/home/m262f99/KOEPF/worksheetsV.4/hw3ansm.mws

Math 262a, Fall 1999, Glenn Tesler
Homework 3

```
> read `hsum.mpl`;
```

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Koepf # 5.8(a)

```
> gosper(1/(k*(k+10)),k);
```

$$-\frac{1}{10}(2053152\,k+362880+3518100\,k^2+2894720\,k^3+1346625\,k^4+379638\,k^5+66150\,k^6+6960\,k^7+405\,k^8+10\,k^9)/((k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)\,k)$$

Check it:

```
> sk := ": simplify(subs(k=k+1,sk)-sk);
```

$$\frac{1}{k\,(k+10)}$$

Conclusion: as an indefinite sum,

```
> Sum(1/(k*(k+10)),k)=sk+C;
```

$$\sum_{k}\frac{1}{k\,(k+10)}=-\frac{1}{10}(2053152\,k+362880+3518100\,k^2+2894720\,k^3+1346625\,k^4+379638\,k^5+66150\,k^6+6960\,k^7+405\,k^8+10\,k^9)/((k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)\,k)+C$$

and as a definite sum,

```
> Sum(1/(k*(k+10)),k=m..n-1) = subs(k=n,sk)-subs(k=m,sk);
```

$$\sum_{k=m}^{n-1}\frac{1}{k\,(k+10)}=-\frac{1}{10}(2053152\,n+362880+3518100\,n^2+2894720\,n^3+1346625\,n^4+379638\,n^5+66150\,n^6+6960\,n^7+405\,n^8+10\,n^9)/((n+9)(n+8)(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)\,n)+\frac{1}{10}(2053152\,m+362880+3518100\,m^2+2894720\,m^3+1346625\,m^4+379638\,m^5+66150\,m^6+6960\,m^7+405\,m^8+10\,m^9)/((m+9)(m+8)(m+7)(m+6)(m+5)(m+4)(m+3)(m+2)(m+1)\,m)$$

Koepf #5.8(d)

```
> gosper(2^k*(k^3-3*k^2-3*k-1)/(k^3*(k+1)^3),k);
```

$$\frac{2^k}{k^3}$$

Koepf #5.8(i)

```
> gosper(binomial(m,k)/binomial(n,k),k);
```

$$\frac{(-n+k-1)\ \mathrm{binomial}(m,k)}{(n+1-m)\ \mathrm{binomial}(n,k)}$$

```
>
```

Koepf #5.13

```
> # Rk2ak(Rk,k) gives the term ak from the certificate Rk,
  using initial term a(0)
  # Rk2ak(Rk,k,false) gives a product formula for it
  instead of a closed form
  # Rk2ak(Rk,k,i) uses initial term a(i)
  # Rk2ak(Rk,k,true/false,i) does both
  Rk2ak := proc(Rk,k)
    local rat,ak,i,cflag,arg;
    i := 0; cflag := true;
    for arg in args[3..nargs] do
      if type(arg,boolean) then cflag := arg
      else i := arg
      fi
    od;

    # term ratio a(k+1)/a(k) = (1+R(k))/R(k+1)
    rat := simplify((1+Rk) / subs(k=k+1,Rk));
    rat := subs(k=n,rat);

    if cflag then
        ak := a(i) * product(rat,n=i..k-1);
    else
        ak := a(i) * Product(rat,n=i..k-1);
    fi;
  end:
> Rk2ak(3,k);
```

$$a(0)\left(\frac{4}{3}\right)^k$$

Koepf #5.13(a)

```
> Rk2ak(alpha/(alpha-1),k);
```

$$a(0)\left(\frac{2\,\alpha-1}{\alpha}\right)^k$$

Koepf #5.13(b)

```
> Rk2ak(k,k);
```

$$a(0)$$

Koepf #5.13(c)

```
> Rk2ak(k^2,k);
```

$$\frac{a(0)\,\Gamma(k-I)\,\Gamma(k+I)}{\Gamma(k+1)^2\,\Gamma(-I)\,\Gamma(I)}$$

```
> Rk2ak(k^2,k,false);
```

$$a(0)\left(\prod_{n=0}^{k-1}\frac{1+n^2}{(n+1)^2}\right)$$

Koepf #5.13(d)

```
> Rk2ak(1/k,k);
```

$$a(0)\left(\prod_{n=0}^{k-1}\frac{(n+1)^2}{n}\right)$$

It’s unhappy about the division by 0. Instead we should do

```
> Rk2ak(1/k,k,1);
```

