Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 3 answers — October 22, 1999

5.1

$$\begin{aligned}\Delta k^{\underline{m}} &= (k+1)(k)\cdots(k-m+2) - k(k-1)\cdots(k-m+2)(k-m+1)\\ &= [(k+1)-(k-m+1)]\cdot k(k-1)\cdots(k-m+2)\\ &= m\cdot k(k-1)\cdots(k-m+2) \;=\; m\cdot k^{\underline{m-1}}\end{aligned}$$

so $\Delta k^{\underline{m+1}} = (m+1)k^{\underline{m}}$, and provided $m \neq -1$, dividing by $m+1$ gives

$$k^{\underline{m}} = \Delta \frac{k^{\underline{m+1}}}{m+1} = \frac{(k+1)^{\underline{m+1}} - k^{\underline{m}}}{m+1}$$

Sum this for $k = a, a+1, \ldots, b-1$:

$$\begin{aligned}\sum_{k=a}^{b-1} k^{\underline{m}} &= \sum_{k=a}^{b-1} \frac{(k+1)^{\underline{m+1}} - k^{\underline{m}}}{m+1}\\ &= \frac{1}{m+1}\left(-a^{\underline{m}} + (a+1)^{\underline{m}} \quad - (a+1)^{\underline{m}} + (a+2)^{\underline{m}} \quad - \cdots \quad - (b-1)^{\underline{m}} + b^{\underline{m}}\right)\\ &= \frac{b^{\underline{m}} - a^{\underline{m}}}{m+1}\end{aligned}$$

5.2 We have

$$u_k\ \Delta v_k + v_{k+1}\ \Delta u_k = u_k v_{k+1} - u_k v_k \ + \ v_{k+1}u_{k+1} - v_{k+1}u_k = -u_k v_k + v_{k+1}u_{k+1}$$

and summing for $k = a, \ldots, b-1$ [*Note:* the book has a typo] gives

$$\sum_{k=a}^{b-1} (u_k\ \Delta v_k + v_{k+1}\ \Delta u_k) = u_b v_b - u_a v_a = u_k v_k \Big|_{k=a}^{b}$$

by telescoping. This rearranges into

$$\sum_{k=a}^{b-1} u_k\ \Delta v_k = u_k v_k \Big|_{k=a}^{b} - \sum_{k=a}^{b-1} v_{k+1}\ \Delta u_k$$

All antidifferences of $H_k = \sum_{j=1}^{k} \frac{1}{j}$ are the same up to additive constant, so we take

$$\begin{aligned}s_n &= \sum_{k=1}^{n-1} H_k = \sum_{k=1}^{n-1}\sum_{j=1}^{k} \frac{1}{j} = \sum_{j=1}^{n-1}\sum_{k=j}^{n-1} \frac{1}{j}\\ &= \sum_{j=1}^{n-1} \frac{n-j}{j} = n\sum_{j=1}^{n-1}\frac{1}{j} - \sum_{j=1}^{n-1} 1 = \boxed{\mathbf{1 - n + nH_{n-1}}}\end{aligned}$$

Actually, this didn't make use of summation by parts at all ...

5.10 The desired antidifference has the form

$$s_k = C + \sum_{n=0}^{k-1} a_n = C + \sum_{n=0}^{k-1} (t_{n+m} - t_n) = C + (t_{k-1} + t_k + \cdots + t_{k+m-1}) - (t_0 + t_1 + \cdots + t_{m-1})$$

for some constant C. We are given that t_k is hypergeometric in k, and we want s_k to be hypergeometric in k as well. The terms $t_{k-1}, \ldots, t_{k+m-1}$ are in the same "rational similarity class" (their quotients are rational functions of k). The terms $C + t_0 + \cdots + t_{m-1}$ are constant w.r.t. k, and their ratio with t_k etc. will in general be hypergeometric but not rational, so the total sum will not be hypergeometric w.r.t. k unless we eliminate these terms. To do this, set $C = -(t_0 + \cdots + t_{m-1})$, and conclude

$$\boxed{\boldsymbol{s_k = t_{k-1} + t_k + \cdots + t_{k+m-1}}}$$

is the desired hypergeometric antidifference of a_k. (In the event that t_k is itself rational, we may add a constant to s_k and it will remain hypergeometric.) Since Gosper's algorithm is guaranteed to find this if it exists, or to recognize that it doesn't exist, we conclude that a_k is Gosper summable.

