Math 262a — Topics in Combinatorics — Fall 1999 — Glenn Tesler

Homework 1 answers — October 6, 1999

Some of these problems are also on a maple worksheet, available on euclid:

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cd ~gptesler/homepage/math262/KOEPF/worksheetsV.4
xmaple hw1.mws &
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1. **Koepf 2.11(b)** The sum starts at $k = 1$ instead of $k = 0$. Shift it so the $k = 0$ term is nonzero:

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

Now the initial term is $t_0 = x$, and the term ratio is

$$r(k) = \frac{(-1)^{k+1}\, x^{k+2}}{k+2} \Big/ \frac{(-1)^k\, x^{k+1}}{k+1} x^{k+1} = \frac{k+1}{k+2}(-x) = \frac{(k+1)(k+1)}{(k+2)(k+1)}(-x)$$

so

$$\boxed{\ln(1+x) = x \cdot {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix}\,\middle|\, -x\right]}$$

Koepf 2.11(f) The initial term is $t_0 = x$, and the term ratio is

$$r(k) = \frac{(-1)^{k+1}\, x^{2k+3}}{2k+3} \Big/ \frac{(-1)^k\, x^{2k+1}}{2k+1} = -\frac{2k+1}{2k+3}x^2 = \frac{(k+1/2)(k+1)}{(k+3/2)(k+1)}(-x^2)$$

so

$$\boxed{\arctan(x) = x \cdot {}_2F_1\left[\begin{matrix}1/2,1\\3/2\end{matrix}\,\middle|\, -x^2\right]}$$

Note that $t_k \neq 0$ for integer $k < 0$. The bound $k = 0$ is unnatural; this manifests itself in our having to multiply by $\frac{k+1}{k+1}$ and then include a 1 as an upper parameter.

Koepf 2.9(d) The first binomial coefficient is 0 unless $0 \le k \le n$, and the second is 0 unless $n \le 2k$, so the summation range is actually $\lceil n/2 \rceil \le k \le n$.

The general term is $t_k = \dfrac{n!\,(2k)!}{k!\,(n-k)!\,n!\,(2k-n)!} = \dfrac{(2k)!}{k!\,(n-k)!\,(2k-n)!}$ so the term ratio is

$$r(k) = \frac{(2k+2)!}{(2k)!}\frac{k!}{(k+1)!}\frac{(n-k)!}{(n-k-1)!}\frac{(2k-n)!}{(2k+2-n)!}$$
$$= \frac{(2k+2)(2k+1)(n-k)}{(2k+2-n)(2k+1-n)(k+1)} = -\frac{(k+1/2)(k-n)}{(k+1-n/2)(k+(1-n)/2)}$$

Answer for non-integer n. This is what is produced by the software on the worksheet. The 0th term is $t_0 = \binom{n}{0}\binom{0}{n} = \binom{0}{n}$ since $\binom{n}{0} = \frac{n!}{n!\,0!} = 1$ for all $n \in \mathbb{C}$, but $\binom{0}{n}$ varies:

$$\binom{0}{n} = \frac{0!}{n!\,(-n)!} = \frac{1}{\Gamma(n+1)\,\Gamma(1-n)} = \frac{1}{n\,\Gamma(n)\,\Gamma(1-n)} = \frac{\sin(\pi n)}{\pi n}$$

using the reflection formula for Γ (Koepf, p. 6, (1.9)). Then the sum is

$$\boxed{\binom{0}{n}\; {}_3F_2\left[\begin{matrix}-n, 1/2, 1\\(1-n)/2, 1-n/2\end{matrix}\,\middle|\, -1\right]} \quad \text{for } n \in \mathbb{C}\setminus\mathbb{N}\,.$$

Of course, we're mainly interested in positive integer n, so this won't do. It's wrong for positive integers n because we multiply by $\binom{0}{n} = 0$, and then one of the denominator factors in the hypergeometric series will divide by 0 to compensate.

