k-3)/(2*k+1)
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   4*k+1
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   4*k+2
sumtools[gosper]   degreebound:=   0
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper’s algorithm successful
sumtools[gosper]   f:=   -1
```

$$-2\,\frac{(2\,k+1)\,k!}{(2\,k+1)!}$$

Attempt the inverse problem: verify that this function at k minus at k-1 gives the summand.

```
> sumk := k -> -k*binomial(n,k)*(-1)^k/n;
```

$$sumk := k \rightarrow -\frac{k\,\mathrm{binomial}(n,k)\,(-1)^k}{n}$$

```
> dsumk := simplify(sumk(k)-sumk(k-1));
```

$$dsumk := -\frac{(-1)^k \,(k \operatorname{binomial}(n,k) + \operatorname{binomial}(n,k-1)\,k - \operatorname{binomial}(n,k-1))}{n}$$

```
> expand(");
```

$$-\frac{k \operatorname{binomial}(n,k)\,(-1)^k}{n} - \frac{(-1)^k\,k^2 \operatorname{binomial}(n,k)}{n\,(n-k+1)} + \frac{(-1)^k\,k \operatorname{binomial}(n,k)}{n\,(n-k+1)}$$

```
> simplify(");
```

$$\frac{k \operatorname{binomial}(n,k)\,(-1)^k}{-n+k-1}$$

```
> combine(");
```

$$\frac{k \operatorname{binomial}(n,k)\,(-1)^k}{-n+k-1}$$

Maple is stubborn, many commands may have to be used to get it to the form we want.

```
> convert(",GAMMA);
```

$$\frac{k\,\Gamma(n+1)\,(-1)^k}{\Gamma(k+1)\,\Gamma(n-k+1)\,(-n+k-1)}$$

```
> combine(");
```

$$\frac{k\,\Gamma(n+1)\,(-1)^k}{\Gamma(k+1)\,\Gamma(n-k+1)\,(-n+k-1)}$$

```
> simplify(");
```

$$-\frac{(-1)^k\,\Gamma(n+1)}{\Gamma(n-k+2)\,\Gamma(k)}$$

```
> convert(",binomial);
```

$$-(-1)^k \operatorname{binomial}(n,k-1)$$

```
=======================================================================

====

>
> sumtools[gosper](k!/k^2,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    1/k*(k-1)^2
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   (k-1)^2
sumtools[gosper]   r:=   k
sumtools[gosper]   degreebound:=   -2
sumtools[gosper]   Gosper's algorithm: no closed form antidifference exists
```

FAIL

```
> sumtools[gosper]((k-3)*k!,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    (k-3)/(k-4)*k
sumtools[gosper]   Gosper's algorithm applicable
```

```
sumtools[gosper]   p:=   k-3
sumtools[gosper]   q:=   k
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   0
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm: no closed form antidifference exists
```

$$FAIL$$

A randomly chosen summand as above will not always be gosper-summable, but if we start with a hypergeometric sum s(k) and compute the difference a(k)=s(k+1)-a(k), then s(k) = const + sum up to k-1 of a(k), so we're guaranteed success.

```
> sk := (k^2+2*k+5)*(2*k+3)!/(3*k+4)!;
```

$$sk := \frac{(k^2 + 2\,k + 5)\,(2\,k + 3)!}{(3\,k + 4)!}$$

```
> ak := simplify(subs(k=k+1,sk)-sk);
```

$$ak := -\frac{1}{3}\,\frac{(158\,k^3 + 430\,k^2 + 678\,k + 445 + 27\,k^4)\,\Gamma(2\,k + 4)}{(3\,k + 7)\,\Gamma(3\,k + 6)}$$

```
> sumtools[gosper](ak,k);
sumtools[gosper]   a(   k   )/a(   k   -1):=    2/3*(2*k+3)*(158*k^3+430*k^2+678*k
+445+27*k^4)/(50*k^3+118*k^2+184*k+66+27*k^4)/(3*k+5)/(3*k+7)
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   158*k^3+430*k^2+678*k+445+27*k^4
sumtools[gosper]   q:=   4*k+6
sumtools[gosper]   r:=   3*(3*k+7)*(3*k+5)
sumtools[gosper]   degreebound:=   2
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -8-4*k-k^2
```

$$\frac{(3\,k + 5)\,(k^2 + 2\,k + 5)\,\Gamma(2\,k + 4)}{\Gamma(3\,k + 6)}$$