5.13 In class we showed $\frac{a(k+1)}{a(k)} = \frac{1+y(k)}{y(k+1)}$ which in Koepf's notation is $\frac{a_{k+1}}{a_k} = \frac{1+R_k}{R_{k+1}}$ and therefore in each problem,

$$a_k = a_j \prod_{n=j}^{k-1} \frac{1+R_n}{R_{n+1}}$$

where a_j is a suitable initial value. Then

	R_k	term ratio	a_k
(a)	$\dfrac{\alpha}{\alpha-1}$	$\dfrac{2\alpha-1}{\alpha}$	$a_0 \cdot \left(\dfrac{2\alpha-1}{\alpha}\right)^k$
(b)	k	$\dfrac{1+k}{k+1} = 1$	a_0
(c)	k^2	$\dfrac{1+k^2}{(k+1)^2}$	$a_0 \prod_{n=0}^{k-1} \dfrac{1+n^2}{(n+1)^2}$
(d)	$1/k$	$\dfrac{(k+1)/k}{1/(k+1)} = \dfrac{(k+1)^2}{k}$	$a_1 \prod_{n=1}^{k-1} \dfrac{(n+1)^2}{n} = a_1 \dfrac{k!^2}{(k-1)!} = a_1 \cdot k \cdot k!$ *Note:* The denominator at $n=0$ is 0, so we start with a_1.
(e)	$(k-1)/k$	$\dfrac{(2k-1)(k+1)}{k^2}$	$a_1 \prod_{n=1}^{k-1} \dfrac{(2n-1)(n+1)}{n^2} = a_1 \dfrac{(2k-3)!!\, k!}{(k-1)!^2} = a_1 \dfrac{(2k-3)!!\, k}{(k-1)!}$
(f)	$(k+1)/k$	$\dfrac{(2k+1)(k+1)}{k(k+2)}$	$a_1 \cdot \dfrac{(2k-1)!!/3 \ \cdot\ k!}{(k-1)!\,(k+1)!/3} = a_1 \cdot \dfrac{(2k-1)!!\,(2k)!!/2^k}{(k-1)!\,(k+1)!}$ $= a_1 \cdot \dfrac{(2k)!/2^k}{(k-1)!\,(k+1)!} = a_1 \cdot \binom{2k}{k-1} \Big/ 2^k$

where we used the notation $(2n)!! = 2{\cdot}4\cdots(n-2)n$ and $(2n+1)!! = 1{\cdot}3{\cdot}5\cdot\cdots\cdot(2n-1)(2n+1)$ for integer n.

5.20 In Example 5.3, page 71 of Koepf, the term ratio for $a(k) = \binom{n}{k}$ is computed:

$$\frac{a(k+1)}{a(k)} = \frac{n-k}{k+1}\,.$$

Since n is a symbol rather than a specific number, the numerator and denominator do not have roots differing by an integer, so Gosper's algorithm finds "p, q, r" quickly and terminates. If n represents a specific integer, however, the algorithm proceeds differently.

Initially we choose $p(k) = 1$, $q(k) = n - k + 1 = -(k - (n+1))$, $r(k) = k$. Consider $g_j(k) := \gcd(q(k), r(k+j)) = \gcd(k-(n+1), k+j)$. This is $k+j \neq 1$ when $j = -(n+1)$; for j to be a nonnegative integer requires $n = -1, -2, -3, \ldots$. Write this as $n = -N$, $j = N-1$, $g_j(k) = k + N - 1$ with N positive. The algorithm computes a different p, q, r, as follows:

$$\begin{aligned} p'(k) &= p(k) \cdot g_j(k) g_j(k-1) \cdots g_j(k-j+1) = (k+N-1)(k+N-2)...(k+1) \\ q'(k) &= q(k)/g_j(k) = -1 \\ r'(k) &= r(k)/g_j(k-j) = 1 \end{aligned}$$

and these new q, r are relatively prime at all shifts so this is final.