$$\frac{a(1)\,\Gamma(k+1)^2}{\Gamma(k)}$$

```
> simplify(");
```

$$a(1)\,\Gamma(k+1)\,k$$

```
> convert(",factorial);
```

$$\frac{a(1)\,(k+1)!\,k}{k+1}$$

It’s very stubborn... it should be a(1) * k * k!

```
>
```

Koepf #5.13(e)

```
> Rk2ak((k-1)/k,k);
```

$$a(0)\left(\prod_{n=0}^{k-1}\frac{(2n-1)(n+1)}{n^2}\right)$$

Same problem with division by 0, try again.

```
> Rk2ak((k-1)/k,k,1);
```

$$\frac{1}{2}\,\frac{a(1)\,2^k\,\Gamma\left(k-\frac{1}{2}\right)\Gamma(k+1)}{\Gamma(k)^2\sqrt{\pi}}$$

```
> simplify(");
```

$$\frac{2^{(k-1)}\,a(1)\,\Gamma\left(k-\frac{1}{2}\right)k}{\Gamma(k)\sqrt{\pi}}$$

```
> Rk2ak((k-1)/k,k,1,false);
```

$$a(1)\left(\prod_{n=1}^{k-1}\frac{(2n-1)(n+1)}{n^2}\right)$$

```
>
```

Koepf #5.13(f)

```
> Rk2ak((k+1)/k,k);
```

$$a(0)\left(\prod_{n=0}^{k-1}\frac{(2n+1)(n+1)}{n(n+2)}\right)$$

```
> Rk2ak((k+1)/k,k,1);
```

$$2\,\frac{a(1)\,2^k\,\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)}{\Gamma(k)\,\Gamma(k+2)\sqrt{\pi}}$$

```
>
>
```

Koepf #5.20

```
> gosper(binomial(n,k),k);
Error, (in gosper) no hypergeometric term antidifference exists
> gosper(binomial(-n,k),k);
```

```
Error, (in gosper) no hypergeometric term antidifference exists
> gosper(binomial(5,k),k);
Error, (in gosper) no hypergeometric term antidifference exists
> infolevel[sum]:=3;
```

$$infolevel_{sum} := 3$$

Note that binomial(-n,k) = (-n)(-n-1)...(-n-k+1)/k! = (-1)^k * binomial(n+k-1,k).
In the following output from gosper, using binomial(-5,k), say, seems to give incorrect results (they are inconsistent with the computations I did by hand in the other part of the answer key), probably because it's confused about the top number being negative. But using the alternate formula gives results consistent with what I said.

```
> bin2 := (n,k) -> (-1)^k * binomial(-n+k-1,k);
```

$$bin2 := (n, k) \to (-1)^k \operatorname{binomial}(-n + k - 1, k)$$

```
>
> for nn from -1 to -5 by -1 do
    print(‘ gosper‘(binomial(nn,k),k) =
            sumtools[gosper](binomial(nn,k),k));
    print(‘‘=sumtools[gosper](bin2(nn,k),k));
    print(‘-----------------------------------‘);
  od;
sumtools[gosper]   a(   k   )/a(   k   -1):=    1
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   1
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper’s algorithm successful
sumtools[gosper]   f:=   k
```

$$\operatorname{gosper}(\operatorname{binomial}(-1, k), k) = (k - 1) \operatorname{binomial}(-1, k)$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    -1
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   1
sumtools[gosper]   q:=   -1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   0
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper’s algorithm successful
sumtools[gosper]   f:=   -1/2
```

$$= -\frac{1}{2} (-1)^k$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    (k+1)/k
sumtools[gosper]   Gosper’s algorithm applicable
sumtools[gosper]   p:=   k+1
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   2
```

```
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   1/2*k*(k+3)
```

$$\mathrm{gosper}(\mathrm{binomial}(\text{-}2, k), k) = \frac{1}{2}\frac{(k-1)(k+2)\,\mathrm{binomial}(\text{-}2, k)}{k+1}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    -(k+1)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   k+1
sumtools[gosper]   q:=   -1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   1
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -3/4-1/2*k
```

$$= -\frac{1}{4}(1+2k)(\text{-}1)^k$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    (k+2)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+1)*(k+2)
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   3
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   1/3*k*(11+6*k+k^2)
```