```
> sum(ak,k);
sum/indefnew:   indefinite summation
sum/extgosper:   applying Gosper algorithm to a(   k   ):=   -1/3*(158*k^3+430*k^2
+678*k+445+27*k^4)/(3*k+7)/GAMMA(3*k+6)*GAMMA(2*k+4)
sum/gospernew:   a(   k   )/a(   k   -1):=    2/3*(2*k+3)*(158*k^3+430*k^2+678*k+4
45+27*k^4)/(50*k^3+118*k^2+184*k+66+27*k^4)/(3*k+5)/(3*k+7)
sum/gospernew:   Gosper's algorithm applicable
sum/gospernew:   p:=   158*k^3+430*k^2+678*k+445+27*k^4
sum/gospernew:   q:=   4*k+6
sum/gospernew:   r:=   3*(3*k+7)*(3*k+5)
sum/gospernew:   degreebound:=   2
sum/gospernew:   solving equations to find f
sum/gospernew:   Gosper's algorithm successful
sum/gospernew:   f:=   -8-4*k-k^2
sum/indefnew:   indefinite summation finished
```

$$\frac{(3\,k + 5)\,(k^2 + 2\,k + 5)\,\Gamma(2\,k + 4)}{\Gamma(3\,k + 6)}$$

```
> sumtools[gosper](1/(k*(k+1)),k);
```

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    1/(k+1)*(k-1)
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   k-1
sumtools[gosper]   r:=   k+1
sumtools[gosper]   degreebound:=   1
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -1
```

$$\frac{-k-1}{k\,(k+1)}$$

```
> simplify(");
```

$$-\frac{1}{k}$$

q version

```
> read `qsum.mpl`;
```

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Konrad-Zuse-Zentrum Berlin

```
> qgosper((-1)^k * q^binomial(k,2) *
  qbinomial(n,k,q),q,k);
```

$$-\frac{(-1+q^k)\,(-1)^k\,q^{\mathrm{binomial}(k,\,2)}\,\mathrm{qbinomial}(n,k,q)}{q^n-1}$$

```
> qsk := ":
```

This means:

```
> Sum((-1)^t * q^binomial(t,2) *
  qbinomial(n,t,q),t=c..k-1)=qsk;
```

$$\sum_{t=c}^{k-1}(-1)^t\,q^{\mathrm{binomial}(t,\,2)}\,\mathrm{qbinomial}(n,t,q)=$$
$$-\frac{(-1+q^k)\,(-1)^k\,q^{\mathrm{binomial}(k,\,2)}\,\mathrm{qbinomial}(n,k,q)}{q^n-1}$$

Check it: make sure the difference in the supposed sum at k+1 & k is the original summand

```
>
> qd:=qsimplify(subs(k=k+1,qsk)-qsk);
```

$$qd:=\frac{(q^n-1)\,\mathrm{qpochhammer}\left(\frac{1}{q^n},q,k\right)q^{(n\,k)}\,\mathrm{qpochhammer}(q^k,q,n)}{(-1+q^k)\,\mathrm{qpochhammer}(q^n,q,k)\,\mathrm{qpochhammer}(q,q,n)}$$

```
> Fnk := (-1)^k * q^binomial(k,2)*qbinomial(n,k,q);
```

$$Fnk := (-1)^k q^{\mathrm{binomial}(k,2)}\ \mathrm{qbinomial}(n,k,q)$$

```
> qsimplify(");
```

$$\frac{\mathrm{qpochhammer}\left(\frac{1}{q^n}, q, k\right) q^{(n\,k)}}{\mathrm{qpochhammer}(q,q,k)}$$

