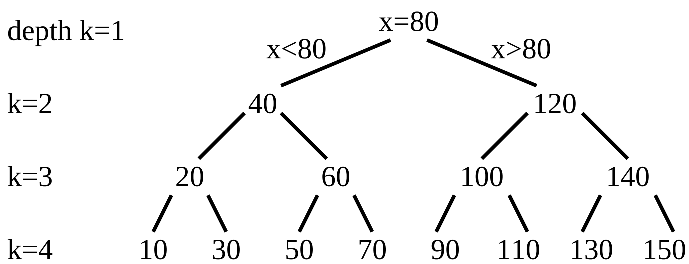
Math 188, Winter 2001, Prof. Tesler
Homework #3, Due January 31

Appendix A# 38, 39
Chapter 2# 1, 2, 6, 8, ~~9,~~ 19; **#9 is cancelled**
and the problem below: H-1

Problem H-1. In this problem, we will analyze the depth of the decision tree in Binary Search (Algorithms 1.5 and 2.1):
Inputs: Positive integer n, a sorted array of keys $S[1..n]$, and a key x.
Output: The location of x in the array, or 0 if it's not in the array.

Here is the decision tree for the input array $S = \langle 10, 20, 30, \cdots, 150\rangle$ with $n = 15$:



Let m be the maximum depth of the tree (in the example, $m = 4$). Answer all the questions on this page for the general case, not for this specific example.

The number of comparisons required to find a key x is its depth k if it's in the tree, or m or $m-1$ if it's not. The running time is roughly proportional to this number.

We will only consider searches for keys that are in S, and in (a)–(e) we will assume the bottom level is "full" (all leaves occur on level m; if it's not full, some are on level m and some on level $m-1$).

(a) What is the number of nodes at depth k, where $1 \le k \le m$ are as indicated in the diagram?

(b) Express the total number of nodes n as a function of m by summing up the answers for (a). Convert it to closed form by plugging suitable values into the geometric series

$$1 + r + r^2 + \cdots + r^j = \frac{1 - r^{j+1}}{1 - r}.$$

(c) Assume you are searching for a key that *is* present in the list S, and that all n keys are equally likely. What is the expected depth of the node containing x? You may use the sum

$$1 + 2r + 3r^2 + \cdots + j\,r^{j-1} = \frac{d}{dr}(1 + r + r^2 + \cdots + r^j) = \frac{d}{dr}\,\frac{1 - r^{j+1}}{1 - r} = \frac{r^j\,(j(r-1) - 1) + 1}{(1-r)^2}\ .$$

(The values plugged into it may differ from part (b).) Your final answer should be an exact answer.

(d) Reexpress your answer to (c) solely in terms of n (use the answer to (b) to determine how to substitute for m in terms of n).

(e) Give a simplified asymptotic form of the final answer to (d).

(f) Next we compute the expected depth recursively, for all n, without restricting to when the bottom level is full. Prove that the expected number of comparisons to locate a key is given by the recursion

$$\begin{aligned} A(0) &= 0 \\ A(1) &= 1 \\ A(n) &= \frac{1}{n}\left(\left\lfloor\frac{n-1}{2}\right\rfloor\left(1 + A\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right) + \left\lfloor\frac{n}{2}\right\rfloor\left(1 + A\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right) + 1\right) \qquad \text{for } n > 1. \end{aligned}$$