$$\mathrm{gosper}(\mathrm{binomial}(\text{-}3, k), k) = \frac{1}{3}\frac{(k-1)(6+4k+k^2)\,\mathrm{binomial}(\text{-}3, k)}{(k+1)(k+2)}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    -(k+2)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+1)*(k+2)
sumtools[gosper]   q:=   -1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   2
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -7/4-2*k-1/2*k^2
```

$$= -\frac{1}{4}\frac{(1+4k+2k^2)(\text{-}1)^k\,\mathrm{binomial}(k+2, k)}{(k+1)(k+2)}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    (k+3)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+3)*(k+2)*(k+1)
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   4
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   1/4*k*(k+5)*(k^2+5*k+10)
```

$$\mathrm{gosper}(\mathrm{binomial}(-4,k),k) = \frac{1}{4}\,\frac{(k-1)\,(k+4)\,(k^2+3\,k+6)\,\mathrm{binomial}(-4,k)}{(k+3)\,(k+2)\,(k+1)}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    -(k+3)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+3)*(k+2)*(k+1)
sumtools[gosper]   q:=   -1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   3
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -1/8*(2*k+5)*(2*k^2+10*k+9)
```

$$= -\frac{1}{8}\,\frac{(2\,k+3)\,(2\,k^2+6\,k+1)\,(-1)^k\,\mathrm{binomial}(k+3,k)}{(k+3)\,(k+2)\,(k+1)}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    (k+4)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+4)*(k+3)*(k+2)*(k+1)
sumtools[gosper]   q:=   1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   5
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   1/5*k*(274+225*k+85*k^2+15*k^3+k^4)
```

$$\mathrm{gosper}(\mathrm{binomial}(-5,k),k) =$$

$$\frac{1}{5}\,\frac{(k-1)\,(120+96\,k+46\,k^2+11\,k^3+k^4)\,\mathrm{binomial}(-5,k)}{(k+4)\,(k+3)\,(k+2)\,(k+1)}$$

```
sumtools[gosper]   a(   k   )/a(   k   -1):=    -(k+4)/k
sumtools[gosper]   Gosper's algorithm applicable
sumtools[gosper]   p:=   (k+4)*(k+3)*(k+2)*(k+1)
sumtools[gosper]   q:=   -1
sumtools[gosper]   r:=   1
sumtools[gosper]   degreebound:=   4
sumtools[gosper]   solving equations to find f
sumtools[gosper]   Gosper's algorithm successful
sumtools[gosper]   f:=   -93/4-42*k-25*k^2-6*k^3-1/2*k^4
```

$$= -\frac{1}{4}\,\frac{(3+32\,k+40\,k^2+16\,k^3+2\,k^4)\,(-1)^k\,\mathrm{binomial}(k+4,k)}{(k+4)\,(k+3)\,(k+2)\,(k+1)}$$

```
> infolevel[sum]:=0;
```

$$infolevel_{sum} := 0$$

Koepf #5.21

```
> gosper(1/k^2,k);
Error, (in gosper) no hypergeometric term antidifference exists
```

```
>
> read ‘qsum.mpl‘;
```

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Koepf #5.25

```
> q525j := qgosper(q^(j*k),q,k);
```

$$q525j := \frac{q^{(j\,k)}}{q^j - 1}$$

```
> for jj from 1 to 5 do
      print(‘ qgosper‘(q^(jj*k),q,k) =
            qgosper(q^(jj*k),q,k))
  od;
```

$$\mathrm{qgosper}(q^k, q, k) = \frac{\_C1\, q^2 - \_C1\, q + q^k}{q - 1}$$

$$\mathrm{qgosper}(q^{(2\,k)}, q, k) = \frac{\_C1\, q^4 - \_C1\, q^2 + q^{(2\,k)}}{(q - 1)\,(q + 1)}$$

$$\mathrm{qgosper}(q^{(3\,k)}, q, k) = \frac{-\_C1\, q^3 + \_C1\, q^6 + q^{(3\,k)}}{(q - 1)\,(q^2 + q + 1)}$$

$$\mathrm{qgosper}(q^{(4\,k)}, q, k) = \frac{q^{(4\,k)} + \_C1\, q^8 - \_C1\, q^4}{(q - 1)\,(q + 1)\,(q^2 + 1)}$$

$$\mathrm{qgosper}(q^{(5\,k)}, q, k) = \frac{q^{(5\,k)} + \_C1\, q^{10} - \_C1\, q^5}{(q - 1)\,(q^4 + q^3 + q^2 + q + 1)}$$

