Math 262a, Fall 1999, Glenn Tesler
Homework 2

Problem 4a solved the "humanoid way", but with maple assistance:

```
> Fnk := (n,k) -> binomial(n,k)*x^k;
```

$$Fnk := (n, k) \to \text{binomial}(n, k)\, x^k$$

Form the desired recurrence, with unknown coefficients:

```
> rec := a*Fnk(n,k) + b*Fnk(n+1,k) + c*Fnk(n,k+1) +
  d*Fnk(n+1,k+1);
```

$$rec := a\, x^k \,\text{binomial}(n, k) + b\, x^k \,\text{binomial}(n+1, k) + c\, x^{(k+1)} \,\text{binomial}(n, k+1) + d\, x^{(k+1)} \,\text{binomial}(n+1, k+1)$$

Divide through by the unshifted function:

```
> rec/Fnk(n,k);
```

$$\left(a\, x^k \,\text{binomial}(n, k) + b\, x^k \,\text{binomial}(n+1, k) + c\, x^{(k+1)} \,\text{binomial}(n, k+1) + d\, x^{(k+1)} \,\text{binomial}(n+1, k+1)\right) \Big/ \left(x^k \,\text{binomial}(n, k)\right)$$

Reduce all the ratios to rational functions of n and k:

```
> simplify(");
```

$$(a\, n\, k + a\, n + a - a\, k^2 + b\, n\, k + b\, n + b\, k + b - 2\, c\, x\, k\, n - c\, x\, k + c\, x\, k^2 + c\, x\, n^2 + c\, x\, n + d\, x\, n^2 + 2\, d\, x\, n - d\, x\, n\, k + d\, x - d\, x\, k) / ((n + 1 - k)\, (k + 1))$$

Clear denominators (or take the numerator):

```
> numer(");
```

$$a\, n\, k + a\, n + a - a\, k^2 + b\, n\, k + b\, n + b\, k + b - 2\, c\, x\, k\, n - c\, x\, k + c\, x\, k^2 + c\, x\, n^2 + c\, x\, n + d\, x\, n^2 + 2\, d\, x\, n - d\, x\, n\, k + d\, x - d\, x\, k$$

This is a polynomial in k. We want it to be true for all integer values of k, so it must be the 0 polynomial. Collect it in powers of k.

```
> collect(",k);
```

$$(-a + c\, x)\, k^2 + (b\, n - c\, x + b - 2\, c\, x\, n + a\, n - d\, x\, n - d\, x)\, k + d\, x\, n^2 + a\, n + a + d\, x + c\, x\, n + b\, n + 2\, d\, x\, n + b + c\, x\, n^2$$

Separate the coefficients of the powers of k.

```
> coeffs(",k);
```

$$d\, x\, n^2 + a\, n + a + d\, x + c\, x\, n + b\, n + 2\, d\, x\, n + b + c\, x\, n^2, -a + c\, x,$$

$$b\,n - c\,x + b - 2\,c\,x\,n + a\,n - d\,x\,n - d\,x$$

This gives a system of equations for the unknowns a,b,c,d. They'll potentially depend on n, but there are no k's in the system, so they'll be k-free.

```
> sols := solve({"},{a,b,c,d});
```

$$sols := \{c = -d,\ a = -d\,x,\ b = 0,\ d = d\}$$

```
> subs(sols,rec);
```

$$-d\,x\,x^k\,\mathrm{binomial}(n, k) - d\,x^{(k+1)}\,\mathrm{binomial}(n, k+1) + d\,x^{(k+1)}\,\mathrm{binomial}(n+1, k+1)$$

```
> recF:=subs(sols,a*F(n,k)+b*F(n+1,k)+c*F(n,k+1)+d*F(n+1,k
  +1));
```

$$recF := -d\,x\,\mathrm{F}(n, k) - d\,\mathrm{F}(n, k+1) + d\,\mathrm{F}(n+1, k+1)$$

```
> recF := subs(d=1,recF);
```