The next step is to find a polynomial $f(k)$ satisfying $q(k+1)f(k) - r(k)f(k-1) = p(k)$, which becomes

$$-f(k) - f(k-1) = (k+N-1)(k+N-2)...(k+1) \tag{1}$$

This is a constant coefficient recursion with a polynomial inhomogeneity, so we expect that $f(k)$ should be a polynomial of degree $N-1$. However, we will do this as in Gosper's algorithm. Since q and r have different leading terms, we are in "Case 1." The degree bound for $f(k)$ is then $D = \deg(p) - \max\{\deg(q), \deg(r)\} = (N-1) - 0 = \boxed{\boldsymbol{N-1=-n-1}}$. So we set $f(k)$ to a generic polynomial of this degree:

$$f(k) = \sum_{i=0}^{N-1} c_i \, k^i$$

and plug that into (1), collect in powers of k, set the coefficients of the powers of k on both sides equal, and solve for the c's.

Another way to do this, not pertinent to Gosper's algorithm in general but pertinent to solving recursions, is to rewrite (1) as follows:

$$\begin{aligned} f(k+1) + f(k) = (E+1)f(k) = (\Delta+2)f(k) = 2(1+\Delta/2)f(k) \\ = -(k+2)(k+3)\cdots(k+N) = (k+N)^{\underline{N-1}} \end{aligned}$$

so that a particular solution is given by

$$f(k) = -\frac{1}{2}\frac{1}{1+\Delta/2}(k+2)_{N-1} = -\frac{1}{2}\sum_{r=0}^{\infty}\frac{\Delta^r}{(-2)^r}(k+N)^{\underline{N-1}}$$

Since Δ decreases the degree of a nonzero polynomial by 1, the sum terminates at $r = N-1$. By suitably modifying problem 5.1,

$$\Delta^r (k+a)^{\underline{b}} = b(b-1)\cdots(b-r+1)(k+a)^{\underline{b-r}} = b^{\underline{r}}(k+a)^{\underline{b-r}}$$

so

$$f(k) = -\frac{1}{2}\sum_{r=0}^{N-1}\frac{(N-1)^{\underline{r}}(k+N)^{\underline{N-1-r}}}{(-2)^r}.$$

Finally,

$$s(k) = \frac{r(k)f(k-1)}{p(k)}a(k) = \frac{f(k-1)}{(k+1)_{N-1}}\binom{-N}{k}.$$

5.21 We are given $a(k) = \frac{1}{k^2} \in \mathbb{Q}(k)$. If it has a hypergeometric antidifference $s(k)$, then $s(k)$ is a rational multiple of $a(k)$ by Gosper's algorithm, and hence $s(k)$ is rational too; $s(k) \in \mathbb{Q}(k)$. The most general antidifference of $a(k)$ is then $s(k) + C$. For an arbitrary application of Gosper's algorithm, $s(k) + C$ would not be hypergeometric unless $C = 0$, but since $s(k)$ is rational, so is $s(k)+C$, so Gosper's algorithm can return many possible functions, all differing by a constant.

By polynomial division, an arbitrary rational function $s(k) \in \mathbb{Q}(k)$ can be written $\gamma(k) + A\alpha(k)/\beta(k)$ for polynomials $\gamma(k), \alpha(k), \beta(k) \in \mathbb{Q}(k)$; $A \in \mathbb{Q}$; and α, β monic with $\deg(\alpha) < \deg(\beta)$. Since we know $s(k)$ approaches a limit as $k \to \infty$, the polynomial $\gamma(k)$ is just a constant C: $s(k) = C + A\alpha(k)/\beta(k)$. All other possible antidifferences of $a(k)$ are obtained just by modifying C.

Then

$$\sum_{k=1}^{n-1} a(k) = s(n) - s(1) = (C - C) + A\left(\frac{\alpha(n)}{\beta(n)} - \frac{\alpha(1)}{\beta(1)}\right)$$

Since $\deg(\alpha) < \deg(\beta)$, $\lim_{n\to\infty} \frac{\alpha(n)}{\beta(n)} = 0$ so this gives

$$\sum_{k=1}^{\infty} a(k) = -A\frac{\alpha(1)}{\beta(1)} \in \mathbb{Q}$$

But here, the sum is known to be the irrational number $\pi^2/6$. There's a contradiction; the assumption that a hypergeometric term antidifference $s(k)$ exists is false.

5.25

$$s(k) = C + \sum_{n=0}^{k-1} q^{jn} = C + \frac{1 - q^{kj}}{1 - q^j} \ .$$

For any fixed value $j = 1, 2, 3, \ldots$ and any constant C (constant w.r.t. k; so $C \in \mathbb{Q}(q)$), this expression is a polynomial (over $\mathbb{Q}(q)$) in q^k of degree j, so it's q-hypergeometric. Gosper's algorithm is shown on the worksheet.