Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 7 answers — November 19, 1999

1–2. See Maple worksheet.

3. (a) Let $D_1 = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$ and $D_2 = \frac{(-1)^{n(n-1)/2}}{a_n} \operatorname{Res}(f, f', x)$.

View D_1 and D_2 as polynomials in the roots α_i of f and β_j of f'. Let the degree of each α_i and β_j be 1, and the degree of a_n be 0. Then D_1 has degree $2\binom{n}{2} = n(n-1)$ because it's a product indexed by pairs $1 \le i < j \le n$, each factor having degree 2. The resultant $\operatorname{Res}(f, f', x)$ is the product of $n(n-1)$ factors $\alpha_i - \beta_j$, each with degree 1. So D_1 and D_2 have the same degree.

The product $\prod_{i<j} (\alpha_i - \alpha_j)^2$ is a symmetric function of the α's, homogeneous of degree $2\binom{n}{2} = n(n-1)$, so it may be expressed as a polynomial in $e_k(\alpha_1, \ldots, \alpha_n) = -a_{n-k}/a_n$ for $k = 1, \ldots, n$. Thus, D_1 is a polynomial in the a_k's divided by some power of a_n, and it vanishes iff $f(x)$ has a repeated root.

The second definition using the resultant also is a polynomial in the a_k's, divided by a_n; but the bottom row of the Sylvester matrix is divisible by a_n, leaving just a polynomial in a_n. The resultant vanishes iff $f(x)$ and $f'(x)$ have a common root iff $f(x)$ has a repeated root, so the same applies to D_2.

Since D_1 and D_2 have the same degrees and vanishing conditions, $D_2 = p \cdot D_1$ for a function p of the a's that never vanishes (provided $a_n \neq 0$). So $p = C{a_n}^k$ for suitable constants C and k. Suppose $\hat{f}$ is monic and $f = a_n \hat{f}$. Let $\hat{D}_1$ and $\hat{D}_2$ be the D_1 and D_2 for $\hat{f}$. Then $\operatorname{Res}(f, f', x) = {a_n}^{2n-1} \operatorname{Res}(\hat{f}, \hat{f}', x)$ and $D_2 = a_n^{2n-2} \hat{D}_2$, and we see that D_1 scales this way too. So we chose the correct exponent, and $D_2 = C\, D_1$ for a constant C, which turns out to be as above. [INCOMPLETE]

(b) The resultant is

$$\operatorname{Res}(ax^2 + bx + c, 2ax + b, x) = \det \begin{bmatrix} 2a & 0 & a \\ b & 2a & b \\ 0 & b & c \end{bmatrix} = -a(b^2 - 4ac)$$

so the discriminant is $b^2 - 4ac$.

4. (a) In Koepf # 5.1 it was shown that for $n^{\underline{k}} = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$ and $\Delta = \Delta_n$, $\Delta n^{\underline{k}} = k \cdot n^{\underline{k-1}}$.

Iterating, $\Delta^r n^{\underline{k}} = k^{\underline{r}} \cdot n^{\underline{k-r}}$. Note that $n^{\underline{r-1}} = 0$ when $r > n$. Thus

$$\Delta^r n^{\underline{k}} \Big|_{n=0} = \begin{cases} 0 & \text{if } r \neq k; \\ k! & \text{if } r = k. \end{cases}$$

The functions $\left\{ n^{\underline{0}}, \ldots, n^{\underline{D}} \right\}$ span all polynomials of degrees $\le D$, so we may represent any polynomial in the form

$$f(n) = \sum_{k=0}^{\infty} b_k\, n^{\underline{k}}$$

for suitable constants b_k, only finitely many of which are nonzero. Apply Δ^r and set $n = 0$ to get $(\Delta^k f)(0) = k!\, b_k$. Then a more convenient way to write $f(n)$ is

$$f(n) = \sum_{k=0}^{\infty} a_k \frac{n^{\underline{k}}}{k!} \quad \text{where} \quad a_k = (\Delta^k f)(0)\ .$$

To compute the given sum, let $f(n) = 5n^3 + 4n^2$. We have

k	$\Delta^k f(n)$	$(\Delta^k f)(0)$
0	$5n^3 + 4n^2$	0
1	$15n^2 + 23n + 9$	9
2	$30n + 38$	38
3	30	30
$k \geq 4$	0	0

so

$$f(n) = \frac{9}{1!}n^{\underline{1}} + \frac{38}{2!}n^{\underline{2}} + \frac{30}{3!}n^{\underline{3}}$$

and an antidifference $F(n)$ s.t. $\Delta F(n) = f(n)$ is

$$F(n) = \frac{9}{2!}n^{\underline{2}} + \frac{38}{3!}n^{\underline{3}} + \frac{30}{4!}n^{\underline{4}} .$$

