2. Koepf # 9.11. We do the order d case:

$$\sum_{i=0}^{d} P_i(n)s(n+i) = 0$$

We found that the choices for $q(n)$ are the monic factors of $P_0(n-1)$ and the choices for $r(n)$ are the monic factors of $P_J(n-J)$. In the worst case that there are no repeated roots, this gives $2^{\deg P_0}$ choices for $q(n)$, and $2^{\deg P_J}$ choices for $r(n)$.

For each pair (q,r) we form a degree d polynomial in C (I called it z in class) based on the coefficients of the highest power of n. There are then up to d choices of C.

So in total there can be up to $d \cdot 2^{\deg P_0 + \deg P_J}$ triples $(q(n), r(n), C)$, resulting in that number of calls to Poly (Algorithm 9.1). This is as much as the question asked.

The whole story is much worse ... the calls to Poly will take varying amounts of time depending on the degrees of the recursions formed for each (q,r,C). Also, if the roots of P_0, P_J, or the polynomial in C, are not rational or simple complex numbers, the computations either have to be approximate, or be carried out using an extension field, represented, for example, as $\mathbb{Q}[x]/(x^5+x+1)$, a 5-dimensional vector space, and this will make the computations worse still.

3(a). The given equation is

$$\sum_{i=0}^{d} p_i(n)f(n+i) = 0\ .$$

Since $f(n) = u(n)$ and $f(n) = u(n)v(n)$ are both solutions,

$$\sum_{i=0}^{d} p_i(n)u(n+i) = 0 \qquad (1)$$

$$\sum_{i=0}^{d} p_i(n)u(n+i)v(n+i) = 0 \qquad (2)$$

Multiply (1) by $v(n)$:

$$\sum_{i=0}^{d} p_i(n)u(n+i)v(n) = 0 \qquad (3)$$

Now

$$v(n+i) - v(n) = \left[(1+\Delta)^i - 1\right] v(n) = \sum_{j=1}^{i} \binom{i}{j} \Delta^j v(n). \qquad (4)$$

(Notice the Δ^0 vanishes.) Subtracting (3) from (2) and plugging in (4) gives

$$\sum_{i=0}^{d} p_i(n)u(n+i) \sum_{j=1}^{i} \binom{i}{j} \Delta^j v(n) = 0$$

or

$$\sum_{j=1}^{d} q_j(n)\Delta^j v(n) = 0$$

where

$$q_j(n) = \sum_{i=j}^{d} \binom{i}{j} p_i(n)u(n+i)$$

and setting $w(n) = \Delta v(n)$ and shifting j by 1 gives the recursion

$$\boxed{\sum_{j=0}^{d-1} \boldsymbol{q_{j+1}(n)\ \Delta^j w(n) = 0}} \qquad (5)$$

for $w(n)$.

Suppose a solution $w(n)$ has been found. Since $v(n) = \Delta w(n)$, we use any given initial value $v(a)$ to compute $v(n) = v(a) + \sum_{k=a}^{n-1} w(k)$, and then multiply by $u(n)$ to get a solution $v(a)u(n) + u(n)\sum_{k=a}^{n-1} w(k)$. But $v(a)u(n)$ is a constant multiple of the known solution $u(n)$, and linear combinations of solutions are solutions, so we may drop this term.