Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 9 — December 10, 1999

1. **Guessing sequences.** We have considered how to take a given formula and find a closed form for its sum; an ideal of operators that annihilate it; and so forth.

   But what if you don't have a formula? If, by brute force, you determine the first few terms of a sequence are a_1, a_2, a_3, a_4, a_5, say, what can you do?

   (a) Let $f(n)$ be the number of triangulations of an n-sided polygon. Draw n-gons for small values of n, and draw all triangulations by brute force. You discover that $f(3) = 1$, $f(4) = 2$, $f(5) = 5$, $f(6) = 14$, $f(7) = 42$, and then there are too many to continue. Go to

   `http://www.research.att.com/~njas/sequences/index.html`

   and enter in the sequence 1,2,5,14,42. Back comes a list of formulas, articles, etc., that have been catalogued about this sequence. You may find that there are many different interpretations of the sequence (in fact, this sequence is the well-known Catalan numbers). The more terms you provide, the fewer sequences you'll get back, because an infinite number of possible sequences begin this way.

   Next time you encounter a sequence you do not know, play it dumb: submit it here first.

   (b) Or, you could "guess" a recurrence or generating function for these numbers, or a differential equation whose solution has these numbers as the coefficients of its Taylor series, etc. Software to do such things includes

   `gfun` Maple program available at `http://pauillac.inria.fr/algo/libraries/`

   `Rate` Mathematica program available at
   `http://radon.mat.univie.ac.at/People/kratt/rate/rate.html`.

   These and similar things can be found through the $A = B$ homepage as well.

   These guessing algorithms are based on the method of undetermined coefficients. If you have found the initial terms $f(0), \ldots, f(k)$ of a sequence by a brute force method as above, you may "guess" that it satisfies a recurrence

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij} n^i f(n+j) = 0 \qquad (1)$$

   as follows. Let $I = J = 0$. Plugging $n = 0, 1, \ldots, k - J$ into (1). and try to solve the $k - J + 1$ equations for the $(I+1)(J+1)$ unknown coefficients a_{ij}'s. If this fails, increase the values of I and J repeatedly. If you succeed with $(I+1)(J+1) \ll k - J + 1$ (number of unknowns greatly exceeded by the number of equations, so it's a vastly overdetermined system), there is a "good chance" you have found the correct recurrence. With this empirical recurrence so discovered, you can then try to prove it by other means.

   Find a recurrence satisfied by the sequence $f(0), f(1), \ldots$ whose first few terms are 1, 2, 21, 211, 3476, 69864, 1741060, 51222620, 1743410160, 67363596160, 2913873706640, 139508695484400, 7324686442350400, 418478888788169600, 25847105490242145600, 1716199939670189406400, 121907633727244649312000.