Thus

$$\begin{aligned}\sum_{n=0}^{m} 5n^3 + 4n^2 &= F(m+1) - F(0) \\ &= \frac{5}{4}(m+1)^4 - \frac{7}{6}(m+1)^3 - \frac{3}{4}(m+1)^2 + \frac{2}{3}(m+1) \\ &= \frac{5}{4}m^4 + \frac{23}{6}m^3 + \frac{13}{4}m^2 + \frac{2}{3}m\end{aligned}$$

(b) Let $\mathbf{b}_k(n) = \Gamma(n)/\Gamma(n+k+\alpha)$.

First evaluate Δ and n applied to this basis:

$$\Delta\mathbf{b}_k(n) = \mathbf{b}_k(n) \cdot \left(\frac{n}{n+k+\alpha} - 1\right) = \mathbf{b}_k(n)\left(1 - \frac{k+\alpha}{n+k+\alpha} - 1\right) = -(k+\alpha)\mathbf{b}_{k+1}(n)$$

$$n\mathbf{b}_k(n) = \frac{n}{n+k+\alpha-1}\mathbf{b}_{k-1}(n) = \mathbf{b}_{k-1}(n) - (k+\alpha-1)\mathbf{b}_k(n)$$

Combine these to get

$$\begin{aligned}n\Delta\mathbf{b}_k(n) &= -(k+\alpha)n\mathbf{b}_{k+1}(n) \\ &= -(k+\alpha)\left(\mathbf{b}_k(n) - (k+1+\alpha-1)\mathbf{b}_{k+1}(n)\right) \\ &= -(k+\alpha)\mathbf{b}_k(n) + (k+\alpha)^2\mathbf{b}_{k+1}(n)\end{aligned}$$

Set

$$f(n) = \sum_{k=0}^{\infty} c_k\mathbf{b}_k(n)$$

and plug this into $n\Delta f(n) = f(n)$ to obtain

$$\sum_{k=0}^{\infty} c_k\left(-(k+\alpha)\mathbf{b}_k(n) + (k+\alpha)^2\mathbf{b}_{k+1}(n)\right) = \sum_{k=0}^{\infty} c_k\mathbf{b}_k(n) .$$

Collecting coefficients in terms of this basis gives

$$0 = \sum_k \left(-(k+\alpha)c_k + (k-1+\alpha)^2 c_{k-1} - c_k\right)\mathbf{b}_k(n) = \sum_k \left(-(k+\alpha+1)c_k + (k+\alpha-1)^2 c_{k-1}\right)\mathbf{b}_k(n)$$

where we sum over all k with the understanding that $c_k = 0$ for $k \leq 0$. This gives the recursion

$$-(k+\alpha+1)c_k + (k+\alpha-1)^2 c_{k-1} = 0 . \tag{1}$$

Now get the indicial equation: at $k = 0$ this becomes

$$0 = -(0+\alpha+1)c_0 + (0+\alpha-1)^2 c_{-1} = -(\alpha+1)c_0 + (\alpha-1)^2 \cdot 0 = -(\alpha+1)c_0$$

where $c_{-1} = 0$ and $c_0 \neq 0$; thus, $\alpha = -1$ is the root of the indicial equation. So the final recursion for the c's is

$$-k\, c_k + (k-2)^2 c_{k-1} = 0 \qquad \text{so} \qquad c_k = \frac{(k-2)^2}{k} c_{k-1} \quad \text{for } k \geq 1.$$

Iterating gives $c_1 = c_0$, $c_2 = c_3 = \cdots = 0$. So the final answer is

$$f(n) = c_0(\mathbf{b}_0(n) + \mathbf{b}_1(n)) = c_0 \left(\frac{\Gamma(n)}{\Gamma(n-1+0)} + \frac{\Gamma(n)}{\Gamma(n-1+1)} \right) = c_0\,((n-1)+(1)) = \boxed{\boldsymbol{c_0\, n}}\ .$$

Alternately, rewrite the original equation as

$$n(f(n+1) - f(n)) = f(n) \quad \Rightarrow f(n+1) = \frac{n+1}{n} f(n) \quad \Rightarrow \boxed{\boldsymbol{f(n) = n\, f(1)}}\ .$$