$$recF := -x\,\mathrm{F}(n, k) - \mathrm{F}(n, k+1) + \mathrm{F}(n+1, k+1)$$

Let f(n)=sum(F(n,k),k=-infinity..infinity). This gives

```
> recf := subs(''F(n+i,k+j)=f(n+i)'$i=0..1'$j=0..1,recF);
```

$$recf := -x\,\mathrm{f}(n) - \mathrm{f}(n) + \mathrm{f}(n+1)$$

```
> rsolve(recf=0,f(n));
```

$$\mathrm{f}(0)\,(x+1)^n$$

Now get the initial value f(0). Maple won't do the sum correctly for -infinity..infinity

```
> sum(Fnk(0,k),k=-infinity..infinity);
```

$$\sum_{k=-\infty}^{\infty} x^k\,\mathrm{binomial}(0, k)$$

But if we recognize F(n,k)=0 for k<0, we can get it to do the sum.

```
> sum(Fnk(0,k),k=0..infinity);
```

$$1$$

```
>
>
```

Problem 4a solved with Koepf's software:

```
> read `hsum.mpl`;
```

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```
> rec := fasenmyer(binomial(n,k)*x^k,k,f(n),0);
Error, (in kfreerec) no kfree recurrence equation of order (, 0, 0, ) exists
> rec := fasenmyer(binomial(n,k)*x^k,k,f(n),1);
```

$$rec := \mathrm{f}(n+1) - \mathrm{f}(n)\,(x+1) = 0$$

```
> rsolve(rec,f(n));
```

$$\mathrm{f}(0)\,(x+1)^n$$

```
> rsolve({rec,f(0)=1},f(n));
```

$$(x+1)^n$$

```
>
>
```

Koepf # 4.1 (2.2)

```
> fasenmyer((-1)^k*binomial(n,k),k,f(n),0);
Error, (in kfreerec) no kfree recurrence equation of order (, 0, 0, ) exists
> fasenmyer((-1)^k*binomial(n,k),k,f(n),1);
```

$$\mathrm{f}(n+1)=0$$

(2.3)

```
> fasenmyer(binomial(n,k)^2,k,f(n),0);
Error, (in kfreerec) no kfree recurrence equation of order (, 0, 0, ) exists
> fasenmyer(binomial(n,k)^2,k,f(n),1);
Error, (in kfreerec) no kfree recurrence equation of order (, 1, 1, ) exists
> fasenmyer(binomial(n,k)^2,k,f(n),2);
```

$$(n+2)\,\mathrm{f}(n+2)-2\,\mathrm{f}(n+1)\,(2\,n+3)=0$$

```
> rsolve(",f(n));
```

$$\frac{4^n\,\Gamma\left(n+\frac{1}{2}\right)\mathrm{f}(0)}{\Gamma(n+1)\sqrt{\pi}}$$

```
>
```

(2.4)

```
> fasenmyer((-1)^k*binomial(n,k)^2,k,f(n),1);
Error, (in kfreerec) no kfree recurrence equation of order (, 1, 1, ) exists
> fasenmyer((-1)^k*binomial(n,k)^2,k,f(n),2);
```

$$(n+2)\,\mathrm{f}(n+2)+4\,\mathrm{f}(n)\,(n+1)=0$$

```
> rsolve(",f(n));
```

$$\mathrm{rsolve}((n+2)\,\mathrm{f}(n+2)+4\,\mathrm{f}(n)\,(n+1)=0,\mathrm{f}(n))$$

It’s not going to do it for us too easily.