Non-negative integer n, Method I. The initial term evaluates to $t_0 = 0$, so we must shift the sum. (We could have done this in advance, but this will show how to do it algorithmically.) There are two cases: n even and n odd. When n is even, we should shift the sum down by $n/2$, and when it's odd, by $(n+1)/2$. Algorithmically, step 4 of Koepf page 21 says to let m be the smallest integer β and then shift by $1-m$. When n is even, $m = 1 - n/2$ and we shift by $n/2$ (i.e., the new sum index k' satisfies $k = k' + n/2$), and when n is odd, $m = (1-n)/2$ and we shift by $(n+1)/2$. (In both cases, we have shifted by $\lceil n/2 \rceil$ as we expect).

even n: Set $k = k' + n/2$. The sum is

$$\sum_{k'=0}^{n/2} \binom{n}{k'+n/2}\binom{2k'+n}{n}$$

and the upper bound is natural, so we can replace it by $\sum_{k'=0}^{\infty}$. The term ratio r_e in terms of the original ratio is

$$r_e(k') = r(k'+n/2) = -\frac{(k'+(n+1)/2)(k'-n/2)}{(k'+1)(k'+1/2)}$$

(notice the shift caused a $k'+1$ denominator factor). The initial term is $\binom{n}{n/2}\binom{n}{n} = \binom{n}{n/2}$, so the sum is

$$\boxed{\boldsymbol{\binom{n}{n/2}\ {}_2F_1\left[\begin{matrix}\frac{n+1}{2}, -\frac{n}{2}\\ 1/2\end{matrix}\ \middle|\ -1\right]}} \quad \text{for even } n \in \mathbb{N}\,.$$

odd n: Set $k = k' + (n+1)/2$. The sum is

$$\sum_{k'=0}^{(n+1)/2} \binom{n}{k'+(n+1)/2}\binom{2k'+n+1}{n}$$

and the upper bound is natural, so we can replace it by $\sum_{k'=0}^{\infty}$. The term ratio r_o is

$$r_o(k') = r(k'+(n+1)/2) = -\frac{(k'+n/2+1)(k'+(1-n)/2)}{(k'+3/2)(k'+1)}$$

and the initial term is $\binom{n}{(n+1)/2}\binom{n+1}{n} = n\binom{n}{(n+1)/2}$, so the sum is

$$\boxed{\boldsymbol{n\binom{n}{(n+1)/2}\ {}_2F_1\left[\begin{matrix}1+\frac{n}{2}, \frac{1-n}{2}\\ 3/2\end{matrix}\ \middle|\ -1\right]}} \quad \text{for odd } n \in \mathbb{N}\,.$$

Method II. There were two cases for the lower bound of the sum, but the upper bound was always n; so make a new sum $\sum u_{k'}$ where $u_{k'} = t_{n-k'}$:

$$\sum_{k'=0}^{\lfloor n/2 \rfloor} \binom{n}{n-k'}\binom{2(n-k')}{n}$$

instead. This sum can be taken as $\sum_{k'=0}^{\infty}$. The new term ratio $\tilde{r}$ in terms of the original is $\tilde{r}(k') = u_{k'+1}/u'_k = t_{n-k'-1}/t_{n-k'} = 1/r(n-k'-1)$, so

$$\tilde{r}(k') = -\frac{(n-k'-1+(1-n)/2)(n-k'-1+1-n/2)}{(n-k'-1-n)(n-k'-1+1/2)} = -\frac{(k'+(1-n)/2)(k'-n/2)}{(k'+1/2-n)(k'+1)}$$

The new initial term is $u_0 = \binom{2n}{n}$. So the whole sum is

$$\boxed{\boldsymbol{\binom{2n}{n}\ {}_2F_1\left[\begin{matrix}\frac{1-n}{2}, -\frac{n}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\ -1\right]}} \quad \text{for } n \in \mathbb{N}.$$

2. **Koepf 2.3**

$$F(x) = {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| x \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \; k!} x^k$$

Now $\theta x^k = x \cdot k x^{k-1} = k x^k$, so $(\theta + c)x^k = (k+c)x^k$. Thus every term of the sum is an eigenvector of $\theta + c$ for any constant c. The left side of (2.23) is

$$\begin{aligned} \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) F(x) &= \sum_{k=0}^{\infty} k \cdot (\beta_1 + k - 1) \cdots (\beta_q + k - 1) \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \; k!} x^k \\ &= \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_{k-1} \cdots (\beta_q)_{k-1} \; (k-1)!} x^k \end{aligned}$$

and the right side of (2.23) is

$$\begin{aligned} x(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) F(x) &= \sum_{k=0}^{\infty} (\alpha_1 + k) \cdots (\alpha_p + k) \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \; k!} x^{k+1} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1} \cdots (\alpha_p)_{k+1}}{(\beta_1)_k \cdots (\beta_q)_k \; k!} x^{k+1} \; . \end{aligned}$$

Just shift k by 1 to make them agree.

