**Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler**
Homework 8 answers — November 24, 1999

I will use $AB$ to denote multiplication of operators, and $A * f$ to denote the action of the operator $A$ applied to the function $f$. If $f$ is a function, it may be viewed as the operator of multiplication by that function. So for $D = \frac{d}{dx}$, we have $Df = fD + D * f$.

1. **Factorization of operators.** Consider the operator $L = E^2 - 2E + 1$ in the rational shift algebra $\mathbb{C}(n)[E; E, 0]$. The solutions of $L * f(n) = 0$ are $f(n) = an + b$ for any constants $a, b$.

(a) Find a monic operator of order 1, $B = E + \alpha(n)$, annihilating $an + b$: $B * (an + b) = 0$.
*Answer:* The monic first order annihilator of $f(n) = an + b$ is

$$\boxed{\boldsymbol{B = E - (E * f(n))/f(n) = E - \frac{an + a + b}{an + b}}}.$$

(b) Factorize $L = AB$. Do we have unique factorization?
*Answer:* We could use the division algorithm. Instead, set $A = E + \beta(n)$,

$$L = AB = E^2 + (\alpha(n+1) + \beta(n))E + \alpha(n)\beta(n) = E^2 - 2E + 1,$$

so $\beta(n) = 1/\alpha(n) = -\frac{an+b}{an+a+b}$.
(Indeed, $\alpha(n+1) + \beta(n) = -\frac{an+2a+b}{an+a+b} - \frac{an+b}{an+a+b} = -\frac{2an+2a+2b}{an+a+b} = -2$.)

$$\boxed{\boldsymbol{L = \left(E - \frac{an + b}{an + a + b}\right)\left(E - \frac{an + a + b}{an + b}\right)}}$$

for all constants $a, b$, with the denominators not vanishing, i.e., $a$ and $b$ aren't both 0. Apparently we do not have unique factorization.

(c) In the Weyl algebra, find all factorizations of $D^2 - 2D + 1$.
*Answer:* Let $L = D^2 - 2D + 1$. The solutions of $L * f(x) = 0$ are $ax + b$. The monic annihilator of $ax + b$ is $B = D - \frac{(ax+b)'}{ax+b} = D - \frac{a}{ax+b}$, and we get

$$\boxed{\boldsymbol{L = \left(D - \frac{ax + b}{a}\right)\left(D - \frac{a}{ax + b}\right)}}$$

for all constants $a, b$ with $a \neq 0$.

2. **Ore algebras.** Consider the difference operator whose action is $\Delta * f(n) = f(n+1) - f(n)$. Put the operator $\Delta f(n)$ into normal form ($\Delta$ on right). Use the $\mathbb{K}[\partial; \sigma, \delta]$ notation to express the Ore Algebra of polynomials in $\Delta$ whose coefficients are rational functions of $n$ placed on the left.
*Answer:*

$$\Delta f(n) = (E-1)f(n) = f(n+1)E - f(n) = f(n+1)(1+\Delta) - f(n) = \boxed{\boldsymbol{f(n+1)\Delta + (f(n+1) - f(n))}}$$

or $\Delta f(n) = (E * f(n))\Delta + (\Delta * f(n))$. Thus we are in the Ore algebra $\mathbb{K}[n][\Delta; E, \Delta]$ (or $\mathbb{K}(n)$, etc.).

Now do the same for the $q$-analogue $\Delta_x^{(q)} * f(x) = \frac{f(xq) - f(x)}{x(q-1)}$.
*Answer:* We define the $q$-dilation operator $H_x^{(q)} * f(x) = f(qx)$. Then

$$\Delta_x^{(q)} = (x(1-q))^{-1}(H_x^{(q)} - 1) \quad \text{and} \quad H_x^{(q)} = x(1-q)\Delta_x^{(q)} + 1.$$

So

$$\begin{aligned}
\Delta_x^{(q)} f(x) &= \frac{1}{x(1-q)}\left(H_x^{(q)} - 1\right) f(x) = \frac{1}{x(1-q)}\left(f(qx)H_x^{(q)} - f(x)\right) \\
&= \frac{1}{x(1-q)}\left(f(qx)H_x^{(q)} - f(x)\right) = \frac{1}{x(1-q)}\left(f(qx)(x(1-q)\Delta_x^{(q)} + 1) - f(x)\right) \\
&= \boxed{\boldsymbol{f(qx)\Delta_x^{(q)} + \frac{f(qx) - f(x)}{x(1-q)}}} = (H_x^{(q)} * f(x))\Delta_x^{(q)} + (\Delta_x^{(q)} * f(x))
\end{aligned}$$

and we are in the Ore algebra $\mathbb{K}(q,x)[\Delta_x^{(q)}; H_x^{(q)}, \Delta_x^{(q)}]$.