```
> rec := "";
```

$$rec:=(n+2)\,\mathrm{f}(n+2)+4\,\mathrm{f}(n)\,(n+1)=0$$

Even n:

```
> subs(n=2*m,rec); subs(f = proc(k) g(k/2) end,");
```

$$(2\,m+2)\,\mathrm{f}(2\,m+2)+4\,\mathrm{f}(2\,m)\,(2\,m+1)=0$$

$$(2\,m+2)\,(\mathbf{proc}(k)\ \mathrm{g}(1/2*k)\ \mathbf{end})(2\,m+2)$$
$$+4\,(\mathbf{proc}(k)\ \mathrm{g}(1/2*k)\ \mathbf{end})(2\,m)\,(2\,m+1)=0$$

```
> eval(");
```

$$(2\,m+2)\,g(m+1)+4\,g(m)\,(2\,m+1)=0$$

```
> rsolve(",g(m));
```

$$\frac{(-1)^m\,4^m\,\Gamma\left(m+\frac{1}{2}\right)g(0)}{\Gamma(m+1)\,\sqrt{\pi}}$$

Use the initial condition g(0)=f(2*0)=f(0)=1.

Odd n:

```
> subs(n=2*m+1,rec); eval(subs(f=proc(k) g((k-1)/2)
  end,"));
```

$$(2\,m+3)\,f(2\,m+3)+4\,f(2\,m+1)\,(2\,m+2)=0$$

$$(2\,m+3)\,g(m+1)+4\,g(m)\,(2\,m+2)=0$$

```
> rsolve(",g(m));
```

$$\frac{1}{2}\,\frac{\Gamma(m+1)\,\sqrt{\pi}\,g(0)\,(-1)^m\,4^m}{\Gamma\left(m+\frac{3}{2}\right)}$$

```
>
>
>
```

Koepf # 4.5

```
> kfreerec(binomial(n-k,k),k,n,0,0,F,a);
Error, (in kfreerec) no kfree recurrence equation of order (, 0, 0, ) exists
> kfreerec(binomial(n-k,k),k,n,1,1,F,a);
Error, (in kfreerec) no kfree recurrence equation of order (, 1, 1, ) exists
> kfreerec(binomial(n-k,k),k,n,2,2,F,a);
```

$$a_{0,0}\,F(n,k)-a_{2,2}\,F(n,k+1)+a_{0,0}\,F(n+1,k+1)-a_{0,0}\,F(n+2,k+1)$$
$$-\,a_{2,2}\,F(n+1,k+2)+a_{2,2}\,F(n+2,k+2)=0$$

```
> nkrec:=":
> nkrec1 := subs(a[0,0]=1,a[2,2]=0,nkrec);
  nkrec2 := subs(a[2,2]=0,a[0,0]=1,nkrec);
```

$$nkrec1 := F(n,k)+F(n+1,k+1)-F(n+2,k+1)=0$$

$$nkrec2 := F(n,k)+F(n+1,k+1)-F(n+2,k+1)=0$$

```
> nrec:=subs('('F(n+i,k+j)=s(n+i)'$j=0..2)'$i=0..2,nkrec1)
  ;
```

$$nrec := s(n)+s(n+1)-s(n+2)=0$$

Or, we could do all of that with one command:

```
> fasenmyer(binomial(n-k,k),k,s(n),2);
```

$$-s(n) - s(n+1) + s(n+2) = 0$$

What we just did is not quite legal. We want k=0..floor(n/2), but we just used the theory for k=-infinity..infinity. However, the function does not vanish ouside the range 0..floor(n/2):

```
> Fnk := (n,k)->binomial(n-k,k);
```

$$Fnk := (n, k) \to \text{binomial}(n - k, k)$$

```
> 'F(2,k)=Fnk(2,k)'$k=-5..5;
```

$$F(2, -5) = 0, F(2, -4) = 0, F(2, -3) = 0, F(2, -2) = 0, F(2, -1) = 0, F(2, 0) = 1,$$
$$F(2, 1) = 1, F(2, 2) = 0, F(2, 3) = -1, F(2, 4) = 5, F(2, 5) = -21$$

It does vanish for k=floor(n/2)+1,...,n, so let s(n)=sum(binomial(n-k,k),k=0..n).
Then sum up nkrec1 for k=0..n. The three terms give:

sum(F(n,k),k=0..n) = s(n)

sum(F(n+1,k+1),k=0..n) = f(n+1) - F(n+1,0) + F(n+1,n+1) = s(n+1)-1+0 = s(n+1)-1

sum(F(n+2,k+1),k=0..n) = f(n+2) - F(n+2,0) + F(n+2,n+1) = s(n+2)-1 for n>0

(for n=0 we have F(n+2,n+1)=F(2,1)=binomial(2-1,1)=1 so it's false)

Combining the three gives s(n)+s(n+1)-s(n+2) = 0.

So the correct recursion was derived by invalid reasoning.
Next we need initial conditions.

