Math 262a, Fall 1999, Glenn Tesler
Homework 4
Alternate solution of Koepf 7.21 using
Zeilberger's EKHAD

Koepf # 7.21

For nonnegative integer n, prove
```
> Sum(binomial(n-k,k)*x^k,k=0..floor(n/2))=Sum(binomial(n+1,2*k+1)*(
  1+4*x)^k / 2^n,k=0..floor(n/2));
```

$$\sum_{k=0}^{\text{floor}(1/2\,n)} \text{binomial}(n-k,\,k)\,x^k = \sum_{k=0}^{\text{floor}(1/2\,n)} \frac{\text{binomial}(n+1,\,2\,k+1)\,(1+4\,x)^k}{2^n}$$

The bounds on both sides are the natural bounds, so we may sum k=-infinity..infinity.
We will show that both sides satisfy the same recursion and initial conditions.
(It is NOT necessary that the algorithm discover the same recursion for both sides, even if they are equal; later we'll learn about noncommutative LCM's and GCD's and the Euclidean algorithm for dealing with that.)
```
> read EKHAD;
```

Version of Feb 25, 1999

This version is much faster than previous versions, thanks to

a remark of Frederic Chyzak. We thank him SO MUCH!

The penultimate version, Feb. 1997,

corrected a subtle bug discovered by Helmut Prodinger

Previous versions benefited from comments by Paula Cohen,

Lyle Ramshaw, and Bob Sulanke.

This is EKHAD, One of the Maple packages

accompanying the book

"A=B"

(published by A.K. Peters, Welesley, 1996)

by Marko Petkovsek, Herb Wilf, and Doron Zeilberger.

The most current version is available on WWW at:

http://www.math.temple.edu/~zeilberg .

Information about the book, and how to order it, can be found in

http://www.central.cis.upenn.edu/~wilf/AeqB.html .

Please report all bugs to: zeilberg@math.temple.edu .

All bugs or other comments used will be acknowledged in future

```
> F1 := (n,k)->binomial(n-k,k)*x^k;
  zeilpap(F1(n,k),k,n);
```

$$F1 := (n, k) \rightarrow \text{binomial}(n - k, k)\, x^k$$

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let   F(n,k)   be given by

$$\text{binomial}(n - k, k)\, x^k$$

and let   SUM(n)   be the sum of   F(n,k)   with
respect to   k   .

SUM(n)   satisfies the following linear recurrence equation

$$-x\,\text{SUM}(n) - \text{SUM}(n + 1) + \text{SUM}(n + 2)$$
$$=0.$$

PROOF: We cleverly construct   G(n,k)   :=

$$\frac{(-n - 1 + k)\, k\, \text{binomial}(n - k, k)\, x^k}{(2\,k - n - 1)\,(-n - 2 + 2\,k)}$$

with the motive that

$$-x\,\text{F}(n, k) - \text{F}(n + 1, k) + \text{F}(n + 2, k)$$
=   G(n,k+1)-G(n,k)   (check!)

and the theorem follows upon summing with respect to   k   .QED.

```
> F2 := (n,k) -> binomial(n+1,2*k+1)*(1+4*x)^k / 2^n;
  zeilpap(F2(n,k),k,n);
```

$$F2 := (n, k) \rightarrow \frac{\text{binomial}(n + 1, 2\,k + 1)\,(1 + 4\,x)^k}{2^n}$$

A PROOF OF A RECURRENCE

By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu

Theorem:Let   F(n,k)   be given by

$$\frac{\text{binomial}(n + 1, 2\,k + 1)\,(1 + 4\,x)^k}{2^n}$$

and let   SUM(n)   be the sum of   F(n,k)   with
respect to   k   .

SUM(n)   satisfies the following linear recurrence equation

$$x\,\text{SUM}(n) + \text{SUM}(n + 1) - \text{SUM}(n + 2)$$
$$=0.$$

```
PROOF: We cleverly construct   G(n,k)   :=
```

$$\frac{1}{2}\,\frac{k\,(2\,k+1)\,\text{binomial}(n+1,\,2\,k+1)\,(1+4\,x)^{k}}{(2\,k-n-1)\,(-n-2+2\,k)\,2^{n}}$$

```
with the motive that
```

$$x\,\mathrm{F}(n,\,k)+\mathrm{F}(n+1,\,k)-\mathrm{F}(n+2,\,k)$$

```
=   G(n,k+1)-G(n,k)    (check!)

and the theorem follows upon summing with respect to   k   .QED.
```

Notice both sides satisfy the same recursion of order 2! The coefficient of SUM(n+2) is 1, so there are no singularities in this equation. Thus, as long as both sums agree for n=0 and n=1 (i.e., the first 2 values), iterating the recursion will make all future values equal as well. Here we check f1(0),...,f1(10) to demonstrate this (but to prove it, only f1(0), f1(1) are needed):

```
> f1 := n -> expand(sum(F1(n,k),k=0..n/2));
               # maple truncates n/2 to floor(n/2) on its own
  'f1(nn)'$'nn'=0..10;
```

$$f1 := n \rightarrow \text{expand}\!\left(\sum_{k=0}^{1/2\,n} \mathrm{F}1(n,\,k)\right)$$

$1,\,1,\,1+x,\,1+2\,x,\,1+3\,x+x^{2},\,1+4\,x+3\,x^{2},\,1+5\,x+6\,x^{2}+x^{3},\,1+6\,x+10\,x^{2}+4\,x^{3},$

$\quad 1+7\,x+15\,x^{2}+10\,x^{3}+x^{4},\,1+8\,x+21\,x^{2}+20\,x^{3}+5\,x^{4},\,1+9\,x+28\,x^{2}+35\,x^{3}+15\,x^{4}+x^{5}$

And the same for f2:

```
> f2 := n -> expand(sum(F2(n,k),k=0..n/2));
  'f2(nn)'$'nn'=0..10;
```

$$f2 := n \rightarrow \text{expand}\!\left(\sum_{k=0}^{1/2\,n} \mathrm{F}2(n,\,k)\right)$$

$1,\,1,\,1+x,\,1+2\,x,\,1+3\,x+x^{2},\,1+4\,x+3\,x^{2},\,1+5\,x+6\,x^{2}+x^{3},\,1+6\,x+10\,x^{2}+4\,x^{3},$

$\quad 1+7\,x+15\,x^{2}+10\,x^{3}+x^{4},\,1+8\,x+21\,x^{2}+20\,x^{3}+5\,x^{4},\,1+9\,x+28\,x^{2}+35\,x^{3}+15\,x^{4}+x^{5}$

And we can have them compared directly, too:

```
> 'expand(f1(nn)-f2(nn))'$nn=0..10;
```

$$0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0$$

```
> 
```