Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 7 — November 19, 1999

1. Suppose that $f(n)$, $g(n)$ are nonzero solutions to
$$f(n+2) - (n+3)f(n+1) + 2n\,f(n) = 0 \tag{1}$$
$$g(n+2) - (2n+1)g(n+1) + n^2\,g(n) = 0\ . \tag{2}$$
   (a) Find a single nontrivial recurrence of which both $f(n)$ and $g(n)$ are solutions.
   (b) If $f(n) = g(n)$ for all n, then what is $f(n)$?
   (c) What specific finite amount of additional data needs to be supplied to determine that $f(n) = g(n)$?
   (d) Express the above recurrences in $f(n)$, $g(n)$ in operator notation, with the operators fully factorized.

2. You have learned two different techniques applicable to the following problems.
   (a) Find a nonzero homogeneous differential equation
$$\sum_{i=0}^{I} p_i(x) y^{(i)}(x) = 0$$
   of minimal order whose solutions are spanned by $\sin(x)$ and x. Now do the same if the $p_i(x)$'s must be polynomials in x. differential equation must be polynomials in x.
   (b) Find a nonzero homogeneous recursion whose solutions are spanned by $n!$ and the Fibonacci numbers: first the minimal equation, then the minimal equation whose coefficients are polynomials in n.

3. The *discriminant* of a polynomial
$$f(x) = a_n x^n + \cdots + a_0 = a_n(x-\alpha_1)\cdots(x-\alpha_n)$$
(where for a field $\mathbb{K}$ and its algebraic closure $\overline{\mathbb{K}}$, $a_i \in \mathbb{K}$ and $\alpha_i \in \overline{\mathbb{K}}$) is defined as
$$\mathrm{disc}(f,x) = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 = \frac{(-1)^{n(n-1)/2}}{a_n} \mathrm{Res}(f, f', x)$$
The first definition implies that the discriminant vanishes iff $f(x)$ has a repeated root over the extension field $\overline{\mathbb{K}}$, while the second implies that the discriminant is simply a polynomial in the coefficients of $f(x)$ in $\mathbb{K}$.
   (a) Prove these definitions are the same.
   (b) Compute $\mathrm{disc}(ax^2 + bx + c, x)$.

Turn the page for more questions.

4. **Further methods of solving recurrence equations.** Many of the methods you learned for solving differential equations have counterparts for recurrence equations. Last week's homework had reduction of order. Now we do series solutions.

(a) Review Koepf # 5.1. Then show that any polynomial $f(n)$ can be given by a variation of Taylor/Frobenius series:

$$f(n) = \sum_{k=0}^{\infty} a_k \frac{n^{\underline{k}}}{k!} \quad \text{where} \quad a_k = (\Delta^k f)(0)$$

and use this to compute

$$\sum_{n=0}^{m} (5n^3 + 4n^2).$$

(b) Solve

$$n\Delta f(n) = f(n) \tag{3}$$

by the following series method.

(i) Form a basis $\mathbf{b}_k(n) = \Gamma(n)/\Gamma(n+k+\alpha)$. Plug $f(n) = \sum_{k=0}^{\infty} c_k \mathbf{b}_k(n)$ into (3) and express both sides in terms of this basis with the coefficients free of n. Collect terms with respect to this basis and find a recurrence for the c_k's. This is just like plugging into a generic Taylor series with basis $\mathbf{b}'_k(x) = x^k$, reindexing if necessary to collect powers of x, and finding a recursion for the coefficients.

(ii) The coefficients $c_k = 0$ for negative integers k, while $c_0 \neq 0$ is the first nonzero term. This condition gives the *indicial equation*, used to solve for α. Do this, solve for the c_k's, and give the final value of $f(n)$.

From our perspective, we are solving a recurrence by getting another recurrence for the coefficients, which may seem circular. But see the two references from last week's homework for a fuller description of the series method; one use of this is to use just the first few terms of this series to determine the asymptotics of the solutions.