```
>
>
> sumfn := n -> sum(Fnk(n,k),k=0..floor(n/2));
```

$$sumfn := n \to \sum_{k=0}^{\text{floor}(1/2\, n)} \text{Fnk}(n, k)$$

```
> 'sumfn(n)'$n=0..10;
```

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$$

Since it's the same 2nd order recurrence as the Fibonacci numbers and the first two numbers agree, the sum does in fact give the Fibonacci numbers.

```
>
```

Koepf # 4.15(a)

```
> Fnk := (n,k)->(-1)^k * binomial(n,k);
  fasenmyer(Fnk(n,k),k,f(n),1);
```

$$Fnk := (n, k) \to (-1)^k \,\text{binomial}(n, k)$$

$$f(n+1) = 0$$

That's wrong because the sum is supposed to be over k=0..m, but this is k=-infinity..infinity.

The bound m is not natural, so additional (unwanted) terms are included.
Instead, get the k-free recurrence, and sum it manually.

```
> nkrec := kfreerec(Fnk(n,k),k,n,1,1,F,a);
```

$$nkrec := a_{1,1}\,\mathrm{F}(n,k) - a_{1,1}\,\mathrm{F}(n,k+1) + a_{1,1}\,\mathrm{F}(n+1,k+1) = 0$$

```
> nkrec := subs(a[1,1]=1,nkrec);
```

$$nkrec := \mathrm{F}(n,k) - \mathrm{F}(n,k+1) + \mathrm{F}(n+1,k+1) = 0$$

Let f(n)=sum(F(n,k),k=0..m). Compute the sum of nkrec for k=0..m as we did in #4.5:

```
> nrec := f(n) - (f(n)-Fnk(n,0)+Fnk(n,m+1)) +
  (f(n+1)-Fnk(n+1,0)+Fnk(n+1,m+1)) = 0;
```

$$nrec := -(-1)^{(m+1)}\,\mathrm{binomial}(n, m+1) + \mathrm{f}(n+1) + (-1)^{(m+1)}\,\mathrm{binomial}(n+1, m+1) = 0$$

```
> solve(nrec,{f(n+1)});
```

$$\{\mathrm{f}(n+1) = (-1)^{(m+1)}\,\mathrm{binomial}(n, m+1) - (-1)^{(m+1)}\,\mathrm{binomial}(n+1, m+1)\}$$

We see Pascal’s triangle there. Can we get maple to see it too? Yes, but not automatically with this software..

```
> simplify(");
```

$$\{\mathrm{f}(n+1) = -(-1)^{m}\,\mathrm{binomial}(n, m+1) + (-1)^{m}\,\mathrm{binomial}(n+1, m+1)\}$$

```
> rec := ";
```

$$rec := \{\mathrm{f}(n+1) = -(-1)^{m}\,\mathrm{binomial}(n, m+1) + (-1)^{m}\,\mathrm{binomial}(n+1, m+1)\}$$

```
> convert(rec,GAMMA);
```

$$\{\mathrm{f}(n+1) = -\frac{(-1)^{m}\,\Gamma(n+1)}{\Gamma(m+2)\,\Gamma(n-m)} + \frac{(-1)^{m}\,\Gamma(n+2)}{\Gamma(m+2)\,\Gamma(n+1-m)}\}$$

```
> simplify(");
```

$$\{\mathrm{f}(n+1) = \frac{(-1)^{m}\,\Gamma(n+1)}{\Gamma(m+1)\,\Gamma(n+1-m)}\}$$

