Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler
Homework 2 — October 13, 1999

1. Facts about q-hypergeometric series. (They are all easy to prove.)
   (a) Here's a reason for the factor $\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}$. Prove the "confluence process"
   $$\lim_{\alpha_r \to \infty} {}_r\phi_s \left[\begin{matrix} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{matrix} \middle| q, \frac{x}{\alpha_r} \right] = {}_{r-1}\phi_s \left[\begin{matrix} \alpha_1, \ldots, \alpha_{r-1} \\ \beta_1, \ldots, \beta_s \end{matrix} \middle| q, x \right] .$$
   (b) Define
   $$e_q(x) = {}_1\phi_0 \left[\begin{matrix} 0 \\ - \end{matrix} \middle| q, x \right] = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}$$
   $$E_q(x) = {}_0\phi_0 \left[\begin{matrix} - \\ - \end{matrix} \middle| q, -x \right] = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} x^k = (-x;q)_\infty$$
   How do these follow from the q-binomial theorem, series form?
   (c) Also define
   $$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i} \qquad \mathrm{Sin}_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}$$
   $$\cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2} \qquad \mathrm{Cos}_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}$$
   Prove
   $$\lim_{q \to 1^-} e_q\left(x(1-q)\right) = \lim_{q \to 1^-} E_q\left(x(1-q)\right) = e^x$$
   $$e_q(x)\, E_q(-x) = 1$$
   $$\sin_q(x)\, \mathrm{Sin}_q(x) + \cos_q(x)\, \mathrm{Cos}_q(x) = 1$$
   $$\sin_q(x)\, \mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\, \cos_q(x) = 0$$

2. (a) Find the complete solution to the recurrence equation
   $$f(n+3) - 8f(n+2) + 21f(n+1) - 18f(n) = 3^n \quad (n \in \mathbb{N}) \tag{1}$$
   (b) Find the solution when $f(0) = 0$, $f(1) = \frac{1}{6}$, $f(2) = 2$.
   (c) What is the complete solution to (1) if $n \in \mathbb{R}$?
   (d) What is the complete solution (for $n \in \mathbb{N}$) if both sides of (1) are multiplied by $n - 100$?

3. Sister Celine's method.
   (a) Find the k-free recurrence for the summand of
   $$\sum_{k=0}^{n} \binom{n}{k} x^k$$
   and apply Sister Celine's method to evaluate the sum.
   (b) Use Koepf's software, as demonstrated in the worksheets `chap4.mws`, `exer4.mws`, and in the text of Chapter 4, to do these problems.
   Koepf # 4.1(2.2,2.3), 4.5, 4.11(a), 4.15(a,c), 4.18.