Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 8 — November 24, 1999

I will use AB to denote multiplication of operators, and $A * f$ to denote the action of the operator A applied to the function f. If f is a function, it may be viewed as the operator of multiplication by that function. So for $D = \frac{d}{dx}$, we have $Df = fD + D * f$.

1. **Factorization of operators.** Consider the operator $L = E^2 - 2E + 1$ in the rational shift algebra $\mathbb{C}(n)[E; E, 0]$. The solutions of $L * f(n) = 0$ are $f(n) = an + b$ for any constants a, b.
   (a) Find a monic operator of order 1, $B = E + \alpha(n)$, annihilating $an + b$: $B * (an + b) = 0$.

   (b) Factorize $L = AB$. Do we have unique factorization?

   (c) In the Weyl algebra, find all factorizations of $D^2 - 2D + 1$.
   Some further information on factoring is in Koepf's book, and in
   M. Bronshteĭn and M. Petkovshek, *Ore rings, linear operators and factorization*, Programmirovanie **1994**, no. 1, 27–44.

2. **Ore algebras.** Consider the difference operator whose action is $\Delta * f(n) = f(n+1) - f(n)$. Put the operator $\Delta f(n)$ into normal form (Δ on right). Use the $\mathbb{K}[\partial; \sigma, \delta]$ notation to express the Ore Algebra of polynomials in Δ whose coefficients are rational functions of n placed on the left.
   Now do the same for the q-analogue $\Delta_x^{(q)} * f(x) = \frac{f(xq) - f(x)}{x(q-1)}$.

3. **"D"-finite functions.** If $(nE - 1)f(n) = 0$ and $(E^2 - n)g(n) = 0$, find a homogeneous recurrence equation with $\mathbb{Q}[n]$ coefficients satisfied by $h(n) = f(n)g(n) - g(n)$. (Certain initial conditions may allow smaller recurrences, but we're not concerned with that: this single recurrence should hold for all possible $f(n), g(n)$ satisfing the given equations.)

4. **Gröbner bases.** These problems and further information about Gröbner bases in the commutative case can be found in
   D. Cox, J. Little and D. O'Shea, *Ideals, varieties, and algorithms*, Second edition, Springer, New York, 1997.
   (a) Let $f(x, y, z) = 2x + 3y + 4z + 5x^2 + 6xy + 7z^3$. Write f with terms in decreasing order; $\mathrm{LT}(f)$; $\mathrm{LC}(f)$; $\mathrm{LM}(f)$; and $\mathrm{multideg}(f)$, for each of these orders: lex order with $x > y > z$; lex order with $z > y > x$; and grlex order with $x > y > z$.

   (b) Let $f = x^3 - x^2y - x^2z + x$, $f_1 = x^2y - z$, $f_2 = xy - 1$.
      (i) In grlex order with $x > y > z$, compute
      $r_1 =$ remainder of f on division by (f_1, f_2);
      $r_2 =$ remainder of f on division by (f_2, f_1).

      (ii) Is $r = r_1 - r_2$ in the ideal $\langle f_1, f_2 \rangle$? If so, find an explicit expression $r = Af_1 + Bf_2$; if not, say why not.

      (iii) Compute the remainder of r on division by (f_1, f_2). Why could you have predicted the answer in advance?

      (iv) Does the division algorithm give us a solution for the "ideal membership problem" for the ideal $\langle f_1, f_2 \rangle$?