It represented them differently, so we can’t tell by inspection... so try again.

```
> qsimplify(Fnk-qd);
```

$$0$$

What are the "p,q,r"? Call them "P,Q,R" instead, due to the double use of Q:

```
> trace(qgosper);
```

$$qgosper$$

```
> qgosper(Fnk,q,k);
{--> enter qgosper, args = (-1)^k*q^binomial(k,2)*qbinomial(n,k,q), q, k
```

(other trace information deleted; K stands for q^k)

$$PQR := [1, K - q^n, -1 + q\,K]$$

```
<-- exit qgosper (now at top level) =
-(-1+q^k)/(q^n-1)*(-1)^k*q^binomial(k,2)*qbinomial(n,k,q)}
```

$$-\frac{(-1+q^k)\,(-1)^k q^{\mathrm{binomial}(k,2)}\ \mathrm{qbinomial}(n,k,q)}{q^n - 1}$$

```
> untrace(qgosper);
```

$$qgosper$$

```
>
```

Now try the q-binomial theorem, series form, truncated:

```
> Sum(qpochhammer(a,q,k1)/qpochhammer(q,q,k1)*x^k1,k1=0..k
  -1)=‘?‘;
```

$$\sum_{k1=0}^{k-1} \frac{\mathrm{qpochhammer}(a,q,k1)\,x^{k1}}{\mathrm{qpochhammer}(q,q,k1)} = ?$$

```
> qgosper(qpochhammer(a,q,k)/qpochhammer(q,q,k)*x^k,q,k);
Error, (in qgosper) No q-hypergeometric antidifference exists.
>
```

So we can’t truncate the series. Can we do the infinite sum?

```
> Sum(qpochhammer(a,q,k)/qpochhammer(q,q,k)*x^k,k=0..infin
  ity)=qpochhammer(a*x,q,infinity)/qpochhammer(x,q,infinit
  y)  * ‘?‘;
```

$$\sum_{k=0}^{\infty} \frac{\mathrm{qpochhammer}(a,q,k)\,x^k}{\mathrm{qpochhammer}(q,q,k)} = \frac{\mathrm{qpochhammer}(a\,x,q,\infty)\ ?}{\mathrm{qpochhammer}(x,q,\infty)}$$

```
> qkfreerec(qpochhammer(a,q,k)/qpochhammer(q,q,k)*x^k,q,k,
  n,0,1);
```

$$-a_{0,1}\,F(n,k)+a_{0,1}\,F(1+n,k)=0$$

Oops, there's no n! So try it for the special case a=q^n.

```
>
> qgosper(qpochhammer(q^n,q,k)/qpochhammer(q,q,k)*x^k,q,k)
  ;
Error, (in qgosper) No q-hypergeometric antidifference exists.
```

So we still can't do the indefinite truncated sum. Use Sister Celine's algorithm to do the infinite sum instead.

```
>
> qkfreerec(qpochhammer(q^n,q,k)/qpochhammer(q,q,k),q,k,n,
  1,1);
```

$$-a_{1,1}\,q^n\,F(1+n,k)-a_{1,1}\,F(n,k+1)+a_{1,1}\,F(1+n,k+1)=0$$

```
> qfasenmyer(qpochhammer(q^n,q,k)/qpochhammer(q,q,k)*x^k,q
  ,k,s(n),1,1);
```

$$(x\,q^n-1)\,s(1+n)+s(n)=0$$

```
> rsolve(",s(n));
sum/indefnew:   indefinite summation
sum/extgosper:   applying Gosper algorithm to a(   _n1   ):=   ln(x*q^_n1-1)
sum/gospernew:   a(   _n1   )/a(   _n1   -1):=    -ln(x*q^_n1-1)/(-ln(x*q^_n1-q)+l
n(q))
sum/gospernew:   is not rational
sum/gospernew:   Gosper's algorithm not applicable
sum/indefnew:   indefinite summation finished
```

$$(-1)^n\left(\prod_{\_n1=0}^{n-1}\frac{1}{x\,q^{-n1}-1}\right)s(0)$$

(we never turned off involevel[sum]:=3)
The initial value s(0) has to be computed separately. Completing this is nearly as hard as the "humanoid" proof we gave before.

```
>
```