It added a constant to the antidifference: _C1*(q^10-q^5)/(q^5-1)=_C1*q^5 is constant with respect to k. The original input summand is rational w.r.t. k, so there is not a unique q-hypergeometric antidifference, but rather an additive "constant" rational w.r.t. q is contained in the answer.

```
> simplify(qgosper(q^(5*k),q,k) - subs(j=5,q525j));
```

$$\_C1\, q^5$$

```
>
>
>
```

Koepf #5.26(a)

```
> qgosper(q^(j*k)*qpochhammer(n,q,k),q,k);
Error, (in qgosper) No q-hypergeometric antidifference exists.
```

```
> fna := j -> q^(j*k) * qpochhammer(n,q,k);
```

$$fna := j \to q^{(j\,k)}\ \mathrm{qpochhammer}(n, q, k)$$

```
> for jj from 1 to 5 do
      printf(`at j=%d`,jj);
      print(qgosper(fna(jj),q,k));
  od;
at j=1
```

$$-\frac{\mathrm{qpochhammer}(n, q, k)}{n}$$

```
at j=2
```

$$\frac{(-q + 1 - n\,q^k)\ \mathrm{qpochhammer}(n, q, k)}{n^2\,q}$$

```
at j=3
```

$$\frac{(-q^3 + q + q^2 - 1 - n\,q^{(k+2)} + n\,q^k - n^2\,q^{(1+2\,k)})\ \mathrm{qpochhammer}(n, q, k)}{n^3\,q^3}$$

```
at j=4
```

$$-(q^2 + q^6 - q^5 + q - 1 - q^4 - n\,q^{(k+2)} + n\,q^{(k+5)} + n\,q^k - n\,q^{(k+3)} - n^2\,q^{(1+2\,k)} + n^2\,q^{(4+2\,k)} + n^3\,q^{(3\,k+3)})\ \mathrm{qpochhammer}(n, q, k) \Big/ (n^4\,q^6)$$

```
at j=5
```

$$-\mathrm{qpochhammer}(n, q, k)\ (-q^2 + q^{10} - q^9 + 2\,q^5 - q + 1 - q^8 - n\,q^{(k+6)} + n\,q^{(k+2)} + n\,q^{(k+9)} - n\,q^{(k+5)} + n\,q^{(k+4)} - n\,q^k - n\,q^{(k+7)} + n\,q^{(k+3)} - n^2\,q^{(5+2\,k)} + n^2\,q^{(1+2\,k)} + n^2\,q^{(8+2\,k)} - n^2\,q^{(4+2\,k)} + n^3\,q^{(7+3\,k)} - n^3\,q^{(3\,k+3)} + n^4\,q^{(6+4\,k)}) \Big/ (n^5\,q^{10})$$

```
>
```

Koepf #5.26(c)

```
> qgosper((-1)^k*q^binomial(k,2)*qbinomial(n,k,q),q,k);
```

$$-\frac{(q^k - 1)\,(-1)^k\,q^{\mathrm{binomial}(k,\,2)}\ \mathrm{qbinomial}(n, k, q)}{q^n - 1}$$

Koepf #5.26(d)

```
> qgosper(q^(j*k)*qbrackets(k,q),q,k);
```

$$-\frac{(-1 + q^{(j+1)} - q^{(k+j)} + q^k)\,q^{(j\,k)}\ \mathrm{qbrackets}(k, q)}{(q^j - 1)\,(-1 + q^{(j+1)})\,(q^k - 1)}$$

```
> qgosper(q^(2*k)*qbrackets(k,q),q,k);
```

$$\mathrm{qbrackets}(k, q)$$

$$(\_C1\,q^6 + \_C1\,q^5 - \_C1\,q^3 - q^{(2\,k+2)} - \_C1\,q^2 + q^{(1+3\,k)} - q^{(1+2\,k)} + q^{(3\,k)} - q^{(2\,k)}) \Big/$$

$$((q^k-1)(q-1)(q+1)(q^2+q+1))$$

```
>
```

Koepf #5.26(e)

```
> qgosper(q^binomial(k,2)*qbrackets(k,q),q,k);
```

$$\frac{q^{\mathrm{binomial}(k,\,2)}\,\mathrm{qbrackets}(k,\,q)}{q^k-1}$$

Koepf #5.26(f)

```
> qgosper(q^(-k*(k+1)/2)*qbrackets(k,q),q,k);
```

$$-\frac{q^{(k-1/2\,k\,(k+1))}\,\mathrm{qbrackets}(k,\,q)}{q^k-1}$$