Koepf 2.4 Let n denote α_1, and let $\alpha_2, \dots, \alpha_p, \beta_1, \dots, \beta_q$ be given. We have

$$F_n(x) = {}_pF_q \left[\begin{matrix} n, \alpha_2, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| x \right] \quad \text{and} \quad \theta F_n(x) = \sum_{k=0}^{\infty} \frac{(n)_k \, (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \; k!} k \, x^k$$

while

$$n(F_{n+1}(x) - F_n(x)) = n \sum_{k=0}^{\infty} ((n+1)_k - (n)_k) \, \frac{(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \; k!} x^k$$

Now $(n+1)_k = (n+1)(n+2) \cdots (n+k) = (n+k)(n+1)_{k-1}$ and $(n)_k = n \cdot (n+1)_{k-1}$, so $(n+1)_k - (n)_k = k \cdot (n+1)_{k-1}$. Combine this with the additional factor of n to get the result.

The proof for other α's is the same, and the proof for $n = \beta_i$ is similar.

Koepf 2.5 Let's get a recursion for α_1. We may expand the left side of (2.23) as

$$\sum_{r=1}^{q+1} e_{r-1}(\beta_1 - 1, \dots, \beta_q - 1) \theta^r \, F_n(x) \tag{*}$$

where e_k is the elementary symmetric function. The right side may be expanded similarly.

Let E be the shift operator E_n. Then we make replacements

$$\theta^1 \, F_n = n(E - 1) \, F_n$$

$$\begin{aligned} \theta^2 \, F_n = n(\theta F_{n+1} - \theta F_n) &= n\Big((n+1)(F_{n+2} - F_{n+1}) - n(F_{n+1} - F_n) \Big) \\ &= n\Big((n+1) F_{n+2} - (2n+1) F_{n+1} + n \, F_n \Big) = \Big(n(n+1)E^2 - (2n+1)E + n \Big) \, F_n \end{aligned}$$

and so forth into (*). The resulting equation will have order $q+1$ on the left side and order p on the right side; combining all the terms onto one side will give an equation of order $\max(p, q+1)$. If $p = q+1$ it might seem that the highest order terms could cancel, but on the left we have θ^{q+1} and on the right, $x\theta^p$, so the leading coefficients don't cancel.

The denominator formulas are obtained similarly. We use E^{-1} instead of E, but the same orders will be obtained.

3. $\boldsymbol{e^x}$ Let $D = \frac{d}{dx}$, so $\theta = xD$. 2.3 gives $\theta\, F(x) = x\, F(x)$, so $xD\, F(x) = x\, F(x)$, and $D\, F(x) = F(x)$ unless $x = 0$. Thus, $\boxed{\boldsymbol{(D-1)F(x) = 0}}$; the $x = 0$ case is included by analyticity.

sin $\boldsymbol{x}$ Let $G(x) = \sin x = x\ {}_0F_1\left[\begin{matrix} - \\ 3/2 \end{matrix}\middle| -\frac{x^2}{4}\right]$. Now, consider $z = Ax^r$ and $\theta = z\frac{d}{dz}$ (here $z = (-\frac{1}{4})x^2$). By the chain rule,

$$\theta = z\frac{d}{dz} = \frac{z}{dz/dx}\frac{d}{dx} = \frac{A\,x^r}{r\,A\,x^{r-1}}\frac{d}{dx} = \frac{x}{r}\frac{d}{dx}$$

which here is $\theta = \frac{x}{2}\frac{d}{dx}$. Then by Koepf #2.3, $F(x) = G(x)/x$ satisfies

$$\theta(\theta + 3/2 - 1)F = z\,F(x).$$

Put it all in terms of x.

$$(x/2)D(xD/2 + 1/2)\,F = -\frac{x^2}{4}F \qquad \text{so} \quad xD(xD+1)\,F = -x^2\,F$$

The left side is tedious to simplify, but straightforward. Move all x's left and D's right, using the rules $Dx = xD + 1$ and more generally, $Df = fD + f'$, where x and f are multiplication operators. Later we may use software for dealing with noncommutative operators, but this isn't described in the textbooks we're now using. Here's the brute force way.