```
> convert(",binomial);
```

$$\{\mathrm{f}(n+1) = (-1)^{m}\,\mathrm{binomial}(n, m)\}$$

```
> subs(n=n-1,");
```

$$\{\mathrm{f}(n) = (-1)^{m}\,\mathrm{binomial}(n-1, m)\}$$

Koepf # 4.15(c)

```
> Fnk := (n,k)->binomial(k,n);
  fasenmyer(Fnk(n,k),k,f(n),1);
```

$$Fnk := (n,k) \to \mathrm{binomial}(k,n)$$

```
Error, (in fasenmyer) wrong number (or type) of parameters in function normal
>
> kfreerec(Fnk(n,k),k,n,1,1,F,a);
```

$$a_{0,0}\,\mathrm{F}(n,k) + a_{0,0}\,\mathrm{F}(n+1,k) - a_{0,0}\,\mathrm{F}(n+1,k+1) = 0$$

```
> nkrec := subs(a[0,0]=1,");
```

$$nkrec := \mathrm{F}(n,k) + \mathrm{F}(n+1,k) - \mathrm{F}(n+1,k+1) = 0$$

Let f(n) = sum(binomial(k,n),k=0..m) = sum(Fnk(n,k),k=0..m).
The bounds are not natural, so they can't be extended to -infinity..+infinity.
Summing nkrec for k=0..m gives

```
> nrec := f(n) + f(n+1) - (f(n+1)-Fnk(n+1,0)+Fnk(n+1,m+1))
  = 0;
```

$$nrec := \mathrm{f}(n) + \mathrm{binomial}(0, n+1) - \mathrm{binomial}(m+1, n+1) = 0$$

```
> solve(",f(n));
```

$$-\mathrm{binomial}(0, n+1) + \mathrm{binomial}(m+1, n+1)$$

```
>
>
```

Koepf # 4.11(a)

```
> fasenmyer(hyperterm([-n,1+beta],[1,1+alpha],x,k),k,f(n),
  3);
```

$$-(2\,n\,\alpha + 15\,n + 5\,\alpha - x\,n - x\,\beta - 3\,x + 3\,n^2 + 19)\,\mathrm{f}(n+2)$$
$$+ (n+2)\,(3\,n - x + 6 + \alpha)\,\mathrm{f}(n+1) - (n+2)\,(n+1)\,\mathrm{f}(n)$$
$$+ (n+3)\,(n + \alpha + 3)\,\mathrm{f}(n+3) = 0$$

```
>
```

Koepf # 4.18

q-Chu-Vandermonde

```
> qfasenmyer(qphihyperterm([q^(-n),b],[c],q,c*q^n/b,k),q,k
  ,S(n),1,1);
Error, (in qfasenmyer) No k-free recurrence equation of order (1,1) exists.
> qfasenmyer(qphihyperterm([q^(-n),b],[c],q,c/q^(-n)/b,k),
  q,k,S(n),1,2);
```

$$-b\,(c\,q^{(n+1)} - 1)\,\mathrm{S}(n+2)$$
$$+ (-c\,b\,q^{(2+2\,n)} + c\,q^{(n+1)} - b\,q - b + c\,b\,q^{(n+1)} + b\,q^{(n+2)})\,\mathrm{S}(n+1)$$
$$- q\,(b - c\,q^n)\,(-1 + q^{(n+1)})\,\mathrm{S}(n) = 0$$

q-Pfaff-Saalschuetz

```
> qfasenmyer(qpochhammer(q^(-n),q,k)*qpochhammer(a,q,k)*qp
  ochhammer(b,q,k)/qpochhammer(q,q,k)/qpochhammer(c,q,k)/q
  pochhammer(a*b/c/q^(n-1),q,k)*q^k,q,k,S(n),1,1);
Error, (in qkfreerec) no kfree recurrence equation of order (, 1, 1, ) exists
```

for higher orders the computations timings are very poor and were interrupted

[>