3. **"D"-finite functions.** If $(nE-1)f(n)=0$ and $(E^2-n)g(n)=0$, find a homogeneous recurrence equation with $\mathbb{Q}[n]$ coefficients satisfied by $h(n)=f(n)g(n)-g(n)$. (Certain initial conditions may allow smaller recurrences, but we're not concerned with that: this single recurrence should hold for all possible $f(n), g(n)$ satisfing the given equations.)
*Answer:* See the maple worksheet.

4. **Gröbner bases.**
   (a) Let $f(x,y,z)=2x+3y+4z+5x^2+6xy+7z^3$. Write $f$ with terms in decreasing order; LT($f$); LC($f$); LM($f$); and multideg($f$), for each of these orders: lex order with $x>y>z$; lex order with $z>y>x$; and grlex order with $x>y>z$.
   *Answer:*

| order | $f$ in that order | LT($f$) | LC($f$) | LM($f$) | multideg($f$) |
|---|---|---|---|---|---|
| lex order, $x>y>z$ | $5x^2+6xy+2x+3y+7z^3+4z$ | $5x^2$ | 5 | $x^2$ | $(2,0,0)$ |
| lex order, $z>y>x$ | $7z^3+4z+6xy+3y+5x^2+2x$ | $7z^3$ | 7 | $z^3$ | $(0,0,3)$ |
| grlex order, $x>y>z$ | $7z^3+5x^2+6xy+2x+3y+4z$ | $7z^3$ | 7 | $z^3$ | $(0,0,3)$ |

   (b) Let $f=x^3-x^2y-x^2z+x$, $f_1=x^2y-z$, $f_2=xy-1$.
   (i) In grlex order with $x>y>z$, compute

$$r_1 = \text{remainder of } f \text{ on division by } (f_1,f_2);$$
$$r_2 = \text{remainder of } f \text{ on division by } (f_2,f_1).$$

*Answer:* First compute $r_1$:
$x^3-x^2y-x^2z+x$: $x^3$ goes into the remainder; $r=x^3$.
$-x^2y-x^2z+x$: the lead term $=-\mathrm{LT}(f_1)$ so $q_1=-1$ and we subtract $-f_1$.
$-x^2z+x-z$: No term is divisible by LT($f_1$) or LT($f_2$), so all remaining terms are put in the remainder, giving $r=x^3-x^2z+x-z$.
Thus $f=q_1f_1+q_2f_2+r_1$ with $q_1=-1$, $q_2=0$, $r_1=\boxed{\boldsymbol{x^3-x^2z+x-z}}$.

Compute $r_2$:
$x^3-x^2y-x^2z+x$: $x^3$ goes into the remainder; $r=x^3$.
$-x^2y-x^2z+x$: the lead term $=-x\,\mathrm{LT}(f_2)$ so $q_2=-x$ and we subtract $-xf_2$.
$-x^2z$: Neither lead term goes into this, so it is contributed to the remainder.
This division gives $f=q_1'f_1+q_2'f_2+r_2$ with $q_1'=0$, $q_2'=-x$, $r_2=\boxed{\boldsymbol{x^3-x^2z}}$.

   (ii) Is $r=r_1-r_2$ in the ideal $\langle f_1,f_2\rangle$? If so, find an explicit expression $r=Af_1+Bf_2$; if not, say why not.
*Answer:* $r=r_1-r_2=x-z$ is in the ideal because $r_1-r_2=(q_1'f_1+q_2'f_2)-(q_1f_1+q_2f_2)=(q_1'-q_1)f_1+(q_2'-q_2)f_2=\boxed{\boldsymbol{(1)f_1+(-x)f_2}}$.

   (iii) Compute the remainder of $r$ on division by $(f_1,f_2)$. Why could you have predicted the answer in advance?
*Answer:* No term of $r_1$ or $r_2$ is divisible by LT($f_1$) or LT($f_2$), so no term of $r_1-r_2$ is either, and thus all terms of this will be contributed to the remainder. So $\overline{x-z}^{(f_1,f_2)}=x-z$.

   (iv) Does the division algorithm give us a solution for the "ideal membership problem" for the ideal $\langle f_1,f_2\rangle$?
*Answer:* No, because $r$ is in the ideal, but $\overline{r}^{(f_1,f_2)}\neq 0$.