We have the operator

$$xDxD + xD = x(xD+1)D + xD = x^2D^2 + xD + xD = x^2D^2 + 2xD$$

giving $(x^2D^2 + 2xD + x^2)F = 0$. We actually want an equation for G where $F = x^{-1}G$, so $(x^2D^2 + 2xD + x^2)x^{-1}\,G = 0$. As operators,

$$Dx^{-1} = x^{-1}D - \frac{1}{x^2}$$

$$D^2x^{-1} = Dx^{-1}D - D\frac{1}{x^2} = (x^{-1}D - \frac{1}{x^2})D - \frac{1}{x^2}D + \frac{2}{x^3} = \frac{1}{x}D^2 - \frac{2}{x^2}D + \frac{2}{x^3}$$

Then as operators,

$$\left(\left[x^2D^2\right] + [2xD] + \left[x^2\right]\right)x^{-1} = \left[xD^2 - 2D + \frac{2}{x}\right] + \left[2D - \frac{2}{x}\right] + [x] = xD^2 + x$$

so $(xD^2 + x)G(x) = x(D^2+1)G(x) = 0$. So when $x \neq 0$ we have $\boxed{\boldsymbol{(D^2+1)G(x) = 0}}$, and by analyticity this holds at $x = 0$ too.

4. **Koepf 2.21**

(a) $(aq^n;q)_\infty = \prod_{j=n}^{\infty}(1-aq^j)$ and $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$, so $\dfrac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{j=0}^{n-1}(1-aq^j) = (a;q)_n$.

(b) $\dfrac{(q\sqrt{a};q)_n}{(\sqrt{a};q)_n} = \dfrac{1-\sqrt{a}\cdot q^n}{1-\sqrt{a}}$. The same holds with $\sqrt{a} \to -\sqrt{a}$. Multiplying gives

$$\frac{(q\sqrt{a};q)_n\,(-q\sqrt{a};q)_n}{(\sqrt{a};q)_n\,(-\sqrt{a};q)_n} = \frac{(1-\sqrt{a}\cdot q^n)(1+\sqrt{a}\cdot q^n)}{(1-\sqrt{a})(1+\sqrt{a})} = \frac{1-a\,q^{2n}}{1-a}\;.$$

(c)

$$(a;q)_n\,(-a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)(1+aq^j) = \prod_{j=0}^{n-1}(1-a^2(q^2)^j) = (a^2;q^2)_n$$

(d)

$$\begin{aligned}(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) &= \prod_{j=0}^{n-1}(-aq^j)(1-\frac{1}{aq^j}) \\ &= \left(\prod_{j=0}^{n-1}(-aq^j)\right)\prod_{j=0}^{n-1}(1-\frac{q^j}{aq^{n-1}}) = (-a)^n\, q^{\sum_{j=0}^{n-1} j}(q^{1-n}/a;q)_n \\ &= (q^{1-n}/a;q)_n(-a)^n q^{\binom{n}{2}}\end{aligned}$$

Koepf 3.15(b) The initial term is $t_0 = 1$. The term ratio is

$$R(k) = \frac{\begin{bmatrix} n \\ k+1\end{bmatrix}_q^2 x^{k+1}}{\begin{bmatrix} n \\ k\end{bmatrix}_q^2 x^k} = \left(\frac{(q;q)_n}{(q;q)_n}\frac{(q;q)_k}{(q;q)_{k+1}}\frac{(q;q)_{n-k}}{(q;q)_{n-k-1}}\right)^2\frac{x^{k+1}}{x^k} = \left(\frac{1-q^{n-k}}{1-q^{k+1}}\right)^2 x$$

Let $Q = q^k$ and express this as a rational function of Q in the proper form.
Since there is a $1-q^{k+1}$ denominator factor already, we don't have to introduce one.

$$R(k) = \left(\frac{1-q^n/Q}{1-qQ}\right)^2 x = \left(\frac{1-q^{-n}Q}{1-qQ}\cdot\frac{-q^n}{Q}\right)^2 x = \frac{(1-aQ)(1-aQ)}{(1-qQ)(1-qQ)}\cdot(q^{2n}x)\cdot(-Q)^{-2}$$

where $a = q^{-n}$. It looks like it may be a ${}_2\phi_1$, but we have the power $(-Q)^{-2}$. It is an ${}_r\phi_s$ with $1+s-r=-2$, so $s=1$ is O.K., but we should increase r to 4 by adding more 0 parameters in the numerator:

$$R(k) = \frac{(1-aQ)(1-aQ)(1-0Q)(1-0Q)}{(1-qQ)(1-qQ)}(q^{2n}x)(-Q)^{-2}$$

So the sum is

$$1\cdot {}_4\phi_1\left[\begin{matrix} q^{-n}, q^{-n}, 0, 0 \\ 1\end{matrix}\,\middle|\, q, q^{2n}x\right]$$