Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 2 answers — October 13, 1999

1. (a) We have

$$ {}_r\phi_s \left[\begin{matrix} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{matrix} \,\middle|\, q, \frac{x}{\alpha_r} \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_{r-1}; q)_k \, x^k}{(\beta_1, \ldots, \beta_s, q; q)_k} \frac{(\alpha_r; q)_k}{{\alpha_r}^k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \tag{1} $$

The second fraction on the right is

$$ \frac{(\alpha_r; q)_k}{{\alpha_r}^k} = \prod_{j=0}^{k-1} \frac{1 - \alpha_r q^{j-1}}{\alpha_r} = \prod_{j=0}^{k-1} \left(\frac{1}{\alpha_r} - q^{j-1} \right) $$

and as $\alpha_r \to \infty$, this tends to

$$ \prod_{j=0}^{k-1} \left(-q^{j-1} \right) = (-1)^k q^{\binom{k}{2}} . $$

The limit as $\alpha_r \to \infty$ of (1) is then

$$ \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_{r-1}; q)_k \, x^k}{(\beta_1, \ldots, \beta_s, q; q)_k} (-1)^k q^{\binom{k}{2}} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} = {}_{r-1}\phi_s \left[\begin{matrix} \alpha_1, \ldots, \alpha_{r-1} \\ \beta_1, \ldots, \beta_s \end{matrix} \,\middle|\, q, x \right] . $$

(b) The series form of the q-binomial theorem is

$$ \sum_{k=0}^{\infty} \frac{(\alpha; q)_k}{(q; q)_k} x^k = \frac{(\alpha x; q)_\infty}{(x; q)_\infty} \tag{2} $$

Setting $\alpha = 0$ gives the equality for $e_q(x)$.
For $E_q(x)$, replace α by $1/\alpha$ and x with $-\alpha x$, and then set $\alpha = 0$.

(c) For the limits, we have that

$$ e_q\left(x(1-q)\right) = \sum_{k=0}^{\infty} \frac{(x(1-q))^k}{(q; q)_k} = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} $$

and as $q \to 1$ the denominator turns into the ordinary $k!$. The proof for $E_q(x)$ is similar.
The two formulas $e_q(x) = 1/(x; q)_\infty$ and $E_q(x) = (-x; q)_\infty$ imply $e_q(x)\, E_q(-x) = 1$.
For the trig function identities, we have

$$ \begin{aligned} &\sin_q(x) \operatorname{Sin}_q(x) + \cos_q(x) \operatorname{Cos}_q(x) = \\ &\quad \frac{1}{4}\Big(-e_q(ix)E_q(ix) + e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) - e_q(-ix)E_q(-ix) \\ &\qquad + e_q(ix)E_q(ix) + e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) + e_q(-ix)E_q(-ix) \Big) \\ &\qquad\qquad = \frac{1}{2}\left(e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) \right) = \frac{1}{2}(1+1) = 1 \end{aligned} $$

and the other is proved similarly.

2. (a) To find the complete solution to the recurrence equation

$$ f(n+3) - 8f(n+2) + 21f(n+1) - 18f(n) = 3^n \quad (n \in \mathbb{N}) \tag{3} $$

we first find the homogeneous solution. The left side may be rewritten $(E-3)^2(E-2)f(n)$, so the homogeneous solution is $f_h(n) = (an+b)3^n + c \cdot 2^n$ for some constants a, b, c.

A particular solution of the form $d \cdot 3^n$ won't work because 3^n is included in the homogeneous solution; instead we must try $d \cdot n^2\, 3^n$. (This is guaranteed to work, it's just a matter of finding d now.) Plugging this into the equation gives

$$(3^n \cdot ((n+3)^2 \cdot 3^3 - 8(n+2)^2 \cdot 3^2 + 21(n+1)^2 \cdot 3 - 18n^2)d = 18d \cdot 3^n = 3^n$$

so $d = 1/18$ and a particular solution is $f_p(n) = n^2\, 3^n/18$. The complete solution is

$$\boxed{\boldsymbol{f(n) = \left(\frac{n^2}{18} + an + b\right) 3^n + c \cdot 2^n \qquad \text{for some constants } a, b, c.}}$$

(b) Plug in the initial conditions:

$$\begin{aligned} 0 &= f(0) = b + c \\ \frac{1}{6} &= f(1) = \left(\frac{1}{18} + a + b\right) \cdot 3 + 2c \\ 2 &= f(2) = \left(\frac{4}{18} + 2a + b\right) \cdot 9 + 4c \end{aligned}$$

Solve the equations to get $a = b = c = 0$. The solution is then $\boxed{\boldsymbol{f(n) = \frac{n^2\, 3^n}{18}}}$.

(c) If $n \in \mathbb{R}$, then every "residue class modulo 1" is independent of the others, so the "constants" a, b, c are replaced by any functions $a(n), b(n), c(n)$ that have period 1. They don't even have to be continuous functions.

(d) If both sides of the recurrence are multiplied by $n - 100$, then the original equation needn't hold at $n = 100$. Thus, $f(103)$ may be chosen arbitrarily. We will have the solution as given in (a) for $n = 0, 1, \ldots, 102$, and a solution of the same form for $n \geq 103$ with new constants a', b', c' that will depend upon $a, b, c, f(103)$.

3. Sister Celine's method. These problems are on the maple printout `hw2.mws`.